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A NOTE ON MULTIVARIATE POISSON FLOWS ON STOCHASTIC PROCESSES

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Abstract

In [1], a deterministic counting rate condition is shown to be necessary and sufficient for a counting process induced on a Markov step process Z to be multivariate Poisson. We show here that the result continues to hold without Z being a Markov step process.

MARKOV STEP PROCESS

It was assumed in [1] that a Markov step process Z induces a multivariate counting process $N=(N_1, N_2, \cdots, N_c)$. The infinitesimal generator A of Z was used there to characterize a vector process whose respective components $r_i(Z(t))$ can be heuristically interpreted as the counting rates for the corresponding N_i at time t. It is shown in [1] that if the components of N do not have simultaneous jumps, a determinacy condition based on the sigma algebras $\boldsymbol{N}_t=\sigma\{N(u), u \leqq t\}$ is necessary and sufficient for N to consist of mutually independent Poisson processes. This condition is that for each t we have almost surely

$$E[r(Z(t)) \mid \boldsymbol{N}_t]=E[r(Z(t))]. \tag{1}$$

The above result is extended in the present letter to processes Z that need not be Markov. To this end, let Z be measurable with respect to an increasing family of sigma algebras $\{\boldsymbol{F}_t\}$, and suppose further that Z induces the counting process N (as defined in [2], Chapter 2) in the sense that $\boldsymbol{N}_t \subseteq \boldsymbol{F}_t$ for each t. Let $E[N_i(t)]<\infty$ for each t, $i=1,2,\cdots,c$, with the N_i having the respective $\boldsymbol{F}_t$-intensities (see [2], II.D7) λ_i. It is also presumed that the conditional expectations $E[\lambda_i(\cdot) \mid \boldsymbol{N}]$ have an $\boldsymbol{N}_t$-progressive version, which we can (and shall) assume to be $\boldsymbol{N}_t$-predictable ([2], Theorem II.T13) without loss of generality.

Now if $\boldsymbol{I}$ stands for the indicator function, it is tautologically true that

$$\boldsymbol{I}[N_i(t)-N_i(s)>n_i]=\int_s^t \boldsymbol{I}[N_i(u-)-N_i(s)=n_i]\, dN_i(u) \tag{2}$$

for any $0 \leqq s \leqq t$. (Equation (2), together with its possible implications, were called to the author's attention by Dr B. Melamed.) Moreover, $N_i(t)-\int_0^t \lambda_i(s)\, ds$ is not only an $\boldsymbol{F}_t$-martingale, but also *a fortiori* an $\boldsymbol{N}_t$-martingale. It then follows from the definition of

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intensity that on the right side of (2)

$$(3)\quad E\left[\int_s^t [N_i(u-)-N_i(s)=n_i]\,dN_i(u)\mid \mathbf{N}_s\right]=E\left[\int_s^t [N_i(u)-N_i(s)=n_i]\lambda_i(u)\,du\mid \mathbf{N}_s\right].$$

Equations (2) and (3) may be combined by taking the conditional expectation in (2) respective to $\mathbf{N}_s$, and substituting. If we then also add over $n_i=0,1,2,\cdots$ and apply Fubini's theorem, we obtain

$$(4)\qquad E[N_i(t)-N_i(s)\mid \mathbf{N}_s]=\int_s^t E[\lambda_i(u)\mid \mathbf{N}_s]\,du.$$

This equation effectively generalizes (1.18) of [1]; our λ_i plays the role of the r_i of [1], which in [1] is generated by a Markov step process Z. Indeed, under the assumptions of [1], our (4) specializes precisely to Equation (1.18) in [1].

Condition (3.2) in [1] may be replaced by

$$(5)\qquad E[\lambda_i(t)\mid \mathbf{N}_t]=E[\lambda_i(t)]$$

almost surely with respect to $dt\,dP$ measure. As in [1], this condition (in the presence of the preceding hypotheses on N, $\mathbf{N}_t$, $\mathbf{F}_t$, and $E[\lambda_i(\cdot)\mid \mathbf{N}]$ above) is necessary and sufficient for N to be a multivariate Poisson process respective to $\mathbf{N}_t$. The proofs are easy exercises in the martingale theory of multivariate counting processes.

If (5) is met, we have in (4)

$$(6)\qquad E[\lambda_i(u)\mid \mathbf{N}_s]=E\{E[\lambda_i(t)\mid \mathbf{N}_t]\mid \mathbf{N}_s\}=E[\lambda_i(t)].$$

Thus N is a multivariate Poisson process according to the multichannel Watanabe theorem (see [2], Theorem II.T6). Conversely, let N be multivariate Poisson. From (4) and the $\mathbf{N}_t$-independent increment property it follows that N_i has the predictable $\mathbf{N}_t$-intensity $E[\lambda_i(\cdot)]$. But also, a version of $E[\lambda_i(\cdot)\mid \mathbf{N}]$ is such an intensity (see [2], Theorem II.T14). The uniqueness of predictable intensities ([2], Theorem II.T12) then yields (5), as was desired.

References

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