BULL. AUSTRAL. MATH. SOC. 47B47, 47A30
VOL. 67 (2003) [27–38]

ON THE NORM OF SYMMETRISED TWO-SIDED MULTIPLICATIONS

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The authors provide precise lower bounds for the completely bounded norm of the operator $T_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by $T_{a,b}(x) = axb + bxa$ and the injective norm of the corresponding tensor. Further, they compute the norm of the operator $x \mapsto a^*xb + b^*xa$ acting on the space of all conjugate-linear operators on H.

1. INTRODUCTION

Let H be a complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. An operator

$$\phi : \mathcal{B}(H) \to \mathcal{B}(H), \quad \phi(x) = \sum_{i=1}^{k} a_i x b_i, \tag{1}$$

where $a_i, b_i \in \mathcal{B}(H)$, is called an elementary operator. As proved by Haagerup in an unpublished manuscript and by Smith [**11**], the completely bounded norm of such an operator is equal to the Haagerup norm of $\sum a_i \otimes b_i$. Sometimes the usual norm of ϕ is equal to the completely bounded norm (in particular if we consider the operator ϕ acting on, say, the Calkin algebra, instead of $\mathcal{B}(H)$, see [**5**]), but in general there is no known simple expression for the norm of an elementary operator on $\mathcal{B}(H)$. (See [**9**] for a survey of this problem.) Besides the simplest case when $k = 1$ in (1), the best understood case is that of generalised derivations for which Stampfli [**14**] found an explicit formula for the norm on $\mathcal{B}(H)$ (see also the survey article by Fialkow [**2**] for more references).

For a slightly more general operator

$$T_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H), \quad T_{a,b}(x) = axb + bxa \tag{2}$$

no formula is known for computing the norm. Clearly $\|T_{a,b}\| \leqslant 2\,\|a\|\,\|b\|$, but in estimating the norm of $T_{a,b}$ in the opposite direction, it is not known what is the largest possible constant c such that

$$\|T_{a,b}\| \geqslant c\,\|a\|\,\|b\| \tag{3}$$

Received 18th February, 2002
The first author was supported by the Ministry of Science and Education of Slovenia, the second author was supported in part by the Grant J1-3125.

https://doi.org/10.1017/S0004972700033499 Published online by Cambridge University Press

for all $a, b \in \mathcal{B}(H)$. Mathieu [**7**] proved (3) with $c = 2/3$ and Stachó and Zalar [**12, 13**] improved this to $c = 2(\sqrt{2}-1)$ for general a and b and to $c = 1$ if a and b are self-adjoint. It was conjectured in [**6**, p. 497] that $c = 1$ in general. It turns out that it would be sufficient to prove this conjecture when a and b are 2×2 matrices with a diagonal and positive, but we have not been able to overcome the computational difficulties in this special case. In principle the problem is solvable by the decision procedure of Tarski [**15**] for inequalities involving polynomials of several variables (we are grateful to our colleague Marko Petkovšek and to Professor Adam Strzebonski from Wolfram Research for this information), but practically the problem seems too hard for the current computer implementations of this procedure.

Here we shall prove by a simple argument the estimate (3) with $c = 1$ for the completely bounded norm of $T_{a,b}$ instead of the usual norm. It is known that each positive elementary operator of length 2 (that is, $k = 2$ in (1)) is automatically completely positive (see [**3, 4, 8, 16**]); in contrast to this the completely bounded norm of such an operator can be different from the usual norm even in the case of $T_{a,b}$ (Example 4.9).

For the injective tensor norm $\|\cdot\|_\lambda$ a very simple argument will show us that

$$\|a \otimes b + b \otimes a\|_\lambda \geqslant c \|a\| \, \|b\|$$

with the best possible constant $c = 2(\sqrt{2}-1)$. By the minimality of the injective tensor norm this implies the above mentioned result of [**12**].

When a, b are self-adjoint, and H real or $\dim H = 2$ if H is complex, the norm of $T_{a,b}$ can be computed explicitely. This is a consequence of the main result here (Theorem 4.2) which provides a simple formula for the norm of the *symmetrised two-sided multiplication* operator

$$S_{a,b} : \overline{\mathcal{B}}(H) \to \overline{\mathcal{B}}(H), \quad S_{a,b}(x) = a^* x b + b^* x a, \tag{4}$$

where $\overline{\mathcal{B}}(H)$ is the space of all conjugate-linear bounded operators on H. The operator $S_{a,b}$ seems more accessible and natural than $T_{a,b}$ since it preserves the space of all self-adjoint operators in $\mathcal{B}(H)$.

We conclude this introduction by recalling some notation and definitions. Any map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ induces a family of maps $\phi_n : M_n(\mathcal{B}(H)) \to M_n(\mathcal{B}(H))$, $n \geqslant 1$, defined by

$$\phi_n([x_{ij}]) = [\phi(x_{ij})]$$

for any matrix $[x_{ij}] \in M_n(\mathcal{B}(H))$. If $\sup_n \|\phi_n\|$ is finite then ϕ is said to be completely bounded, and this supremum defines the *completely bounded norm* $\|\phi\|_{cb}$ of ϕ. (Here, of course, the norm in $M_n(\mathcal{B}(H))$ is given via the identification $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$.) (We refer to [**1**] or [**10**] for more on completely bounded mappings.)

The Haagerup norm on the algebraic tensor product $\mathcal{B}(H) \otimes \mathcal{B}(H)$ is defined by

$$\|\phi\|_h = \inf \left\| \sum_{i=1}^{k} a_i a_i^* \right\|^{1/2} \cdot \left\| \sum_{i=1}^{k} b_i^* b_i \right\|^{1/2}, \tag{5}$$

where the infimum is over all possible representations of ϕ in the form $\phi = \sum_{i=1}^{k} a_i \otimes b_i$ (see [1]).

By the natural map

$$\Theta : \mathcal{B}(H) \otimes \mathcal{B}(H) \to \mathcal{CB}(\mathcal{B}(H)), \quad \Theta\Big(\sum\nolimits_i a_i \otimes b_i\Big)(x) = \sum\nolimits_i a_i x b_i$$

we may algebraically identify $\mathcal{B}(H) \otimes \mathcal{B}(H)$ with the space of all elementary operators on $\mathcal{B}(H)$. As we already mentioned, for each $\phi \in \mathcal{B}(H) \otimes \mathcal{B}(H)$ the completely bounded norm of $\Theta(\phi)$ is equal to the Haagerup norm of ϕ.

2. An estimate for the completely bounded norm of $T_{a,b}$

Let M_2 denote the algebra of complex 2×2 matrices.

Theorem 2.1. *The inequality*

$$\|T_{a,b}\|_{cb} \geqslant \|a\| \, \|b\|$$

holds for all $a, b \in \mathcal{B}(H)$.

Proof: First assume that $\dim H = 2$ and identify $\mathcal{B}(H)$ with M_2. Let $\underline{a} = [a, b]$, $\underline{b} = [b, a]^t$. We shall use the notation $\underline{a} \odot \underline{b} = a \otimes b + b \otimes a$. It suffices to prove that the Haagerup norm of $\underline{a} \odot \underline{b}$ satisfies $\|\underline{a} \odot \underline{b}\|_h \geqslant \|a\| \, \|b\|$. Multiplying a by a suitable constant t and b by $1/t$ we may assume first that $\|a\| = \|b\|$ and then (normalising) that $\|a\| = 1 = \|b\|$. Note that $\underline{a}\Lambda^{-1} \odot \Lambda \underline{b} = \underline{a} \odot \underline{b}$ for each invertible matrix $\Lambda \in M_2$; moreover, it follows from [1, Lemma 9.2.3] that

$$\|\underline{a} \odot \underline{b}\|_h = \inf \left\|\underline{a}\Lambda^{-1}\right\| \|\Lambda \underline{b}\|, \tag{6}$$

where the infimum is over all invertible matrices $\Lambda \in M_2$. Furthermore, since for each unitary 2×2 matrix u we have that $\|\underline{a}u\| = \|\underline{a}\|$ and similarly for columns, by using the polar decomposition, it suffices to take in (6) the infimum over all positive matrices Λ only, and clearly we may also assume that $\det \Lambda = 1$. Thus, we have to prove that

$$\left\|\underline{a}\Lambda^{-1}\right\|^2 \|\Lambda \underline{b}\|^2 \geqslant 1 \tag{7}$$

for all positive $\Lambda \in M_2$ with $\det \Lambda = 1$. So let

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix}, \quad \alpha\gamma - |\beta|^2 = 1, \quad \alpha, \gamma \geqslant 0.$$

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Then

$$\underline{a}\Lambda^{-1} \odot \Lambda\underline{b} = (\gamma a - \overline{\beta} b) \otimes (\beta a + \alpha b) + (-\beta a + \alpha b) \otimes (\gamma a + \overline{\beta} b).$$

To simplify the notation put

$$A = |\beta|^2 + \gamma^2, \quad B = \beta(\alpha + \gamma), \quad C = \alpha^2 + |\beta|^2.$$

Then (7) can be written as

$$\|Aaa^* - 2\,\mathrm{Re}(Bab^*) + Cbb^*\| \cdot \|Aa^*a + 2\,\mathrm{Re}(Bb^*a) + Cb^*b\| \geqslant 1. \tag{8}$$

We may assume that $A \geqslant C$ (the case $C \geqslant A$ is treated in the same way). Then, noting that $\|T_{a,b}\|_{cb} = \|T_{uav,ubv}\|_{cb}$ for all unitary $u, v \in M_2$, we may replace a and b by $|a|$ and u^*b, respectively, where $a = u\,|a|$ is the polar decomposition of a. In other words, we may assume that a is positive. So, a and b are of the form $a = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}$ and $b = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}$, where $h \in [0,1]$ and $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{C}$. Then

$$Aaa^* - 2\,\mathrm{Re}(Bab^*) + Cbb^* = \begin{bmatrix} A - 2\,\mathrm{Re}(B\overline{\beta}_1) + C\big(|\beta_1|^2 + |\beta_2|^2\big) & * \\ * & * \end{bmatrix},$$

and

$$Aa^*a + 2\,\mathrm{Re}(Bb^*a) + Cb^*b = \begin{bmatrix} A + 2\,\mathrm{Re}(B\overline{\beta}_1) + C\big(|\beta_1|^2 + |\beta_3|^2\big) & * \\ * & * \end{bmatrix}.$$

The fact that $\det \Lambda = 1$ implies (by a simple computation) that $|B|^2 - AC = -1$, hence, since $C \geqslant 0$, we have that $A \pm 2\,|B|\,|\beta_1| + C\,|\beta_1|^2 \geqslant 0$. Since $A \geqslant C$ and $AC = 1 + |B|^2$, we also have $A \geqslant 1$ and it follows that

$$\begin{aligned}
\Big(A - 2\,\mathrm{Re}(B\overline{\beta}_1) &+ C\big(|\beta_1|^2 + |\beta_2|^2\big)\Big)\Big(A + 2\,\mathrm{Re}(B\overline{\beta}_1) + C\big(|\beta_1|^2 + |\beta_3|^2\big)\Big) \\
&\geqslant \big(A + C\,|\beta_1|^2 - 2\,\mathrm{Re}(B\overline{\beta}_1)\big)\big(A + C\,|\beta_1|^2 + 2\,\mathrm{Re}(B\overline{\beta}_1)\big) \\
&= \big(A + C\,|\beta_1|^2\big)^2 - 4\big(\mathrm{Re}(B\overline{\beta}_1)\big)^2 \geqslant \big(A + C\,|\beta_1|^2\big)^2 - 4\,|B|^2\,|\beta_1|^2 \\
&= \big(A + C\,|\beta_1|^2\big)^2 - 4(AC - 1)\,|\beta_1|^2 = \big(A - C\,|\beta_1|^2\big)^2 + 4\,|\beta_1|^2 \\
&\geqslant A^2\big(1 - |\beta_1|^2\big)^2 + 4\,|\beta_1|^2 \geqslant \big(1 - |\beta_1|^2\big)^2 + 4\,|\beta_1|^2 = \big(1 + |\beta_1|^2\big)^2.
\end{aligned}$$

Since the norm of each matrix always dominates the maximal absolute value of its entries, this proves (8) and the theorem when $\dim H = 2$.

The case when $\dim H > 2$ can be reduced to the case just proved as follows. Let $\varepsilon > 0$ and choose unit vectors $\xi, \eta \in H$ such that $\|a\xi\| \geqslant \|a\| - \varepsilon$ and $\|b\eta\| \geqslant \|b\| - \varepsilon$. Let K_1 be two dimensional space containing ξ and η, and let K_2 be two dimensional space containing $a\xi$ and $b\eta$. Furthermore, let $p \in \mathcal{B}(H)$ be the orthogonal projection onto K_1, and let $q \in \mathcal{B}(H)$ be a partial isometry with the final space K_1 and the initial space K_2. Then $\|qap\| \geqslant \|a\| - \varepsilon$ and $\|qbp\| \geqslant \|b\| - \varepsilon$. It is easy to verify that $\|T_{a,b}\|_{cb} \geqslant \|T_{qap,qbp}\|_{cb}$, hence, regarding qap and qbp as operators on the two dimensional space K_1, it follows from what we have already proved that

$$\|T_{a,b}\|_{cb} \geqslant \|qap\|\,\|qbp\| \geqslant \big(\|a\| - \varepsilon\big)\big(\|b\| - \varepsilon\big).$$

Finally, to complete the proof, let $\varepsilon \to 0$. ∎

3. An estimate for the injective tensor norm of $a \otimes b + b \otimes a$

Recall ([**1**]) that the *injective norm* on the tensor product $E \otimes F$ of Banach spaces is defined for each $w = \sum_{i=1}^{k} a_i \otimes b_i \in E \otimes F$ by

$$\|w\|_\lambda = \sup\Big\{\big|(f \otimes g)(w)\big| : f \in E^*, \|f\| = 1; g \in F^*, \|g\| = 1\Big\},$$

where E^* denotes the dual of E and $(f \otimes g)(w) := \sum_{i=1}^{k} f(a_i)g(b_i)$. In other words, denoting by $(E^*)_1$ the unit ball of E^* and associating with each $w \in E \otimes F$ the (continuous) function $\widehat{w}$ on $(E^*)_1 \times (F^*)_1$ by $\widehat{w}(f,g) = (f \otimes g)(w)$, $\|w\|_\lambda$ is just the supremum norm of $\widehat{w}$.

Since the injective norm is the minimal reasonable tensor cross norm, the following proposition immediately implies the main result of [**12**].

Proposition 3.1. *Let $a, b \in \mathcal{B}(H)$ and let $\tau_{a,b} = a \otimes b + b \otimes a$. Then*

$$\|\tau_{a,b}\|_\lambda \geqslant 2\big(\sqrt{2} - 1\big)\,\|a\|\,\|b\|.$$

Proof: We may assume that $\|a\| = \|b\| = 1$ and regard a, b as functions on $\Delta := \big(\mathcal{B}(H)^*\big)_1$ and $\tau_{a,b}$ as a function on $\Delta \times \Delta$ in the usual way. Multiplying a and b by suitable scalars of modulus 1, we may assume that $a(s_0) = 1$ and $b(t_0) = 1$ for some $s_0, t_0 \in \Delta$. Put $a_1 = a(t_0)$ and $b_1 = b(s_0)$. Then we have

$$\tau_{a,b}(s_0, s_0) = 2b_1, \quad \tau_{a,b}(t_0, t_0) = 2a_1, \quad \tau_{a,b}(s_0, t_0) = 1 + a_1 b_1.$$

If $|a_1|$ or $|b_1|$ is greater or equal than $\sqrt{2} - 1$, we are done. So suppose that $|a_1| < \sqrt{2} - 1$ and $|b_1| < \sqrt{2} - 1$. Then

$$|1 + a_1 b_1| > 1 - \big(\sqrt{2} - 1\big)^2 = 2\big(\sqrt{2} - 1\big),$$

and the proof is completed. ∎

Remark 3.1. It is easy to see that the constant $2\big(\sqrt{2} - 1\big)$ in Proposition 3.1 can not be improved; consider, for example the diagonal matrices

$$a = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} - 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -\big(\sqrt{2} - 1\big) & 0 \\ 0 & 1 \end{bmatrix}.$$

The following proposition (together with Proposition 3.1 and Remark 3.1) implies that in general $T_{a,b}$ does not attain its norm on operators of rank 1.

PROPOSITION 3.2. *For each* $w = \sum_{i=1}^{k} a_i \otimes b_i \in \mathcal{B}(H) \otimes \mathcal{B}(H)$ *we have that*

$$\|w\|_\lambda = \sup\left\{\left\|\sum_{i=1}^{k} a_i x b_i\right\| : x \in \mathcal{B}(H), \|x\| = 1, \operatorname{rank}(x) = 1\right\}.$$

PROOF: Put $w(x) = \sum_{i=1}^{k} a_i x b_i$ and $\|w\|_\beta = \sup\Big\{\|w(x)\| : x \in \mathcal{B}(H), \|x\| = 1, \operatorname{rank}(x) = 1\Big\}$. Since each rank 1 operator $x \in \mathcal{B}(H)$ is of the form $x = \nu \otimes \overline{\zeta}$ for some $\nu, \zeta \in H$, we have

$$\begin{aligned}
\|w\|_\beta &= \sup\left\{\left|\sum_{i=1}^{k} \langle a_i x b_i \xi, \eta\rangle\right| : \|x\| = 1, \operatorname{rank}(x) = 1, \|\xi\| = \|\eta\| = 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^{k} \langle a_i \nu, \eta\rangle \langle b_i \xi, \zeta\rangle\right| : \|\zeta\| = \|\eta\| = \|\nu\| = \|\xi\| = 1\right\} \\
&= \sup\left|\sum_{i=1}^{k} f(a_i) g(b_i)\right|,
\end{aligned}$$

where the last supremum is taken over all functionals of the form $f = \nu \otimes \overline{\eta}$, $g = \xi \otimes \overline{\zeta}$. Since each element in the predual $\mathcal{T}(H)$ of $\mathcal{B}(H)$ is a norm limit of convex combinations of elements of the form $\nu \otimes \overline{\eta}$ and the unit ball of $\mathcal{T}(H)$ is weak* dense in the unit ball of the dual of $\mathcal{B}(H)$, it follows that $\|w\|_\beta$ is equal to the injective norm $\|w\|_\lambda$. ∎

4. THE NORM OF THE OPERATOR $x \mapsto a^* x b + b^* x a$ ON $\overline{\mathcal{B}}(H)$

Let $W(a)$ and $w(a)$ be the spatial numerical range and the numerical radius, respectively, of an operator $a \in \mathcal{B}(H)$.

LEMMA 4.1. *Let H be a finite-dimensional Hilbert space and let $a, b \in \mathcal{B}(H)$. Then*

$$w(a^*a + b^*b) = \min_{t>0} w\left(ta^*a + \frac{1}{t}b^*b\right) \tag{9}$$

if and only if there exists a unit vector $\xi \in H$ such that

$$\|a\xi\|^2 = \|b\xi\|^2 = \frac{1}{2} w(a^*a + b^*b). \tag{10}$$

PROOF: We may assume that $w(a^*a + b^*b) = 1$. If (10) is satisfied, then

$$\left\langle \left(ta^*a + \frac{1}{t}b^*b\right)\xi, \xi \right\rangle = t\,\|a\xi\|^2 + \frac{1}{t}\,\|b\xi\|^2 = \frac{1}{2}\left(t + \frac{1}{t}\right) \geqslant 1.$$

Conversely, let us assume that (9) holds. Put $K = \ker(a^*a + b^*b - 1)$ and let $p \in \mathcal{B}(H)$ be orthogonal projection onto K. Put $s_n = 1/n$, $n \geqslant 2$. By (9) there exists a sequence $\{\eta_n\}$ of unit vectors in H such that

$$\left\langle \left((1 - s_n)a^*a + \frac{1}{1 - s_n} b^*b\right)\eta_n, \eta_n \right\rangle \geqslant w(a^*a + b^*b) = 1 \tag{11}$$

for each $n \geqslant 2$. Put $c = a^*a + b^*b$ and $d = b^*b - a^*a$. Then from (11) we get

$$\langle c\eta_n, \eta_n\rangle + s_n\langle d\eta_n, \eta_n\rangle + \frac{s_n^2}{1 - s_n}\langle b^*b\eta_n, \eta_n\rangle \geqslant 1. \tag{12}$$

Since H is finite dimensional, the unit ball of H is compact and so there is a convergent subsequence of $\{\eta_n\}$. Denote this subsequence again by $\{\eta_n\}$ and let $\eta = \lim_n \eta_n$. Then from (12) and $\langle c\eta, \eta\rangle \leqslant 1$ it follows that $\langle c\eta, \eta\rangle = 1$; hence $c\eta = \eta$ (since $\|c\| = 1$) and so $\eta \in K$. From (12) it also follows that $s_n\langle d\eta_n, \eta_n\rangle + (s_n^2)/(1 - s_n)\langle b^*b\eta_n, \eta_n\rangle \geqslant 0$, so dividing by s_n,

$$\langle d\eta_n, \eta_n\rangle + \frac{s_n}{1 - s_n}\langle b^*b\eta_n, \eta_n\rangle \geqslant 0.$$

Letting $n \to \infty$ we conclude that $\langle d\eta, \eta\rangle \geqslant 0$. In the same way, starting from the sequence $t_n = -(1/n)$ instead of $s_n = 1/n$, we obtain a unit vector $\nu \in K$ such that $\langle d\nu, \nu\rangle \leqslant 0$. Since the numerical range is convex, from $\langle d\eta, \eta\rangle \geqslant 0$ and $\langle d\nu, \nu\rangle \leqslant 0$ it follows that $0 \in W(pd|_K)$. So there exists a unit vector $\xi \in K$ such that $\big\langle (b^*b - a^*a)\xi, \xi \big\rangle = 0$. This together with $(a^*a + b^*b)\xi = \xi$ implies that $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$ and the proof is completed. ∎

Remember that $\overline{\mathcal{B}}(H)$ denotes the space of all bounded conjugate-linear operators on H and $S_{a,b} : \overline{\mathcal{B}}(H) \to \overline{\mathcal{B}}(H)$ is the operator defined by $S_{a,b}(x) = a^*xb + b^*xa$. Denote by $\overline{\mathcal{B}}(H)_{sa}$ self-adjoint operators in $\overline{\mathcal{B}}(H)$.

Theorem 4.2. *For all $a, b \in \mathcal{B}(H)$ we have that*

$$\|S_{a,b}\| = \left\| S_{a,b}|_{\overline{\mathcal{B}}(H)_{sa}} \right\| = \min_{t>0} \left\| ta^*a + \frac{1}{t} b^*b \right\|.$$

Proof: We may assume that $\|a^*a + b^*b\| = 1$. Furthermore, since $S_{a,b} = S_{ta,(1/t)b}$ for all scalars $t \neq 0$, we may assume that $\min_{t>0}\|ta^*a + (1/t)b^*b\| = \|a^*a + b^*b\| = 1$. Suppose first that H is finite-dimensional. Then by Lemma 4.1 there exists a unit vector ξ satisfying $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$, hence on the linear span $\mathcal{L}$ of $\{a\xi, b\xi\}$ we can define a conjugate-linear isometry x by $xa\xi = b\xi$ and $xb\xi = a\xi$. By choosing a conjugate-linear symmetry on $\mathcal{L}^\perp$, we can extend x to a conjugate-linear operator on H such that $x = x^*$ and $x^2 = 1$. Then $\left|\big\langle (a^*xb + b^*xa)\xi, \xi \big\rangle\right| = 1$, hence $\|S_{a,b}\| \geqslant 1$. Since $\|S_{a,b}\| \leqslant \|S_{a,b}\|_{cb} \leqslant \min_{t>0}\|ta^*a + (1/t)b^*b\| = 1$, this completes the proof when H is finite-dimensional.

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If H is infinite-dimensional, let $\{p_n\}$ be a net of finite rank orthogonal projections increasing to the identity. Denote by a_n the restriction of $p_n a$ to the range of p_n, and analogously for b. For each n let t_n be such that $\min_{t>0}\|ta_n^*a_n + (1/t)b_n^*b_n\| = \|t_n a_n^* a_n + (1/t_n)b_n^*b_n\|$. Then we have

$$\begin{aligned}\|S_{a,b}\| &= \sup_{\|x\|=1}\|a^*xb + b^*xa\| \geqslant \sup_{\|x\|=1}\|a^*p_nxp_nb + b^*p_nxp_na\| \\ &\geqslant \sup_{\|x\|=1}\|(p_na^*p_n)(p_nxp_n)(p_nbp_n) + (p_nb^*p_n)(p_nxp_n)(p_nap_n)\| \\ &= \left\|t_na_n^*a_n + \frac{1}{t_n}b_n^*b_n\right\|.\end{aligned}$$

Passing to a subnet, if necessary, assume that $t_n \to t_0$. Then

$$\lim_n\left\|t_na_n^*a_n + \frac{1}{t_n}b_n^*b_n\right\| = \left\|t_0a^*a + \frac{1}{t_0}b^*b\right\| \geqslant \min_{t>0}\left\|ta^*a + \frac{1}{t}b^*b\right\|.$$

Hence,

$$\|S_{a,b}\| \geqslant \min_{t>0}\left\|ta^*a + \frac{1}{t}b^*b\right\|.$$

Since the reverse inequality is clear, the theorem is proved. ∎

Proposition 4.3. *Let $R_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ be (real) linear mapping defined by $R_{a,b}(x) = a^*xb + b^*x^*a$. Then*

$$\|R_{a,b}\| = \min_{t>0}\left\|ta^*a + \frac{1}{t}b^*b\right\|.$$

Proof: The proof is very similar to the previous one, so we shall skip the details. Choose a unit vector ξ satisfying the condition (10) in Lemma 4.1 and a unitary operator x such that $xb\xi = a\xi$. Then $\|R_{a,b}\| \geqslant \langle (a^*xb + b^*x^*a)\xi, \xi\rangle = w(a^*a + b^*b) = \|a^*a + b^*b\|$. For the reverse inequality note that

$$\|R_{a,b}(x)\| = \left\|\begin{bmatrix} a^* & b^* \\ 0 & 0\end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & x^*\end{bmatrix}\begin{bmatrix} b & 0 \\ a & 0\end{bmatrix}\right\| \leqslant \|a^*a + b^*b\|\,\|x\| \quad (x \in \mathcal{B}(H)).$$

∎

Proposition 4.4. *Let $a, b \in \mathcal{B}(H)$ be self-adjoint. If H is real or* $\dim H = 2$, *then*

$$\|T_{a,b}\| = \|T_{a,b}|_{\mathcal{B}(H)_{sa}}\| = \min_{t>0}\left\|ta^2 + \frac{1}{t}b^2\right\|.$$

Proof: If H is real this is just Theorem 4.2 for real scalars. So let H be complex with $\dim H = 2$. Choose an orthonormal basis $\{\eta_1, \eta_2\}$ of H relative to which a is diagonal. Since b is self-adjoint, the diagonal entries of b are real, and the two (in general complex conjugate) off-diagonal entries of b can be made real by replacing η_2 with $\theta\eta_2$ for an appropriate scalar θ of modulus 1. Thus, we may assume that a and

b are real matrices. As in the proof of Theorem 4.2, we may assume that $\min_{t>0}\|ta^2 + (1/t)b^2\| = \|a^2+b^2\| = 1$. Then from Lemma 4.1 we obtain a unit vector ξ such that $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$. Furthermore, ξ is an eigenvector of the real symmetric matrix $(a^2+b^2)\xi = \xi$ (corresponding to the eigenvalue 1), hence ξ is real. Then $\langle a\xi, b\xi\rangle \in \mathbb{R}$ and we can find a unitary self-adjoint matrix x satisfying $xa\xi = b\xi$ and $xb\xi = a\xi$. The rest of the proof is the same as in Theorem 4.2 and will be omitted. ∎

COROLLARY 4.5. *If $a, b \in M_2$ are self-adjoint, then*

$$\|T_{a,b}\|_{cb} = \|T_{a,b}\|.$$

PROOF: By Proposition 4.4 we have

$$\min_{t>0}\left\|ta^2 + \frac{1}{t}b^2\right\| \geqslant \|T_{a,b}\|_{cb} \geqslant \|T_{a,b}\| = \min_{t>0}\left\|ta^2 + \frac{1}{t}b^2\right\|,$$

hence $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$. ∎

The main result in [**13**] states that, whenever $a, b \in \mathcal{B}(H)$ are self-adjoint, $\left\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\right\| \geqslant \|a\|\,\|b\|$. The following estimate is sharper.

COROLLARY 4.6. *Let $a, b \in \mathcal{B}(H)$ be self-adjoint. Then*

$$\left\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\right\| \geqslant \sup_{\substack{p=p^*=p^2\\ \operatorname{rank}(p)=2}} \min_{t>0}\left\|t(pap)^2 + \frac{1}{t}(pbp)^2\right\| \geqslant \|a\|\,\|b\|.$$

PROOF: The first inequality follows immediately from Proposition 4.4 since

$$\left\|T_{a,b}|\mathcal{B}(H)_{sa}\right\| \geqslant \left\|T_{pap,pbp} \mid \mathcal{B}(pH)_{sa}\right\|$$

for each projection $p \in \mathcal{B}(H)$. To prove the second inequality, we may assume that $\|a\| = \|b\| = 1$. Note that if $t \geqslant 1$ then $\left\|t(pap)^2 + (1/t)(pbp)^2\right\| \geqslant \|pap\|^2$ and $\|pap\|$ approximates $\|a\|$ when p is the projection to the span of $\{\xi, a\xi\}$, where ξ is a vector on which a almost achieves its norm. A similar argument is available if $(1/t) \geqslant 1$. ∎

For 2×2 matrices we have a better estimate. Denote by $\|\cdot\|_2$ the Hilbert-Schmidt norm.

COROLLARY 4.7. *If $a, b \in M_2$ are self-adjoint, then*

$$\left\|T_{a,b}|_{(M_2)_{sa}}\right\| \geqslant \|a\|_2\,\|b\|_2.$$

PROOF: We may assume that $\min_{t>0}\|ta^2 + (1/t)b^2\| = \|a^2+b^2\|$. Put $m = \|a^2+b^2\|$. By Lemma 4.1 there exists a unit vector ξ satisfying $(a^2+b^2)\xi = \xi$ and $\|a\xi\|^2 = \|b\xi\|^2 = m/2$. Let $\xi^\perp$ be a unit vector orthogonal to ξ and put $c = \left\|a\xi^\perp\right\|^2$. Since $a^2+b^2 \leqslant m1$, we have $\left\|b\xi^\perp\right\|^2 \leqslant m - c$. From

$$\|a\|_2^2 = \|a\xi\|^2 + \left\|a\xi^\perp\right\|^2 = \frac{1}{2}m + c$$

and

$$\|b\|_2^2 = \|b\xi\|^2 + \|b\xi^\perp\|^2 \leqslant \frac{3}{2}m - c,$$

it follows that

$$\|a\|_2^2\|b\|_2^2 \leqslant \left(\frac{1}{2}m + c\right)\left(\frac{3}{2}m - c\right) \leqslant m^2 = \left\|T_{a,b}|_{(M_2)_s}\right\|^2. \qquad \square$$

Clearly, the inequality $\|T_{a,b}\| \geqslant \|a\|_2\|b\|_2$ can not be generalised to self-adjoint $n \times n$ matrices for $n > 2$. As an example, consider the 3×3 diagonal matrices $a = \operatorname{diag}(1,1,0)$ and $b = \operatorname{diag}(0,0,1)$. (In this example we also have that $\|T_{a,b}\|_{cb} = 1 = \|a\|\,\|b\|$, hence the estimate in Theorem 2.1 can not be improved.)

EXAMPLE 4.8. If H is complex and $\dim H > 2$, then $\min_{t>0} w(ta^2 + b^2/t)$ can be greater than $\left\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\right\|$. To see this, first observe the following.

If H is finite dimensional and $a, b \in \mathcal{B}(H)$ are such that

$$\left\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\right\| = w(a^2 + b^2) = 1,$$

then there exists a unit vector $\xi \in H$ such that $(a^2 + b^2)\xi = \xi$ and $\langle a\xi, b\xi\rangle \in \mathbb{R}$.

Indeed, choose $x = x^* \in \mathcal{B}(H)$ with $\|x\| = 1$ and a unit vector $\xi \in H$ such that $\left|\left\langle (axb + bxa)\xi, \xi\right\rangle\right| = 1$. Using the fact that equality holds in the Schwarz inequality only if the two vectors are linearly dependent, we deduce from

$$\begin{aligned} 1 &= \left|\left\langle (axb + bxa)\xi, \xi\right\rangle\right| = \left|\langle xb\xi, a\xi\rangle + \langle xa\xi, b\xi\rangle\right| \leqslant 2\,\|a\xi\|\,\|b\xi\| \\ &\leqslant \|a\xi\|^2 + \|b\xi\|^2 = \left\langle (a^2 + b^2)\xi, \xi\right\rangle \leqslant 1 \end{aligned}$$

that $(a^2 + b^2)\xi = \xi$ and then $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$ and $xb\xi = \beta a\xi$, $xa\xi = \alpha b\xi$ for some complex numbers α and β of modulus 1. Then from $\left|\langle xb\xi, a\xi\rangle + \langle xa\xi, b\xi\rangle\right| = \left|(1/2)\alpha + (1/2)\beta\right| = 1$ it follows that $\alpha = \beta$. Since x is a contraction, we have $\left\|x(a\xi + \lambda b\xi)\right\| \leqslant \|a\xi + \lambda b\xi\|$ for each complex number λ. But this is equivalent to the condition $\operatorname{Re}\left(\lambda\langle a\xi, b\xi\rangle\right) \leqslant \operatorname{Re}\left(\lambda\langle b\xi, a\xi\rangle\right)$, which implies $\langle a\xi, b\xi\rangle \in \mathbb{R}$.

Now let

$$a = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Then

$$a^2 + b^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{i}{4} \\ 0 & -\frac{i}{4} & \frac{1}{4} \end{bmatrix}$$

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and since the norm of 2×2 matrix in the lower right corner of $a^2 + b^2$ is $(2+\sqrt{2})/4 < 1$, we have $\|a^2 + b^2\| = 1$. If $\left\|T_{a,b}|_{B(H)_{sa}}\right\| = 1$, then by the above observation we would have $\langle a\xi, b\xi\rangle \in \mathbb{R}$, where $\xi = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^t$ is the only eigenvector of $a^2 + b^2$ corresponding to the eigenvalue $1 = w(a^2 + b^2)$. However, in our case $\langle a\xi, b\xi\rangle = -(i/2\sqrt{2}) \notin \mathbb{R}$.

In view of Corollary 4.5 one may ask if $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ for all 2×2 matrices a and b. The following example shows that this is not the case.

EXAMPLE 4.9. Put $a = e - iu$ and $b = (e + iu)/2$, where

$$e = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad u = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

Let $x = [x_{ij}]$. Then

$$T_{a,b}(x) = \begin{bmatrix}x_{11} & x_{21}\\ 0 & 0\end{bmatrix},$$

so $\|T_{a,b}\| = 1$. We shall show that $\|T_{a,b}\|_{cb} = \sqrt{2}$. First we note that $T_{a,b}(x) = axb + bxa = exe + uxu$. Furthermore, as in the proof of Theorem 2.1, denote by $\Lambda = \begin{bmatrix}\alpha & \beta\\ \overline{\beta} & \gamma\end{bmatrix}$ a positive matrix with $\det\Lambda = 1$. Let $A = |\beta|^2 + \gamma^2$, $B = \beta(\alpha + \gamma)$, $C = \alpha^2 + |\beta|^2$ and note that $\det\Lambda = 1$ is equivalent to the condition $AC - |B|^2 = 1$. Then, as in the proof of Theorem 2.1, to compute $\|T_{a,b}\|_{cb}$, that is, the Haagerup norm of $w = e \otimes e + u \otimes u$, it suffices to consider the representations of w of the form

$$w = (\gamma e - \overline{\beta}u) \otimes (\alpha e + \beta u) + (-\beta e + \alpha u) \otimes (\overline{\beta}e + \gamma u).$$

Then by a short computation,

$$\|w\|_h = \inf\left\{\left\|(A + C)e\right\|^{1/2}\left\|Ae^{\perp} + Ce + 2\operatorname{Re}(Bu^*)\right\|^{1/2} : AC - |B|^2 = 1\right\},$$

where $e^{\perp} = 1 - e$. Furthermore,

$$Ae^{\perp} + Ce + 2\operatorname{Re}(Bu^*) = \begin{bmatrix}C & \overline{B}\\ B & A\end{bmatrix}$$

and the norm of the last matrix is equal to $\left(A + C + \sqrt{(A - C)^2 + 4|B|^2}\right)/2$. By symmetry we may assume that $A \geqslant C$, hence

$$\left\|Ae^{\perp} + Ce + 2\operatorname{Re}(Bu^*)\right\| \geqslant \frac{1}{2}\left(A + C + |A - C|\right) = A.$$

Therefore

$$\|T_{a,b}\|_{cb} = \|w\|_h \geqslant (A + C)^{1/2}A^{1/2} \geqslant (2AC)^{1/2} = \left(2(1 + |B|^2)\right)^{1/2} \geqslant \sqrt{2}.$$

In fact $\|T_{a,b}\|_{cb} = \sqrt{2}$, since $\|w\|_h \leqslant \|e^2 + uu^*\|^{1/2}\|e^2 + u^*u\|^{1/2} = \sqrt{2}$.

https://doi.org/10.1017/S0004972700033499 Published online by Cambridge University Press

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