THE EXPANSION PROBLEM WITH BOUNDARY CONDITIONS AT A FINITE SET OF POINTS

RANDAL H. COLE

1. Introduction. The problem of expanding an arbitrary function in a series of characteristic solutions of the ordinary differential equation

$$u^{[n]} + P_1 u^{[n-1]} + \ldots + P_n u = 0 \qquad \left(u^{[j]} = \frac{d^j u}{dx^j}\right) \tag{1.1}$$

and the boundary relations

$$\sum_{\mu=1}^{m} \sum_{j=1}^{n} v_{ij}^{(\mu)} u^{[j-1]}(a_\mu) = 0, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

is well known. The various discussions are distinguished by the manner in which a parameter λ appears in the differential system and by the number of points at which the boundary conditions apply. The case in which the boundary conditions apply at intermediate as well as at the end points of a fundamental interval has been considered by Wilder **(3)**. His investigation was confined to the case where $P_n = P_{n0}(x) + \lambda^n$ and where each coefficient $v_{ij}{}^{(\mu)}$ in the boundary relations is free from λ.

The present discussion treats the case where each coefficient P_k is a polynomial in λ of degree k and each coefficient $v_{ij}{}^{(\mu)}$ in the boundary relations is an arbitrary polynomial in λ. The reduction of the system (1.1) and (1.2) to an equivalent matrix system has been accomplished **(4)**, therefore the results obtained by Langer **(1)** can be applied to the present problem.[1] It will be assumed that the reader is familiar with Langer's paper so that direct reference can be made to some of his formulas. In order to facilitate the use of such formulas, Langer's notation has been used here with only minor modifications.

Langer's development concerns a differential system in the complex domain. His boundary conditions apply at a specified set of m points in this domain. Although his results are valid when the variable is restricted to be real, there are several points of interest attending this restriction. The first of these is the form of Green's matrix. Langer has defined a set of m Green's matrices corresponding to the m boundary points. In the real case, these can be combined to yield a single Green's matrix, $\mathfrak{G}(x, s, \lambda)$, which has a finite discontinuity, with respect to the variable s, at each of the boundary points. In all

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other respects, this matrix has the familiar properties of a Green's function. That is, it has a unit discontinuity when $x = s$, and it is a formal solution of the given boundary system and of the adjoint system. The second point of interest is that the adjoint boundary conditions (3·5b) are simply specifications of finite discontinuities at the boundary points. The discontinuities of the Green's matrix satisfy these adjoint conditions. It is clear, therefore, that these finite jumps are characteristic of the adjoint solution and of Green's matrix and, further, that no system with boundary conditions of the form of (2·1b) can be self-adjoint if $m > 2$.

It should be noted that Haltiner **(5)** specialized Langer's results to the real case for two point boundary conditions and obtained a new definition of adjoint boundary relations. These relations have the advantage of being explicitly defined in terms of the given boundary problem. The same advantage is enjoyed by the m point relations obtained here.

The formal points of interest outlined above are significant, but the primary problem in the subsequent discussion is the determination of the specific regularity conditions on the boundary problem which will ensure the convergence of the expansion of an arbitrary vector. This is accomplished by decomposing the Green's matrices defined by Langer and by finding relations among the parts. These relations are equally valid in the complex case and can be used to broaden the scope of Langer's regularity conditions. This point will be amplified in § 5, but it is appropriate to point out here that Langer's general results have been illuminated by applying them to a special case.

Whyburn **(6)**, **(7)**, **(8)** has considered differential systems in which integral boundary conditions are combined with linear conditions at a countable set of points. In particular **(6)**, he has developed some formal aspects of a system with combined integral and two point conditions. His Green's matrix is consistent with the Green's matrix defined below and will, therefore, lend itself to a reduction similar to that achieved in § 4.

2. The differential system. The basic system is the equation (1.1) and boundary conditions (1.2) with the following assumptions:

(a) $$P_k = \sum_{l=0}^{k} P_{kl}(x)\lambda^l, \quad k = 1, 2, \ldots, n,$$

with $P_{kl}(x)$ free from λ and indefinitely differentiable.

(b) *The algebraic equation $r^n + P_{11}(x)r^{n-1} + \ldots + P_{nn}(x) = 0$ has roots $r_i(x)$, $i = 1, 2, \ldots, n$, which together with their differences, $r_i(x) - r_j(x)$, $i \neq j$, have constant arguments and are bounded from zero for all values of x on a fundamental interval $[a_1, a_m]$.*

(c) *The points $a_1, a_2, \ldots, a_m$, $(a_i < a_{i+1})$, at which the boundary relations apply, are the end points of the fundamental interval and a set of $m - 2$ arbitrary interior points of that interval.*

(d) *Each coefficient $v_{ij}^{(\mu)}$ in (1.2) is an arbitrary polynomial in λ with constant coefficients.*

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This system can be reduced to the matrix system (see **(4)**)

$$\mathfrak{Y}'(x, \lambda) = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x)\}\mathfrak{Y}(x, \lambda) \tag{2.1a}$$

$$\sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(a_\mu, \lambda) = \mathfrak{O}, \tag{2.1b}$$

where $\mathfrak{R}(x)$ is the diagonal matrix $(\delta_{ij}r_j(x))$; the diagonal components of the matrix $\mathfrak{Q}(x)$ are zeros, and the other components are indefinitely differentiable and free from λ; and the components of $\mathfrak{W}^{(\mu)}(\lambda)$ are polynomials in λ.

The above results may be stated as a theorem.

THEOREM 1. *The system* (1.1) *and* (1.2), *satisfying assumptions* (a), (b), (c), *and* (d), *may be reduced to the matrix system* (2.1a) *and* (2.1b).

All the subsequent results are developed for the matrix system which, therefore, can well be regarded as the basic one. The nth order system is preferred in this role because of its classical significance.

Langer has treated the problem associated with the matrix system when x is a complex variable and the boundary points are m specified points in the x-plane. He has obtained asymptotic solutions of the equation, defined the adjoint system and a set of m Green's matrices, developed a biorthogonality relation and a formal expansion of an arbitrary vector in a series of characteristic solutions. He has expressed the expansion as a series of residues of Green's matrices and shown that under appropriate conditions the latter converges to the arbitrary vector. Langer's formal results will be adapted to the present problem. An independent derivation of Green's matrix and of the formal expansion would contribute to the continuity of this discussion but would to some extent duplicate known results. Furthermore, such a derivation can be applied to more general boundary conditions than those considered here and will be made the subject of a separate discussion. The pertinent results from Langer's paper are given below, some of them being stated in the form of theorems.

The characteristic values, $\lambda_1, \lambda_2, \ldots,$ of system (2.1) are the roots of the equation $D(\lambda) = 0$ (cf. **(1**, § 7**)**). $D(\lambda)$ is the determinant of the matrix

$$\mathfrak{D}(\lambda) = \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(a_\mu, \lambda) \tag{2.2}$$

where $\mathfrak{Y}(x, \lambda)$ is any non-singular matrix solution of (2.1a). The characteristic solutions are non-trivial vector solutions of (2.1). They exist when λ is a characteristic value. The Green's matrices, $\mathfrak{G}^{(\mu)}(x, s, \lambda)$, $\mu = 1, 2, \ldots, m$, are defined by (see **(1**, § 9**)**)

$$\mathfrak{G}^{(\mu)}(x, s, \lambda) = \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(a_\mu, \lambda)\mathfrak{Y}^{-1}(s, \lambda), \tag{2.3}$$

where $\mathfrak{Y}(x, \lambda)$ is the non-singular matrix solution used in the definition of $\mathfrak{D}(\lambda)$. Let τ be a non-negative integer and let $\mathfrak{f}(x)$ be any vector (n-tuple of real functions) which has a derivative of order $\tau + 1$. Define the set of vectors, $\mathfrak{f}^{(0)}(x), \mathfrak{f}^{(1)}(x), \ldots, \mathfrak{f}^{(\tau+1)}(x)$, by the relations (**1**, (15.3))

$$\mathfrak{f}^{(0)}(x) = \mathfrak{f}(x),$$
$$\mathfrak{f}^{(h)}(x) = \mathfrak{R}^{-1}(x)\{\mathfrak{f}^{(h-1)\prime}(x) - \mathfrak{Q}(x)\mathfrak{f}^{(h-1)}(x)\}, \quad h = 1, 2, \ldots, \tau + 1.$$

THEOREM 2 (*cf.* (**1**, (15.8))). *The formal expansion of* $\mathfrak{f}^{(l)}(x)$ *may be reduced to the infinite series of residues*

$$\mathfrak{s}^{(l)}(x) = \sum_{\beta=0}^{\infty} \mathrm{res}_\beta \sum_{\mu=1}^{m} \lambda^l \left\{ \int_{a_1}^{a_\mu} \mathfrak{G}^{(\mu)}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds + \mathfrak{G}^{(\mu)}(x, a_\mu, \lambda) \sum_{h=0}^{\tau} \lambda^{-h-1}\mathfrak{f}^{(h)}(a_\mu) \right\}.$$

THEOREM 3. *The partial sum,* $\mathfrak{s}_k^{(l)}(x)$, *of the series of residues associated with the first k characteristic values, is given by*

$$\text{(2.4)} \quad \mathfrak{s}_k^{(l)}(x) = \frac{1}{2\pi i}\int_{\Gamma_k} \sum_{\mu=1}^{m} \left\{ -\int_{a_\mu}^{x} \mathfrak{G}^{(\mu)}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds + \mathfrak{G}^{(\mu)}(x, a_\mu, \lambda) \sum_{h=0}^{\tau} \lambda^{-h-1}\mathfrak{f}^{(h)}(a_\mu) \right\} \lambda^l d\lambda,$$

where Γ_k *is a contour in the* λ*-plane enclosing precisely the first k characteristic values.*

The relation (2.4) is Langer's formula (**1**, (15.10)) except that x_1 has been replaced by s and η_μ by a_μ. It is clear that the expansion depends on the choice of the integer τ and that if $l = 0$, we have an expansion of the vector $\mathfrak{f}(x)$ itself.

3. Green's matrix. The term

$$\sum_{\mu=1}^{m} \int_{a_\mu}^{x} \mathfrak{G}^{(\mu)}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds,$$

appearing in formula (2.4), represents the sum of m integrals. In the complex case, each integral is over a curve joining one of the boundary points to the point x. These curves may be entirely distinct or they may be drawn so that they have segments in common. In the real case, on the other hand, no such option exists. The intervals of integration have, of necessity, points in common. Consequently, in the real case it is notationally convenient to define $G(x, s, \lambda)$, which will be called the *Green's matrix*, by the relation

$$\text{(3.1)} \qquad \mathfrak{G}(x, s, \lambda) = \left\{ \begin{array}{ll} \sum_{\mu=1}^{q} \mathfrak{G}^{(\mu)}(x, s, \lambda), & s < x \\ -\sum_{\mu=q+1}^{m} \mathfrak{G}^{(\mu)}(x, s, \lambda), & s > x \end{array} \right\}, \ s \text{ on } (a_q, a_{q+1}).$$

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With $\mathfrak{G}(x, s, \lambda)$ thus defined by a distinct formula on each of the subintervals into which (a_1, a_m) is subdivided by the point x and the intermediate boundary points, it is easily verified that

$$\sum_{\mu=1}^{m} \int_{a_\mu}^{x} \mathfrak{G}^{(\mu)}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds = \int_{a_1}^{a_m} \mathfrak{G}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds. \tag{3.2}$$

Employing (3.1) and (2.3), the discontinuities of $\mathfrak{G}(x, s, \lambda)$ at the boundary points are seen to be such that

$$\begin{aligned} \mathfrak{G}(x, a_h + 0, \lambda) - \mathfrak{G}(x, a_h - 0, \lambda) &= \mathfrak{G}^{(h)}(x, a_h, \lambda) \\ &= \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(h)}(\lambda) \end{aligned} \tag{3.3}$$

where, as a notational convenience, the symbols $G(x, a_1 - 0, \lambda)$ and $G(x, a_m + 0, \lambda)$ are used to represent the zero matrix. In terms of Green's matrix, then, formula (2.4) becomes

$$\begin{aligned} \mathfrak{s}_k^{(l)}(x) = \frac{1}{2\pi i}\int_{\Gamma_k}\Big[&- \int_{a_1}^{a_m} \mathfrak{G}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}(s)ds \\ &+ \sum_{\mu=1}^{m} \{\mathfrak{G}(x, a_\mu + 0, \lambda) - \mathfrak{G}(x, a_\mu - 0, \lambda)\} \sum_{h=0}^{\tau} \lambda^{-h-1}\mathfrak{f}^{(h)}(a_\mu)\Big]\lambda^l d\lambda. \end{aligned} \tag{3.4}$$

It is of interest to observe that the Green's matrix defined in (3.1) has all the familiar properties of a Green's function in classical boundary problems. Because of its form, each matrix $\mathfrak{G}^{(\mu)}(x, s, \lambda)$, regarded as a function of x, is a solution of equation (2.1a). Since, therefore, $\mathfrak{G}(x, s, \lambda)$ is a sum of such matrices, it is a formal solution of (2.1a). It fails to be a true solution because of a discontinuity at $x = s$. Further, it is easily verified that

$$\sum_{h=0}^{m} \mathfrak{W}^{(h)}(\lambda)\mathfrak{G}(a_h, s, \lambda) = \mathfrak{O}.$$

Thus, $\mathfrak{G}(x, s, \lambda)$ is a formal matrix solution of the boundary problem (2.1).

The boundary problem adjoint to (2.1) may be defined by

$$\mathfrak{Z}'(x, \lambda) = - \mathfrak{Z}(x, \lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x)\} \tag{3.5a}$$

$$\mathfrak{Z}(a_h + 0, \lambda) - \mathfrak{Z}(a_h - 0, \lambda) = \mathfrak{A}(\lambda)\mathfrak{W}^{(h)}(\lambda), \qquad h = 1, 2, \ldots, m, \tag{3.5b}$$

where for convenience the symbols $\mathfrak{Z}(a_1 - 0, \lambda)$ and $\mathfrak{Z}(a_m + 0, \lambda)$ are defined to represent the zero matrix. A matrix $\mathfrak{Z}(x, \lambda)$ is a solution of this system if it satisfies equation (3.5a) and if a parametric matrix $\mathfrak{A}(\lambda)$ exists such that (3.5b) is satisfied. Solutions of the adjoint system, therefore, have discontinuities at the boundary points. This definition of the adjoint system is consistent with Langer's definition (**1**, (10.1)) if η_0 is identified with a_1.

Because of its form, Green's matrix, regarded as a function of s, is seen to be a formal solution of (3.5a). Moreover, recalling (3.3), its discontinuities at the boundary points are evidently precisely those required by (3.5b) with $\mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)$ as the parametric matrix $\mathfrak{A}(\lambda)$. As in the earlier case, it fails to be a true solution of (3.5) because of an additional discontinuity at $s = x$.

The characteristics of Green's matrix will be listed in the form of a theorem.

THEOREM 4. *Green's matrix defined by* (3.1) *has the following properties:*

(i) *It is continuous in x and s except when $x = s$ and when $s = a_\mu$, $\mu = 1, 2, \ldots, m$. The discontinuity when $x = s$ is given by*

$$\mathfrak{G}(s+0, s, \lambda) - \mathfrak{G}(s-0, s, \lambda) = \mathfrak{J}.$$

(ii) *For each fixed s, it is a formal solution of the boundary system* (2.1).

(iii) *For each fixed x, it is a formal solution of the boundary system* (3.5).

The non-homogeneous boundary problem,

$$\mathfrak{y}'(x, \lambda) = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x)\}\mathfrak{y}(x, \lambda) + \mathfrak{f}(x)$$

$$\sum_{h=1}^{m} \mathfrak{W}^{(h)}(\lambda)\mathfrak{y}(a_h, \lambda) = \mathfrak{o},$$

when λ is not a characteristic value, has a vector solution $\mathfrak{u}(x, \lambda)$ given by

$$\mathfrak{u}(x, \lambda) = \int_{a_1}^{a_m} \mathfrak{G}(x, s, \lambda)\mathfrak{f}(s)ds.$$

The corresponding non-homogeneous adjoint problem has the vector solution

$$\mathfrak{v}(x, \lambda) = -\int_{a_1}^{a_m} \mathfrak{f}(s)\mathfrak{G}(s, x, \lambda)ds,$$

with

$$\mathfrak{a}(\lambda) = -\int_{a_1}^{a_m} \mathfrak{f}(s)\mathfrak{Y}(s, \lambda)ds\mathfrak{D}^{-1}(\lambda)$$

as the parametric vector. The verification of these facts is straightforward.

A reduction of formula (3.4) can be achieved by using the fact that $\mathfrak{G}(x, s, \lambda)$ is a formal solution of the adjoint system. It is more convenient, however, to cite the reduction given by Langer in **(1**, § 17**)** which results in his formula (17.3) and to express this latter formula in terms of the matrix $\mathfrak{G}(x, s, \lambda)$. The result is

$$(3.6)\quad \mathfrak{s}_k^{(l)}(x) = \frac{1}{2\pi i}\int_{\Gamma_k} \sum_{h=0}^{\tau} \mathfrak{f}^{(h)}(x)\lambda^{-h+l-1}d\lambda - \frac{1}{2\pi i}\int_{\Gamma_k}\int_{a_1}^{a_m} \lambda^{l-\tau-1}\mathfrak{G}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}^{(\tau+1)}(s)dsd\lambda.$$

Since the first term on the right of (3.6) has the value $\mathfrak{f}^{(l)}(x)$, we may write

$$(3.7)\quad \mathfrak{s}_k^{(l)}(x) = \mathfrak{f}^{(l)}(x) - \frac{1}{2\pi i}\mathfrak{b}_k^{(l)}(x)$$

where

$$(3.8)\quad \mathfrak{b}_k^{(l)}(x) = \int_{\Gamma_k}\int_{a_1}^{a_m} \lambda^{l-\tau-1}\mathfrak{G}(x, s, \lambda)\mathfrak{R}(s)\mathfrak{f}^{(\tau+1)}(s)dsd\lambda.$$

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Thus the problem of showing that $\mathfrak{s}_k^{(l)}(x)$ converges to $\mathfrak{f}^{(l)}(x)$ has been reduced to the problem of showing that

$$\lim_{k\to\infty} \mathfrak{b}_k^{(l)}(x) = \mathfrak{o}.$$

4. The structure of Green's matrix. The synthesis of Green's matrix, achieved by formula (3.1), is notationally convenient in dealing with the formal aspects of the boundary problem. In order to establish the convergence of the formal expansion, however, it is desirable to obtain a decomposition of Green's matrix beyond that exhibited in (3.1). The following lemma will be useful in this reduction.

LEMMA 1. *Let $\mathfrak{U}^{(1)}, \mathfrak{U}^{(2)}, \ldots, \mathfrak{U}^{(m)}$ be a set of $n \times n$ matrices and let their sum, $\mathfrak{D}$, be non-singular. Corresponding to each matrix $\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)}$, there is a set of $(m+1)$ matrices $\mathfrak{H}^{(\mu\nu)}$, $\nu = 1, 2, \ldots, m+1$, such that*

$$\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)} = \sum_{\nu=1}^{m+1} \mathfrak{H}^{(\mu\nu)}, \qquad \mu = 1, 2, \ldots, m,$$

and

$$\mathfrak{H}^{(\mu\nu)} = -\mathfrak{H}^{(\nu\mu)}, \qquad \mu, \nu = 1, 2, \ldots, m.$$

The matrix $\mathfrak{H}^{(\mu,m+1)}$ has zero components except on its diagonal where each component is the corresponding diagonal component of $\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)}$.

Let the symbol $\mathfrak{J}_{hl}$ represent the matrix in which all the components are zero except for a unit component in the hth row and lth column. That is, $\mathfrak{J}_{hl} = (\delta_{ih}\delta_{lj})$, $h, l = 1, 2, \ldots, n$. Also, let the matrix $\mathfrak{J}^{hh}$ be defined by $\mathfrak{J}^{hh} = \mathfrak{J} - \mathfrak{J}_{hh}$. The cofactor of the element in the jth row and the ith column of $\mathfrak{D}$ may be written as $|\mathfrak{D}\mathfrak{J}^{ii} + \mathfrak{J}_{ji}|$. Hence, if D is the determinant of $\mathfrak{D}$,

$$\mathfrak{D}^{-1} = 1/D\,(|\mathfrak{D}\mathfrak{J}^{ii} + \mathfrak{J}_{ji}|)$$

and

$$\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)} = 1/D\left(\sum_{k=1}^{n} |\mathfrak{D}\mathfrak{J}^{ii} + \mathfrak{J}_{ki}|u_{kj}^{(\mu)}\right)$$
$$= 1/D\left(\sum_{k=1}^{n} |\mathfrak{D}\mathfrak{J}^{ii} + u_{kj}^{(\mu)}\mathfrak{J}_{ki}|\right).$$

The general component of the matrix on the right is exhibited as the sum of n determinants which differ from each other only with respect to their ith columns. They may be added, therefore, by replacing the ith column of any one of them by the sum of the ith columns of all of them. Since this column sum is readily seen to be the jth column of $\mathfrak{U}^{(\mu)}$, we have

$$\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)} = 1/D(|\mathfrak{D}\mathfrak{J}^{ii} + \mathfrak{U}^{(\mu)}\mathfrak{J}_{ji}|).$$

The right side of this relation may be expressed as the sum of two matrices, one having zeros on the diagonal and the other having zeros elsewhere. Thus,

$$(4.1)\quad \mathfrak{D}^{-1}\mathfrak{U}^{(\mu)} = \left(\frac{1-\delta_{ij}}{D}\,|\mathfrak{D}\mathfrak{Z}^{ii} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}|\right) + \left(\frac{\delta_{ij}}{D}\,|\mathfrak{D}\mathfrak{Z}^{ii} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}|\right).$$

The second matrix on the right will be represented by the symbol $\mathfrak{H}^{(\mu,m+1)}$. Since it is a diagonal matrix, we may replace the index i by j so that,

$$(4.2)\qquad \mathfrak{H}^{(\mu,m+1)} = \left(\frac{\delta_{ij}}{D}\,|\mathfrak{D}\mathfrak{Z}^{jj} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{jj}|\right).$$

The first matrix on the right of (4.1) may be decomposed into a sum of m matrices by expanding the determinantal factor of the general component as follows.

$$|\mathfrak{D}\mathfrak{Z}^{ii} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}| = \left|\mathfrak{D}\mathfrak{Z}^{ii}\mathfrak{Z}^{jj} + \sum_{\nu=1}^{m} \mathfrak{U}^{(\nu)}\mathfrak{Z}_{jj} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}\right|$$
$$= \sum_{\nu=1}^{m} |\mathfrak{D}\mathfrak{Z}^{ii}\mathfrak{Z}^{jj} + \mathfrak{U}^{(\nu)}\mathfrak{Z}_{jj} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}|.$$

Thus, if we define the matrix $\mathfrak{H}^{(\mu\nu)}$ by

$$(4.3)\quad \mathfrak{H}^{(\mu\nu)} = \left(\frac{1-\delta_{ij}}{D}\,|\mathfrak{D}\mathfrak{Z}^{ii}\mathfrak{Z}^{jj} + \mathfrak{U}^{(\nu)}\mathfrak{Z}_{jj} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}|\right),\ \mu, \nu = 1, 2, \ldots, m,$$

we have

$$\sum_{\nu=1}^{m} \mathfrak{H}^{(\mu\nu)} = \left(\frac{1-\delta_{ij}}{\mathfrak{D}}\,|\mathfrak{D}\mathfrak{Z}^{ii} + \mathfrak{U}^{(\mu)}\mathfrak{Z}_{ji}|\right).$$

Hence,

$$\mathfrak{D}^{-1}\mathfrak{U}^{(\mu)} = \sum_{\nu=1}^{m+1} \mathfrak{H}^{(\mu\nu)}.$$

An examination of (4.3) reveals that, if $\nu = \mu$, the determinantal factor of the general element of $\mathfrak{H}^{(\mu\nu)}$ has two identical columns. Thus,

$$(4.4)\qquad \mathfrak{H}^{(\mu\mu)} = \mathfrak{O}.$$

Again, interchanging the symbols μ and ν in formula (4.3) has the effect of interchanging two columns in the determinantal factor. Since this changes the sign of the determinant, we infer that

$$(4.5)\qquad \mathfrak{H}^{(\mu\nu)} = -\,\mathfrak{H}^{(\nu\mu)}.$$

This proves the lemma.

It may also be noted that

$$(4.6)\qquad \sum_{\mu=1}^{m} \mathfrak{H}^{(\mu,m+1)} = \mathfrak{Z}.$$

This is obtained by summing (4.2).

In anticipation of a notational device to be introduced later, we define the matrix $\mathfrak{H}^{(m+1,\mu)}$ by

$$\mathfrak{H}^{(m+1,\mu)} = -\mathfrak{H}^{(\mu,m+1)} \tag{4.7}$$

and let $\mathfrak{H}^{(m+1,m+1)}$ be the zero matrix of order n. Relations (4.4) and (4.7) are then valid for $\mu = 1, 2, \ldots, m+1$, and relation (4.5) is valid for $\mu, \nu = 1, 2, \ldots, m+1$.

THOEREM 5. *There exist matrices* $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$, $\mu, \nu = 1, 2, \ldots, m+1$, *such that*

$$\mathfrak{G}^{(\mu)}(x, s, \lambda) = \sum_{\nu=1}^{m+1} \mathfrak{G}^{(\mu\nu)}(x, s, \lambda), \mu = 1, 2, \ldots, m, \tag{4.8}$$

and

$$\mathfrak{G}^{(\mu\nu)}(x, s, \lambda) = -\mathfrak{G}^{(\nu\mu)}(x, s, \lambda). \tag{4.9}$$

To prove the theorem, let the matrix $\mathfrak{U}^{(\mu)}$, appearing in Lemma 1, be specified by

$$\mathfrak{U}^{(\mu)} = \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(a_\mu, \lambda), \qquad \mu = 1, 2, \ldots, m. \tag{4.10}$$

The matrix $\mathfrak{D}$ of that lemma becomes, then, the characteristic matrix $\mathfrak{D}(\lambda)$ and will be non-singular if λ is not a characteristic value. Hence,

$$\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(a_\mu, \lambda) = \sum_{\nu=1}^{m+1} \mathfrak{H}^{(\mu\nu)}.$$

Let $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ be defined by

$$\mathfrak{G}^{(\mu\nu)}(x, s, \lambda) = \mathfrak{Y}(x, \lambda)\mathfrak{H}^{(\mu\nu)}\mathfrak{Y}^{-1}(s, \lambda), \qquad \mu, \nu = 1, 2, \ldots, m+1. \tag{4.11}$$

It follows at once from Lemma 1 and relation (4.7) that the relations (4.8) and (4.9) are valid.

As a particular instance of (4.9), it may be noted that

$$\mathfrak{G}^{(\mu\mu)}(x, s, \lambda) = \mathfrak{O}. \tag{4.12}$$

Further, from (4.6) we infer that

$$\sum_{\mu=1}^{m} \mathfrak{G}^{(\mu,m+1)}(x, s, \lambda) = \mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(s, \lambda). \tag{4.13}$$

The asymptotic representation for the solution $\mathfrak{Y}(x, \lambda)$, obtained by Langer (**1**, (6.10) and (6.11)), is

$$\mathfrak{Y}(x, \lambda) = \mathfrak{P}(x, \lambda)\mathfrak{E}(x, \lambda) \tag{4.14}$$

where,

$$\mathfrak{E}(x, \lambda) = (\delta_{ij}e^{\lambda R_j(x)}), \quad \text{with } R_j(x) = \int_{a_1}^{x} r_j(t)dt,$$

and $\mathfrak{P}(x, \lambda)$ has an asymptotic representation of the form

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$$\mathfrak{P}(x, \lambda) = \mathfrak{J} + \sum_{h=1}^{k-1} \lambda^{-h}\mathfrak{P}^{(h)}(x) + \lambda^{-k}\mathfrak{B}_k(x, \lambda).$$

In the latter relation, k is any natural number and $\mathfrak{P}^{(h)}(x)$, $h = 1, 2, \ldots, k-1$, and $\mathfrak{B}_k(x, \lambda)$ are indefinitely differentiable in x, and the components of $\mathfrak{B}_k(x, \lambda)$ are analytic in λ and bounded for $|\lambda|$ large.

In view of the representation (4.14) and the definition of $\mathfrak{U}^{(\mu)}$ in (4.10), it is clear that the components of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ are exponential sums. To the end of deducing the structure of these sums, we prove the following lemma.

LEMMA 2. *The matrix $\mathfrak{H}^{(\mu\nu)}$ has the representation given in formulas* (4.15) *and* (4.16) *below.*

Since the components of $\mathfrak{W}^{(\mu)}(\lambda)$ are polynomials in λ, $\mathfrak{U}^{(\mu)}$ may be expressed as

$$\mathfrak{U}^{(\mu)} = (v_{ij}^{(\mu)} \exp\{\lambda R_j(a_\mu)\}),$$

where $v_{ij}^{(\mu)}$ is asymptotically a polynomial in $1/\lambda$ multiplied by some non-negative integral power of λ. $\mathfrak{H}^{(\mu, m+1)}$ is a diagonal matrix and, from (4.2), its jth diagonal component is seen to be $1/D$ multiplied by a determinant whose jth column is the jth column of $\mathfrak{U}^{(\mu)}$ and whose other columns are corresponding columns of $\mathfrak{D}$. Since $\mathfrak{D} = \mathfrak{U}^{(1)} + \mathfrak{U}^{(2)} + \ldots + \mathfrak{U}^{(m)}$, this determinant may be expanded into the sum of m^{n-1} determinants, each of which contains the jth column of $\mathfrak{U}^{(\mu)}$ as its jth column, and the αth column of one of the matrices $\mathfrak{U}^{(1)}, \mathfrak{U}^{(2)}, \ldots, \mathfrak{U}^{(m)}$ as its αth column, $\alpha \neq j$. Thus,

$$(4.15) \quad \mathfrak{H}^{(\mu, m+1)} = \left(\frac{\delta_{ij}}{D} \sum_{k_\alpha=1, \alpha\neq j}^{m} h_{\{k_\alpha|\alpha\neq j\}}^{(\mu, m+1)} \exp\left\{\lambda\left[R_j(a_\mu) + \sum_{\alpha=1, \alpha\neq j}^{n} R_\alpha(a_{k_\alpha})\right]\right\}\right),$$

where $h_{\{k_\alpha|\alpha\neq j\}}^{(\mu, m+1)}$ is asymptotically a polynomial in $1/\lambda$, multiplied by some power of λ. The subscript symbol $\{k_\alpha|\alpha \neq j\}$ is an abbreviation for the set $k_1, k_2, \ldots, k_{j-1}, k_{j+1}, \ldots, k_n$. The summation operator applies independently to each member of this set. Thus, the jth diagonal component of $\mathfrak{H}^{(\mu, m+1)}$ is exhibited as an exponential sum of m^{n-1} terms.

The matrix $\mathfrak{H}^{(\mu\nu)}$, $\mu, \nu = 1, 2, \ldots, m$, has zeros on its diagonal. From (4.3), the component in the ith row and jth column, $i \neq j$, is seen to be $1/D$ multiplied by a determinant whose ith column is the jth column of $\mathfrak{U}^{(\mu)}$, whose jth column is the jth column of $\mathfrak{U}^{(\nu)}$, and whose other columns are columns of $\mathfrak{D}$. This determinant may be expanded into the sum of m^{n-2} determinants, each of which contains, as its ith and jth columns, the jth columns of $\mathfrak{U}^{(\mu)}$ and $\mathfrak{U}^{(\nu)}$, respectively, and as its αth column, $\alpha \neq i, j$, the αth column of one of the matrices $\mathfrak{U}^{(1)}, \mathfrak{U}^{(2)}, \ldots, \mathfrak{U}^{(m)}$. Hence,

$$(4.16) \quad \mathfrak{H}^{(\mu\nu)} = \left(\frac{1-\delta_{ij}}{D} \sum_{k_\alpha=1, \alpha\neq i,j}^{m} h_{\{k_\alpha|\alpha\neq i,j\}}^{(\mu\nu)} \exp\left\{\lambda\left[R_j(a_\mu) + R_j(a_\nu) + \sum_{\alpha=1, \alpha\neq i,j}^{n} R_\alpha(a_{k_\alpha})\right]\right\}\right).$$

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The notation in this relation is similar to that used in (4.15). This completes the proof of the lemma.

Recalling the definition of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ in relation (4.11) and the representation of $\mathfrak{Y}(x, \lambda)$ in (4.14), we may write

$$\mathfrak{G}^{(\mu\nu)}(x, s, \lambda) = \mathfrak{P}(x, \lambda)\mathfrak{E}(x, \lambda)\mathfrak{H}^{(\mu\nu)}\mathfrak{E}^{-1}(s, \lambda)\mathfrak{P}^{-1}(s, \lambda).$$

Anticipating the form of the product on the right, let the following two relations define their left members.

$$(4.17)\quad \phi^{(\mu,m+1)}_{\{k_\alpha|\alpha\neq j\}}(x, s) = R_j(x) - R_j(s) + R_j(a_\mu) + \sum_{\alpha=1,\alpha\neq j}^{n} R_\alpha(a_{k_\alpha}),\qquad \mu = 1, 2, \ldots, m,$$

$$(4.18)\quad \phi^{(\mu\nu)}_{\{k_\alpha|\alpha\neq i,j\}}(x, s) = R_i(x) - R_j(s) + R_j(a_\mu) + R_j(a_\nu) + \sum_{\alpha=1,\alpha\neq i,j}^{n} R_\alpha(a_{k_\alpha}),\qquad \mu, \nu = 1, 2, \ldots, m, \mu \neq \nu.$$

Both $\mathfrak{E}(x, \lambda)$ and its inverse are diagonal matrices, hence, multiplying each of the relations (4.15) and (4.16) on the left by $\mathfrak{E}(x, \lambda)$ and on the right by $\mathfrak{E}^{-1}(s, \lambda)$, we have

$$\mathfrak{E}(x, \lambda)\mathfrak{H}^{(\mu,m+1)}\mathfrak{E}^{-1}(s, \lambda) = \left(\frac{\delta_{ij}}{D}\sum_{k_\alpha=1,\alpha\neq j}^{m} h^{(\mu,m+1)}_{\{k_\alpha|\alpha\neq j\}} \exp\{\lambda\phi^{(\mu,m+1)}_{\{k_\alpha|\alpha\neq j\}}(x, s)\}\right)$$

and

$$\mathfrak{E}(x, \lambda)\mathfrak{H}^{(\mu\nu)}\mathfrak{E}^{-1}(s, \lambda) = \left(\frac{1-\delta_{ij}}{D}\sum_{k_\alpha=1,\alpha\neq i,j}^{m} h^{(\mu\nu)}_{\{k_\alpha|\alpha\neq i,j\}} \exp\{\lambda\phi^{(\mu\nu)}_{\{k_\alpha|\alpha\neq i,j\}}(x, s)\}\right).$$

The matrix $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ is obtained by multiplying the appropriate one of the above matrices on the left by $\mathfrak{P}(x, \lambda)$ and on the right by $\mathfrak{P}^{-1}(s, \lambda)$. In this connection, we may observe that each component of the product, $\mathfrak{A}\mathfrak{B}\mathfrak{C}$, of three matrices is a linear combination of all the components of $\mathfrak{B}$, and that each coefficient in this linear combination is the product of some component of $\mathfrak{A}$ with some component of $\mathfrak{C}$. From this, and the fact that the components of both $\mathfrak{P}(x, \lambda)$ and $\mathfrak{P}^{-1}(s, \lambda)$ are asymptotically polynomials in $1/\lambda$, it is clear that each component of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ will be an exponential sum containing, in general, all the exponential terms appearing in $\mathfrak{H}^{(\mu\nu)}$. The coefficients of these sums will, moreover, be of the same form as the coefficients in the non-zero components of $\mathfrak{H}^{(\mu\nu)}$, except that they will be functions of x and s. Hence, each component of $\mathfrak{G}^{(\mu,m+1)}(x, s, \lambda)$, $\mu = 1, 2, \ldots, m$, is of the form

$$(4.19)\qquad \lambda^{\theta}/D \sum_{j=1}^{n}\sum_{k_\alpha=1,\alpha\neq j}^{m} g^{(\mu,m+1)}_{\{k_\alpha|\alpha\neq j\}} \exp\{\lambda\phi^{(\mu,m+1)}_{\{k_\alpha|\alpha\neq j\}}(x, s)\}.$$

Similarly, each component of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$, $\mu, \nu = 1, 2, \ldots, m$, is of the form

$$(4.20)\qquad \lambda^{\theta}/D \sum_{i,j=1}^{n}\sum_{k_\alpha=1,\alpha\neq i,j}^{m} g^{(\mu\nu)}_{\{k_\alpha|\alpha\neq i,j\}} \exp\{\lambda\phi^{(\mu\nu)}_{\{k_\alpha|\alpha\neq i,j\}}(x, s)\}.$$

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The non-negative integer θ is defined to be the smallest such integer for which the coefficients, $g_{\{k_\alpha|\alpha\neq j\}}{}^{(\mu,m+1)}$ and $g_{\{k_\alpha|\alpha\neq i,j\}}{}^{(\mu\nu)}$, are asymptotic polynomials in $1/\lambda$ for every admissible value of their various indices. The above results are summarized in the following theorem.

THEOREM 6. *Each component of* $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ *is an exponential sum of the form shown in* (4.19) *or* (4.20). *The coefficient of* λ *in the exponent of* e *in each term of the sum is given by* (4.17) *or* (4.18).

As a useful notational device, we define the square matrix $((\mathfrak{G}))$, whose components are matrices, by the relation

$$((\mathfrak{G})) = ((\mathfrak{G}^{(\mu\nu)}(x, s, \lambda))), \qquad \mu, \nu = 1, 2, \ldots, m+1.$$

Because of relation (4.9) in Theorem 5, this matrix is seen to be skew-symmetric. Further, let the symmetric matrix $\mathfrak{F}$ be defined by

$$\mathfrak{F} = (\phi^{(\mu\nu)}(x, s)), \qquad \mu, \nu = 1, 2, \ldots, m+1.$$

The components of this matrix are the functions defined in (4.17) and (4.18) for all values of μ and ν for which those definitions are valid. The definition of the remaining components is achieved by the relations

$$\phi^{(\mu\nu)}(x, s) \equiv 0, \qquad \text{if } \mu = \nu,$$

$$\phi^{(m+1,\nu)}(x, s) = \phi^{(\nu,m+1)}(x, s), \qquad \nu = 1, 2, \ldots, m+1.$$

Thus, the element in the μth row and νth column of $\mathfrak{F}$ corresponds uniquely to the element in the μth row and νth column of $((\mathfrak{G}))$. That is to say, the exponential sum which constitutes the general component of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ is $1/D$ multiplied by a linear combination of exponential terms of the form $\exp\{\lambda\phi^{(\mu\nu)}(x, s)\}$, where the undesignated parameters in $\phi^{(\mu\nu)}(x, s)$ are allowed to range through all their admissible values. It follows that, when we are concerned with the sum of any specific block of components in $((\mathfrak{G}))$, the exponential sums contained therein will have in their exponents precisely those ϕ-functions which appear in the corresponding block of components in $\mathfrak{F}$.

The sums of certain blocks of components in $((\mathfrak{G}))$ can be concisely represented, if we define the vector $\mathfrak{d}_j$ to be an $(m+1)$-dimensional vector with 1 in the jth place and zeros elsewhere and define the vector $\mathfrak{i}_q$ by the relation

$$\mathfrak{i}_q = \sum_{j=1}^{q} \mathfrak{d}_j.$$

Thus, recalling (4.8),

$$\mathfrak{d}_\mu((\mathfrak{G}))\mathfrak{i}_{m+1} = \mathfrak{G}^{(\mu)}(x, s, \lambda).$$

Hence,

$$\sum_{\mu=1}^{q} \mathfrak{G}^{(\mu)}(x, s, \lambda) = \mathfrak{i}_q((\mathfrak{G}))\mathfrak{i}_{m+1},$$

and it is immediately clear that this notation can be used to rewrite formula (3.1). That is,

$$(4.21) \qquad \mathfrak{G}(x, s, \lambda) = \begin{Bmatrix} \mathfrak{i}_q((\mathfrak{G}))_{m+1}, & s < x \\ -(\mathfrak{i}_m - \mathfrak{i}_q)((\mathfrak{G}))\mathfrak{i}_{m+1}, & s > x \end{Bmatrix} s \text{ on } (a_q, a_{q+1}).$$

THEOREM 7. *Formula* (4.21) *for Green's matrix may be reduced to*

$$(4.22) \quad \mathfrak{G}(x, s, \lambda) = \begin{Bmatrix} \mathfrak{i}_q((\mathfrak{G}))(\mathfrak{i}_{m+1} - \mathfrak{i}_q), & s < x \\ -(\mathfrak{i}_m - \mathfrak{i}_q)((\mathfrak{G}))(\mathfrak{i}_q + \mathfrak{d}_{m+1}), & s > x \end{Bmatrix} s \text{ on } (a_q, a_{q+1}).$$

This result follows immediately when it is recalled that $((\mathfrak{G}))$ is skew-symmetric, and hence, that both $\mathfrak{i}_q((\mathfrak{G}))\mathfrak{i}_q$ and $(\mathfrak{i}_m - \mathfrak{i}_q)((\mathfrak{G}))(\mathfrak{i}_m - \mathfrak{i}_q)$ are zero.

The simplification of Green's matrix achieved by Theorem 7 is of basic significance. In its absence, the definition of regularity would of necessity be made in terms of formula (4.21). Such a definition would not permit the fundamental conclusion stated in Theorem 8 below.

5. Regularity of the boundary problem. In § 4 it was noted that each component of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ is $1/D$ multiplied by an exponential sum. Since D is itself an exponential sum given by **(1**, (11.3)**)**

$$D = D(\lambda) = \sum_{\alpha} A_\alpha(\lambda) e^{\lambda \Omega_\alpha},$$

each component of $\mathfrak{G}^{(\mu\nu)}(x, s, \lambda)$ may be interpreted as the quotient of two exponential sums. A comparison of the exponents of the numerator with those of the denominator is clearly vital to a discussion of the convergence of $\mathfrak{b}_k^{(l)}(x)$ defined in (3.8).

Let the set of exponent coefficients $\{\Omega_\alpha | A_\alpha(\lambda) \not\equiv 0\}$ be represented by the symbol E_D. This set is a subset of the set E defined by

$$\left\{ \sum_{\alpha=1}^{n} R_\alpha(a_{k_\alpha}) \right\},$$

where each member of $k_1, k_2, \ldots, k_n$ is chosen independently from the integers $1, 2, \ldots, m$, **(1**, (11.3) *et seq.***)**. Let the members of the set E_D be plotted on a complex z-plane, and let P_D be the closed region bounded by the convex polygon of smallest area which contains all these points in its interior or on its perimeter. It may be noted for future reference that the members of the set E may be similarly plotted and that they will determine a corresponding closed minimum convex polygonal region P. The region P_D may coincide with P, but if certain members of the set $\{A_\alpha(\lambda)\}$ are identically zero, P_D will be a proper subregion of P.

The exponent coefficients, defined in (4.17) and (4.18), are functions of s for each fixed value of x and each permissible set of values of the parameters involved. If the symbol $\phi^{(\mu\nu)}(x, s)$ is used to represent any one of these functions the relation

$$(5.1) \qquad z = \phi^{(\mu\nu)}(x, s)$$

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will effect a mapping of any s-interval into a complex z-plane. Since $R_j(s)$ has a constant argument and $R_j'(s) \neq 0$, the image is a straight line and the mapping is one-to-one. It may be similarly inferred that, for s fixed, the relation (5.1) will effect a one-to-one mapping of any x-interval into a straight line image. The definition of regularity will be made in terms of the location of the s-interval images relative to the region P_D defined above.

Definition. The boundary problem will be said to be regular relative to a specific value of x if, for all permissible values of the parameters $\{k_\alpha | \alpha \neq j\}$ or $\{k_\alpha | \alpha \neq i, j\}$, as the case may be:

(i) Every ϕ-function in the sum

$$\mathfrak{i}_q \mathfrak{F}(\mathfrak{i}_{m+1} - \mathfrak{i}_q)$$

maps (a_q, a_{q+1}) into P_D for every q such that $a_{q+1} \leqslant x$, and maps (a_q, x) into P_D when $a_q < x < a_{q+1}$;

(ii) Every term in the sum

$$(\mathfrak{i}_m - \mathfrak{i}_q)\mathfrak{F}(\mathfrak{i}_q + \mathfrak{d}_{m+1})$$

maps (a_q, a_{q+1}) into P_D for every q such that $a_q \geqslant x$, and maps (x, a_{q+1}) into P_D when $a_q < x < a_{q+1}$.

The boundary problem will be said to be regular relative to any subinterval of $[a_1, a_m]$, if it is regular relative to every x on that subinterval.

It will be seen, on recalling the representation of $\mathfrak{G}(x, s, \lambda)$ in (4.22), that if a problem is regular, every exponent coefficient in the exponential sum constituting the numerator of each component of $\mathfrak{G}(x, s, \lambda)$ will have values lying in P_D for all values of the variable s.

A sufficient condition for regularity will now be developed by showing that each s-interval mentioned in the definition of regularity is mapped into the region P by the mapping functions associated with it. From this it will follow that if P_D coincides with P the boundary problem is regular.

If, in the mapping relation (5.1), $\phi^{(\mu\nu)}(x, s)$ is the function defined by (4.18), it is clear that the image points $\phi^{(\mu\nu)}(a_1, a_\mu)$ and $\phi^{(\mu\nu)}(a_m, a_\mu)$ belong to the set E and are, therefore, in P. Hence, since P is convex, $\phi^{(\mu\nu)}(x, a_\mu)$ is in P for any x on $[a_1, a_m]$. Similarly, it may be inferred that $\phi^{(\mu\nu)}(x, a_\nu)$ is in P for the same x. This leads to the conclusion contained in the following lemma.

Lemma 3. *The relation* (5.1) *with* $\mu, \nu = 1, 2, \ldots, m$, $(\mu \neq \nu)$ *maps the s-interval* $[a_\mu, a_\nu]$ *into a line in P for any fixed x on* $[a_1, a_m]$. *Moerover, if s is bounded away from the end points of its interval, z is bounded away from the vertices of P.*

If $\nu = m + 1$ in (5.1) and $\phi^{(\mu, m+1)}(x, s)$ is defined by (4.17), it is clear that the images of all pairs of values of x and s lie on the same straight line. Since the points $\phi^{(\mu, m+1)}(a_1, a_\mu)$ and $\phi^{(\mu, m+1)}(a_m, a_\mu)$ lie in P, the point $\phi^{(\mu, m+1)}(x, a_\mu)$ lies in P for any x on $[a_1, a_m]$. Noting, then, that $\phi^{(\mu, m+1)}(x, x)$ is in P, we can state the following lemma.

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LEMMA 4. *The relation* (5.1) *with* $\nu = m + 1$, $\mu = 1, 2, \ldots, m$, *maps the* s*-interval* $[a_\mu, x]$ *into a line in* P *for any* x *on* $[a_1, a_m]$. *If* s *is bounded away from* x *and* a_μ, z *is bounded away from the vertices of* P.

Let (a_q, a_{q+1}) be any s-interval determined by a pair of consecutive boundary points. If $a_{q+1} \leqslant x$, it is readily seen, by employing the above lemmas, that relation (5.1) maps (a_q, a_{q+1}) into P provided that $\mu \leqslant q$ and $\nu \geqslant q + 1$. For, under these conditions on μ and ν, (a_q, a_{q+1}) is contained in $[a_\mu, a_\nu]$ when $\nu \neq m + 1$ and is contained in $[a_\mu, x]$ when $\nu = m + 1$. If $a_q < x < a_{q+1}$, a similar argument shows that (a_q, x) is mapped into P by (5.1) when $\mu \leqslant q$ and $\nu \geqslant q + 1$. These facts can be summarized by saying that each ϕ-function in the sum

$$\mathfrak{i}_q \mathfrak{F}(\mathfrak{i}_{m+1} - \mathfrak{i}_q)$$

maps (a_q, a_{q+1}) into P for every q such that $a_{q+1} \leqslant x$, and maps (a_q, x) into P when $a_q < x < a_{q+1}$. In a similar fashion, it can be inferred that each ϕ-function in the sum

$$(\mathfrak{i}_m - \mathfrak{i}_q)\mathfrak{F}(\mathfrak{i}_q + \mathfrak{d}_{m+1})$$

maps (a_q, a_{q+1}) into P for every q such that $a_q \geqslant x$, and maps (x, a_{q+1}) into P when $a_q < x < a_{q+1}$. Comparing these results with the definition of regularity, the following theorem can be stated.

THEOREM 8. *If* P_D *coincides with* P, *the boundary problem is regular.*

The above theorem establishes the fact that all problems in the category initially specified are regular except possibly those for which the determinant $D(\lambda)$ is degenerate in the sense that P_D is a proper subregion of P. Success in establishing this fact depended on the relations (4.9) and (4.12), by means of which the original form of Green's matrix given in (3.1) was simplified to the form exhibited in (4.22). The relations in question apply equally well in the more general complex case. Consequently, Langer's regularity conditions could, with advantage, be amplified to include a recognition of the simplifying properties of these relations. In this connection, it should be noted that Langer made specific mention of the possibility of a simplification within the formula for a single matrix $\mathfrak{G}^{(\mu)}(x, s, \lambda)$, but that the simplification suggested here occurs between the terms of a sum of such matrices. Hence, in order to take advantage of the relations, the paths of integration, corresponding to those in formula (2.4), need to be chosen so that some of them have segments in common. This will generally be possible and the attendant simplification will be sufficient to admit as regular many problems (the present one is a case in point) which would not be regular according to a literal interpretation of Langer's conditions.

6. Convergence of the expansion. The convergence discussion given in **(1)** is applicable here, but it will be replaced by one which imposes a less restrictive condition on the vector to be expanded.

The matrix $\mathfrak{G}(x, s, \lambda)$ is a sum of matrices whose components are displayed in (4.19) and (4.20). The multiplication of $\mathfrak{G}(x, s, \lambda)$ on the right by $\mathfrak{N}(s)\mathfrak{f}^{(\tau+1)}(s)$ will, therefore, yield a vector whose components are sums of functions of the form

$$\frac{\lambda^{\theta}}{D} h^{(\mu\nu)}(x, s, \lambda) \exp\{\lambda\phi^{(\mu\nu)}(x, s)\},$$

where $h^{(\mu\nu)}(x, s, \lambda)$ is asymptotically a polynomial in $1/\lambda$. Let the integer ρ be chosen as in **(1**, § 12**)** so that, for each α for which Ω_α is a vertex of the polygon bounding P_D, the function

$$\lambda^{-\rho} D e^{-\lambda\Omega_\alpha},$$

(1, (12.2)**)**, is uniformly bounded from zero for λ on the contours of the set $\{\Gamma_k\}$. Define $k^{(\mu\nu)}(x, s, \lambda)$ by

$$\frac{1}{D} h^{(\mu\nu)}(x, s, \lambda) = k^{(\mu\nu)}(x, s, \lambda)\lambda^{\rho} e^{\lambda\Omega_\alpha}$$

so that $k^{(\mu\nu)}(x, s, \lambda)$ is bounded and integrable in s and λ for λ on Γ_k. A typical term in the sum that comprises any component of the vector $\mathfrak{b}_k^{(l)}(x)$, as defined in (3.8), is given by

$$(6.1) \qquad \int_{\Gamma_k}\int_{a_1}^{a_m} \lambda^{l+\theta-\rho-\tau-1} k^{(\mu\nu)}(x, s, \lambda) \exp\{\lambda(\phi^{(\mu\nu)}(x, s) - \Omega_\alpha)\}\, ds d\lambda.$$

If x is a point at which the boundary problem is regular, $\phi^{(\mu\nu)}(x, s)$ lies in the region P_D for every s on (a_1, a_m), with the exception of the boundary points $a_2, a_3, \ldots, a_{m-1}$, and the point x at each of which the integrand is not defined. For any λ, then, the index α can be chosen so that

$$(6.2) \qquad R\{\lambda(\phi^{(\mu\nu)}(x, s) - \Omega_\alpha)\} \leqslant 0$$

for all values of s. There will, moreover, exist a sector on the λ-plane, which may be specified by

$$(6.3) \qquad \xi_\alpha \leqslant \arg\lambda \leqslant \xi'_\alpha$$

such that the inequality (6.2) is maintained for all λ therein. A finite set of such sectors will cover the whole λ-plane and will effect a subdivision of the contour Γ_k into segments. The symbol $\Gamma_{k\alpha}$ will be used to designate that segment which lies in the sector specified by (6.3). The integral (6.1) may be expressed as a sum according to the partition of Γ_k, and a further decomposition is determined by partitioning $[a_1, a_m]$ at the points $a_2, a_3, \ldots, a_{m-1}$, and x, where the integrand is discontinuous. In consequence, we may say that any component of the vector $\mathfrak{b}_k^{(l)}(x)$ consists of a sum of terms of the type

$$(6.4) \qquad \int_{\Gamma_{k\alpha}}\int_{c}^{d} \lambda^{-1}\varphi(x, s, \lambda) ds d\lambda,$$

where c and d are any two consecutive partition points of $[a_1, a_m]$ and

$$(6.5) \qquad \varphi(x, s, \lambda) = \lambda^{l+\theta-\rho-\tau} k^{(\mu\nu)}(x, s, \lambda) \exp\{\lambda(\phi^{(\mu\nu)}(x, s) - \Omega_\alpha)\}.$$

The non-negative integer τ, on which the expansion depends, is assumed to be at least as large as $\theta - \rho$ and sufficiently small to insure the existence of $\mathfrak{f}^{(\tau+1)}(x)$. Let the exponent of λ in (6.5) be written as $l - l_1$, where $l_1 = -\theta + \rho + \tau$. If $l \leqslant l_1$, it is clear that $\varphi(x, s, \lambda)$ is bounded and integrable for λ large.

If $l < l_1$,

$$\lim_{|\lambda| \to \infty} \varphi(x, s, \lambda) = 0$$

uniformly in s on (c, d). Thus it is easily inferred that integral (6.4) converges to zero as $k \to \infty$. From this it follows that $\mathfrak{b}_k^{(l)}(x)$ converges to zero and $\mathfrak{s}_k^{(l)}(x)$ converges to $\mathfrak{f}^{(l)}(x)$ as $k \to \infty$.

If $l = l_1$ and if, for ϵ arbitrary, arg λ and s are restricted by $\xi_\alpha + \epsilon \leqslant \arg \lambda \leqslant \xi_\alpha' - \epsilon$ and $c + \epsilon \leqslant s \leqslant d - \epsilon$, respectively, then, recalling Lemmas 3 and 4, § 5,

$$\lim_{|\lambda| \to \infty} \exp\{\lambda(\phi^{(\mu\nu)}(x, s) - \Omega_\alpha)\} = 0,$$

uniformly in s. At once,

$$\lim_{|\lambda| \to \infty} \varphi(x, s, \lambda) = 0,$$

uniformly in s, for arg λ and s restricted as above. From this it follows easily (see (**2**, Lemma 1, p. 166)) that integral (6.4) converges to zero, and hence that $\mathfrak{s}_k^{(l)}(x)$ converges to $\mathfrak{f}^{(l)}(x)$ as $k \to \infty$.

Combining the two cases, then, it may be stated that the series $\mathfrak{s}^{(l)}(x)$ converges to $\mathfrak{f}^{(l)}(x)$ for $l \leqslant l_1$. The convergence is readily seen to be uniform in x on any closed interval on which the boundary problem is regular.

If $l < l_1$, it is easily inferred (see (**1**, § 17)) that the series arising from the term-by-term differentiation of $\mathfrak{s}^{(l)}(x)$ converges to $\mathfrak{f}^{(l)\prime}(x)$. In particular, $\mathfrak{s}^{(0)}(x)$ converges uniformly to $\mathfrak{f}(x)$, and this series admits of term-by-term differentiation to the order l_1. The following theorem summarizes some of these results.

THEOREM 9. *Let τ be the larger of the integers* 0 *and $\theta - \rho$. If $\mathfrak{f}(x)$ is any vector with a bounded and integrable derivative of order* $(\tau + 1)$ *on a closed subinterval* $[c, d]$ *on which the boundary problem is regular, then the series expansion $\mathfrak{s}^{(0)}(x)$, associated with τ, converges uniformly to $\mathfrak{f}(x)$ on* $[c, d]$. *Moreover, if $\theta - \rho$ is negative, this series admits of term-by-term differentiation to the order of $\rho - \theta$.*

REFERENCES

1. R. E. Langer, *The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc., *46* (1939), 151–190.

2. ——— *Developments associated with a boundary problem not linear in the parameter*, Trans. Amer. Math. Soc., *25* (1923), 155–172.

3. C. E. Wilder, *Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points*, Trans. Amer. Math. Soc., *18* (1917), 415–442.

4. R. H. Cole, *Reduction of an n-th order differential equation and m-point boundary conditions to an equivalent matrix system*, Amer. J. Math. (1), *68* (1946), 179–184.

5. G. H. Haltiner, *The theory of linear differential systems based upon a new definition of the adjoint*, Duke Math. J., *15* (1948), 893–919.

6. W. M. Whyburn, *Differential systems with general boundary conditions*, Seminar Reports in Mathematics, University of California Publ. Math. (1), *2* (1944), 45–61.

7. ——— *Differential equations with general boundary conditions*, Bull. Amer. Math. Soc. (10), *48* (1942), 692–704.

8. ——— *Differential systems with boundary conditions at more than two points*, Proceedings of the Conference on Differential Equations, held at the University of Maryland, March 17–19, 1955 (University of Maryland Bookstore, College Park, Md., 1956), 1–21.

University of Western Ontario