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ON SOME FUNCTIONAL EQUATIONS OF CARLEMAN

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The celebrated Fredholm theory of linear integral equations holds if the kernel $K(x, y)$ or one of its iterates $K^{(n)}$ is bounded. Hilbert utilizing his theory of quadratic form was able to extend the theory to the kernels $K(x, y)$ satisfying

(a) $$\int_a^b |K(x,y)|^2\,dy < \infty$$

(b) $$\int_a^b \int_a^b K(x,y)u(x)\overline{u(y)}\,dx\,dy \le k^2 \int_a^b |u(x)|^2\,dx$$

where k is independent of $u(x)$.

These theories were extended considerably by T. Carleman who deleted condition (b) above.

Equations involving this Carleman kernel have been found useful in connection with Hermitian forms, continued fractions, Schroedinger wave equations (see [1], [2]) and more recently in scattering theory in quantum physics, etc. [3]. See also [5] for a variety of applications and extensions.

This theory was again extended by Carleman [2, p. 138] to kernels $K(x, y)$; merely measurable, symmetric in $[0, 1]\times[0, 1]$ and such that there is associated with the kernel a linear operator $L_x(\xi, f)$, ξ a parameter such that

(1) (i) $$L_x(\xi, K(x,y)) \subset L_2[0,1] \text{ in } y.$$

For approximating kernels K_n,

(ii) $|L_x(\xi, K_n(x,y))| < \gamma(\xi, y), \subset L_2[0,1]$ in y

$\gamma(\zeta, y)$ independent of n.

(iii) $\lim_n L_x(\xi, f_n(x)) = L_x(\xi, f(x))$

if $f_n \subset L_2$ and $f_n \to f$ (weak convergence).

(iv) $$\int_0^1 L_x(\xi, K_n(x,y))\phi(y)\,dy = L_x\left(\xi, \int_0^1 K_n(x,y)\phi(y)\,dy\right)$$

for all ϕ in $L_2[0, 1]$.

In this note we consider the operator equation

(2) $$L_x(\xi, K(x,y))\phi = L_x(\xi, f),$$

i.e.

$$\int_0^1 L_x(\xi, K(x,y))\phi(y)\,dy = L_x(\xi, f(x))$$

with L_x as above, f in $L_2[0, 1]$ and ϕ the sought for solution in L_2.

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DEFINITION. An operator $L_x(\xi, f)$ is called closed if $L_x(\xi, f)=0$ implies $f=0$, a.e. on [0, 1].

The equation $L_x(\xi, K(x, y))\phi=0$ is called closed if $\phi=0$ a.e. is the only L_2 solution.

THEOREM 1. *Let λ_{nv}, $v=1, 2, \ldots$, $\phi_{nv}v=1, 2, 3\ldots$ be the characteristic values and corresponding characteristic functions associated with the approximating kernels, K_n. Suppose $L_x(\xi, K)\phi=0$ is closed and*

$$A_{n_j} = \sum_{v} \lambda^2_{n_j, v} f^2_{n_j, v}, \left[f_{n, v} = \int_0^1 f(x)\phi_{n, v}(x)\, dx, \right] \tag{3}$$

converges for each j while $A_{n_j} < M$, where M is a constant independent of j. Then equation (2) *has a solution. We cite the following lemmas.*

LEMMA 1. *Let f_v be a sequence of L_2 functions on* [0, 1] *s.t.* $\|f_v\| \le M$, $\nu=1, 2, \ldots$; *M a constant. Then there exists a subsequence f_{v_i} and a function f such that*

$$\{f_{v_j}\} \to f \text{ (weak convergence)} \quad \text{and} \quad \|f\|^2 < M.$$

Proof. (See [4].)

LEMMA 2. *Let $\{f_v\}$ be a sequence of L_2 functions on* [0, 1] *such that*

$$\|f_v\| < C, \qquad f_v \to f \text{ (weakly)}$$

while $g_n \to g$ a.e. in [0, 1] *and* $|g_n(x) < \|\gamma\| < \infty$, *then*

$$\lim_n \int_0^1 f_n g_n\, dx = \int_0^1 f \cdot g\, dx.$$

Proof. (See [2, p. 132].)

Proof of theorem. In view of (3) there exists a L_2 solution ϕ of the approximating equation

$$K_{n_j}\phi = f, \quad f \text{ in } L_2[0, 1].$$

By hypothesis

$$L_x(\xi, K_n)\phi_n = L_x(\xi, f).$$

From (3) and Parseval's identity

$$\|\phi_{n_j}\| = A_{n_j} \le M.$$

From Lemma 1 there exists a subsequence $\{\phi_{v_i}\} \to \phi$ (weakly) and $\|\phi\| < M$. From Lemma 2, and properties of L_x we have

$$\lim_{n_j \to \infty} L_x(\xi, K_{n_j})\phi_{n_j} = L_x(\xi, K)\phi = L_x(\xi, f)$$

and we are done.

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As remarked by Carleman [2, p. 139] for kernels with which there can be associated an operator as above, many of the fundamental results of [1, Ch. II], can be extended.

In view of the similarity in the proofs of the following extensions, with those for the less general Carleman kernels cited earlier [2, pp. 55–58], we merely state the theorems below.

THEOREM 2. (See [2, p. 55].) *Suppose*

$$L_x(\xi, \phi) - \lambda L_x(\xi, K)\phi = 0 \tag{4}$$

admits only the zero solution in L_2, *for a particular nonreal value of* λ. *Suppose further that* $L_x(\xi, p)$ *is real for every real* $p(x)$ *and* $\overline{L_x(\xi, p)} = L_x(\xi, \bar{p})$ (*where the bar denotes conjugation*). *Then,* (4) *will have only the zero* L_2 *solution for all nonreal values of* λ.

THEOREM 3. (See [2, p. 58].) *The number of linearly independent* L_2 *solutions of* (4) *is the same for all real values of* λ.

REMARK. If the operator L_x is closed and

$$L_x(\xi, K)\phi = L_x(\xi, K\phi),$$

i.e.

$$\int_0^1 L_x(\xi, K(x, y))\phi(y)\, dy = L_x\left(\xi, \int_0^1 K(x, y)\phi(y)\, dy\right),$$

then every solution of (2) is a solution of the corresponding equation of the first kind, i.e.

$$K\phi = f.$$

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