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GENERALISED RETRACT SEMIGROUPS

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We give a description of the structure of the semigroups for which each principal ideal is a retract. The globally idempotent case is solved quickly using a—suitably modified—construction which has been developed by Tully for the study of semigroups in which each ideal is a retract. The general case can be treated by a naturally obtained semigroup of subsets of the semigroup constructed for the globally idempotent case.

1. INTRODUCTION

Let $I \subseteq S$ be an ideal of the semigroup S. I is a *retract* ideal if there exists a *retraction*, that is, a homomorphism $\phi : S \to I$ such that $\phi|I = \mathrm{id}_I$. In [4], Tully characterised the structure of *retract semigroups*. He gave a general construction of semigroups in which each ideal is a retract. Roughly speaking, a retract semigroup S can be constructed by means of (1) a semilattice X of a very special type, (2) 0-simple semigroups I_α which are indexed by the elements $\alpha \in X$, and (3) partial homomorphisms $f_{\alpha,\beta} : I_\alpha^* \to I_\beta^*$ given for all $\alpha \geqslant \beta$ which define a multiplicative structure on $\bigcup\limits_{\alpha \in X} I_\alpha^*$. The semilattice X is a tree in which each principal ideal is (downwards) well ordered. (Precisely these semilattices are retract semilattices.) An arbitrary retract semigroup S then can be represented as an inflation of a semigroup which is constructed in the way outlined above. The aim of this note is to show that the construction of Tully also can be used—if suitably extended—to characterise the structure of those semigroups for which each *principal* ideal is a retract. These latter semigroups are called *generalised* retract semigroups.

We suppose that an arbitrary semilattice X is ordered by $x \leqslant y$ if and only if $xy = x$. A semilattice X is a *tree* if any two elements which possess a common upper bound in fact are comparable elements. Furthermore, let S be a subsemigroup of the semigroup T. Then T is an *inflation* of S if there exists a mapping $\phi : T \to S$ such that $\phi|S = \mathrm{id}_S$ and $xy = (x\phi)(y\phi)$ for all $x, y \in T$. For an arbitrary semigroup S let S^* be the *non zero part* of S, that is, $S^* = S$ if S has no zero and $S^* = S \setminus \{0\}$ if 0 is the zero of S. For $x \in S$ let $J(x)$ denote the principal ideal (in S) generated by x.

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For $x, y \in S$ let $x \mathcal{J} y \Leftrightarrow J(x) = J(y)$. The $\mathcal{J}$-class of all generators of the principal ideal $J(x)$ is denoted by J_x. It is well known that $I(x) = J(x) \setminus J_x$ is an ideal in $J(x)$. The semigroup $J(x)/I(x)$ is a *principal factor*. A semigroup S is *semisimple* if each of its principal factors is (0-)simple.

The main result of Tully [4] can be formulated as follows:

THEOREM 1. *Let X be a semilattice such that each non empty subset $A \subseteq X$, for which X possesses an upper bound, in fact contains a greatest element. For each $\alpha \in X$ let I_α be a non trivial 0-simple semigroup such that $I_\alpha^* \cap I_\beta^* = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha \geqslant \beta \in X$ let $f_{\alpha,\beta} : I_\alpha^* \to I_\beta^*$ denote a partial homomorphism, subject to the following conditions:*

(1) $f_{\alpha,\alpha} = \mathrm{id}_{I_\alpha^*}$;
(2) $c(xf_{\alpha,\beta}f_{\beta,\gamma}) = c(xf_{\alpha,\gamma})$ *whenever* $c \in I_\gamma^*, x \in I_\alpha^*$ *and* $\alpha \geqslant \beta \geqslant \gamma$;
(3) $(xf_{\alpha,\beta}f_{\beta,\gamma})c = (xf_{\alpha,\gamma})c$ *whenever* $c \in I_\gamma^*, x \in I_\alpha^*$ *and* $\alpha \geqslant \beta \geqslant \gamma$;
(4) *for arbitrary* $x \in I_\alpha^*, y \in I_\beta^*$ *the set*

$$D(x,y) = \{\gamma \leqslant \alpha, \beta \mid (xf_{\alpha,\gamma})(yf_{\beta,\gamma}) \neq 0 \text{ in } I_\gamma\}$$

is not empty and thus has a greatest element to be denoted by $\delta(x,y)$.

Now let $S = \bigcup_{\alpha \in X} I_\alpha^*$ and define a multiplication $*$ on S by the rule

$$x * y = \big(xf_{\alpha,\delta(x,y)}\big)\big(yf_{\beta,\delta(x,y)}\big) \qquad (x \in I_\alpha^*, y \in I_\beta^*)$$

such that the right hand side product is computed in $I_{\delta(x,y)}^*$. Furthermore, let T be an inflation of a semigroup thus constructed. Then T is a retract semigroup. Conversely, every retract semigroup T can be so obtained.

First we notice that conditions (2) and (3) can be simplified to a transitivity condition for the mappings $f_{\alpha,\beta}$.

LEMMA 1. *Let S be a semigroup constructed as in Theorem 1. Then $f_{\alpha,\beta}f_{\beta,\gamma} = f_{\alpha,\gamma}$ for all $\alpha \geqslant \beta \geqslant \gamma$.*

PROOF: Let $x \in I_\alpha^*, \alpha \geqslant \beta \geqslant \gamma \in X$; then $x = uv$ for some $u, v \in I_\alpha^*$ since I_α is 0-simple. Now

$$\begin{aligned} xf_{\alpha,\beta}f_{\beta,\gamma} &= (uv)f_{\alpha,\beta}f_{\beta,\gamma} \\ &= [(uf_{\alpha,\beta})(vf_{\alpha,\beta})]f_{\beta,\gamma} \\ &= (uf_{\alpha,\beta}f_{\beta,\gamma})(vf_{\alpha,\beta}f_{\beta,\gamma}) \\ &= (uf_{\alpha,\gamma})(vf_{\alpha,\beta}f_{\beta,\gamma}) && \text{by (3)} \\ &= (uf_{\alpha,\gamma})(vf_{\alpha,\gamma}) && \text{by (2)} \\ &= (uv)f_{\alpha,\gamma} = xf_{\alpha,\gamma}. \end{aligned}$$

□

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If in the *construction* of Theorem 1 the special tree X is replaced by an *arbitrary semilattice* X then we still get a construction of a semigroup provided that the existence of the greatest element of the set $D(x,y)$ for each $x,y \in S$ be ensured. We therefore give the following

DEFINITION: Let X be a semilattice and to each $\alpha \in X$ associate a non trivial 0-simple semigroup I_α such that $I_\alpha^* \cap I_\beta^* = \emptyset$ if $\alpha \neq \beta$. For all $\alpha \geqslant \beta \in X$ let $f_{\alpha,\beta} : I_\alpha^* \to I_\beta^*$ be a partial homomorphism satisfying

(1) $f_{\alpha,\alpha} = \mathrm{id}_{I_\alpha^*}$;
(2) $f_{\alpha,\beta} f_{\beta,\gamma} = f_{\alpha,\gamma}$ whenever $\alpha \geqslant \beta \geqslant \gamma$;
(3) for arbitrary $x \in I_\alpha^*, y \in I_\beta^*$ the set

$$D(x,y) = \{\gamma \leqslant \alpha, \beta \mid (xf_{\alpha,\gamma})(yf_{\beta,\gamma}) \neq 0 \text{ in } I_\gamma\}$$

has a greatest element, to be denoted by $\delta(x,y)$.

The semigroup S formally defined in the same fashion as in Theorem 1 is a *Tully semigroup*, to be denoted by $S = (X; I_\alpha, f_{\alpha,\beta})$.

It is routine to see that the groupoid $(X; I_\alpha, f_{\alpha,\beta})$ in fact is a semigroup. Furthermore, if X has a least element μ then by definition I_μ^* is closed under multiplication and thus it is a simple semigroup. Tully semigroups appear in different contexts in semigroup literature (see [**1**, **2**] and [**3**]).

Roughly speaking, Tully semigroups are strong semilattices of the partial semigroups I_α^* such that the product of some $x \in I_\alpha^*, y \in I_\beta^*$ is contained in some I_γ^* for a well-defined $\gamma \leqslant \alpha\beta$. In particular, strong semilattices of simple semigroups are Tully semigroups. In what follows we shall show that this construction is an adequate tool for describing the structure of generalised retract semigroups.

2. THE GLOBALLY IDEMPOTENT CASE

In order to treat this problem we first consider the less complicated case when S is globally idempotent, that is, $S^2 = S$. In this case, almost the same ideas as in [**4**] can be used. Given a globally idempotent generalised retract semigroup S, we shall prove that $S = (X; I_\alpha, f_{\alpha,\beta})$ where $X = S/\mathcal{J}$, the semigroups I_α are the principal factors of S and the partial homomorphisms $f_{\alpha,\beta}$ are obtained as restrictions of certain retract homomorphisms. What we have to prove in fact is that in this case S is semisimple and that $S/\mathcal{J}$ is a semilattice.

LEMMA 2. *Let S be a generalised retract semigroup; then S^2 is semisimple.*

PROOF: Let $x = uv \in S^2$; then $x = (uf_{J(x)})(vf_{J(x)})$ where $f_{J(x)}$ denotes a retraction of S onto $J(x)$. Since $J(x) \subseteq J(yf_{J(x)}) \subseteq J(x)$ for $y = u,v$ we have $uf_{J(x)} \,\mathcal{J}\, x \,\mathcal{J}\, vf_{J(x)}$ which implies that no principal factor of S^2 is null. ∎

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Lemma 3. *Let $S = S^2$ be a generalised retract semigroup; then the intersection of two principal ideals is a principal ideal. In particular, $S/\mathcal{J}$ is a semilattice.*

Proof: $S/\mathcal{J}$ is ordered by $J_x \leqslant J_y$ if and only if $J(x) \subseteq J(y)$. Let $x, y \in S, x \neq y$. Since $xy \in J(x) \cap J(y)$, $S/\mathcal{J}$ is downwards directed. Let $I = J(x)$; we consider the element yf_I for some retract homomorphism $f_I : S \to I$. Let $z \in J(x) \cap J(y)$; then $z = uyv$ for certain $u, v \in S$ because S is semisimple. Since $z \in I$ we may write $z = (uf_I)(yf_I)(vf_I)$. Therefore, $J_z \leqslant J_{yf_I}$ for each $z \in J(x) \cap J(y)$ and thus $J(x) \cap J(y) \subseteq J(yf_I)$. On the other hand, by Lemma 2, $yf_I = u(yf_I)v = uyv$ for suitable $u, v \in I$ so that $yf_I \in J(y)$. Hence $J(yf_I) \subseteq J(x) \cap J(y)$ and thus $J(x) \cap J(y) = J(yf_{J(x)})$. ∎

We are ready to formulate :

Theorem 2. *A globally idempotent semigroup S is a generalised retract semigroup if and only if S is isomorphic to a Tully semigroup $(X; I_\alpha, f_{\alpha,\beta})$.*

Proof: We only give a sketch of the proof since the details can be done in the same way as in [4]. First, let S be a globally idempotent generalised retract semigroup. By Lemma 2, each principal factor $J(x)/I(x)$ is (0)-simple and by Lemma 3, the ordered set $S/\mathcal{J}$ is a semilattice. Therefore, let $X = S/\mathcal{J}$, that is, the ordered set of all classes of $\mathcal{J}$-equivalent elements, and for $\alpha \in X$ let $I_\alpha = J(x)/I(x)$ if $x \in \alpha$. For each principal ideal $J(x)$ in S we choose a fixed retraction $f_{J(x)}$ (which, in general is not unique). Now let $\alpha, \beta \in X, \alpha \geqslant \beta$ and let I denote the principal ideal which is generated by an arbitrary element $y \in \beta$. Define the mapping $f_{\alpha,\beta}$ by $xf_{\alpha,\beta} = xf_I$ for all $x \in I_\alpha^*$. The mapping $f_{\alpha,\beta}$ is clearly a partial homomorphism. Since $f_I|I = \mathrm{id}_I$ condition (1) of Theorem 1 follows. In the same way as [4, Lemma 3] it can be seen that $xf_{\alpha,\beta} \in I_\beta^*$, that is, $f_{\alpha,\beta} : I_\alpha^* \to I_\beta^*$. Furthermore, by the same method as [4, Lemma 4] it follows that conditions (2) and (3) of Theorem 1 hold. Thus by Lemma 1, condition (2) of the definition of a Tully semigroup follows. Furthermore, similarly as [4, Lemma 5] it follows that the multiplication in S can be described in terms of the mappings $f_{\alpha,\beta}$ according to the rules of a Tully semigroup.

Conversely, let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a Tully semigroup. It can be seen easily that for $x, y \in S$ we have $x \;\mathcal{J}\; y$ if and only if $x, y \in I_\alpha^*$, for some $\alpha \in X$, that is, the $\mathcal{J}$-classes of S are precisely the sets $I_\alpha^*, \alpha \in X$. Let $x \in I_\alpha^* \subseteq S$ and $I = J(x)$. We have to show that I is a retract ideal. Let y be an arbitrary element of S, say $y \in I_\beta^*$. Naturally, we define $yf_I = yf_{\beta,\alpha\beta}$. Then $yf_I \in I$ for each $y \in S$ since $I = \bigcup_{\gamma \leqslant \alpha} I_\gamma^*$. Also, $f_I|I = \mathrm{id}_I$. So it remains to show that f_I is a homomorphism. Let $y, z \in S$, say

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$y \in I_\beta^*, z \in I_\gamma^*$ and let $\delta = \delta(y, z)$. Then

$$\begin{aligned}(yz)f_I &= [(yf_{\beta,\delta})(zf_{\gamma,\delta})]f_I \\ &= [(yf_{\beta,\delta})(zf_{\gamma,\delta})]f_{\delta,\alpha\delta} \\ &= (yf_{\beta,\alpha\delta})(zf_{\gamma,\alpha\delta}) \\ &= (yf_{\beta,\alpha\beta}f_{\alpha\beta,\alpha\delta})(zf_{\gamma,\alpha\gamma}f_{\alpha\gamma,\alpha\delta}) \\ &= (yf_I)f_{\alpha\beta,\alpha\delta}(zf_I)f_{\alpha\gamma,\alpha\delta} \\ &= (yf_I)(zf_I).\end{aligned}$$

The last identity is a consequence of $\delta(yf_{\beta,\alpha\beta}, zf_{\gamma,\alpha\gamma}) = \alpha\delta$ which is an easy consequence of the rules of multiplication in a Tully semigroup. Thus $S = (X; I_\alpha, f_{\alpha,\beta})$ is a generalised retract semigroup which completes the proof. □

3. The general case

For ordinary retract semigroups S the connection between S and its semisimple part S^2 is particularly uncomplicated: S is an inflation of S^2. For generalised retract semigroups S this connection is not given in such a simple way as the following example demonstrates.

EXAMPLE. Let I be a 0-simple semigroup for which I^* is not closed under multiplication, say $I = B_2$, the combinatorial Brandt semigroup of five elements. For each $n \in \mathbb{N}$ let I_n denote an isomorphic copy of I such that $I_n \cap I_m = \emptyset$ if $m \neq n$. We assume that the elements of I_n are realised as the elements of I which are indexed by n: $I_n = \{x_n \mid x \in I\}$. For $n \geqslant m \in \mathbb{N}$ let $f_{n,m} : I_n^* \to I_m^*$ be defined by $x_n f_{n,m} = x_m$. Let $X = \mathbb{N} \cup \{0\}$ and $I_0^* = \{0\}$, that is, I_0 is the 0-simple semigroup of two elements. Of course we define $f_{n,0} \equiv 0$ for all $n \in \mathbb{N}$. Then $S = (X; I_n, f_{n,m})$ is a Tully semigroup. The multiplication in S is given by

$$x_n y_m = \begin{cases} (xy)_{\min(n,m)} & \text{if } xy \in I^* \\ 0 & \text{if } xy = 0 \text{ in } I. \end{cases}$$

Now let $T = S \cup \{x\}$ where $x \notin S$ and $x \in I^*$ such that $x^2 = 0$. Define a multiplication on T by $xy_n = x_n y_n, y_n x = y_n x_n$ and $x^2 = x_n^2 = x0 = 0x = 0$ for all $n \in \mathbb{N}$. Then $T \neq T^2 = S$. Also, T is a generalised retract semigroup but it is not an inflation of S.

This example shows that the condition "$S = S^2$" in Lemma 3 cannot be omitted. Adjoining a second element $x' \neq x$ to S which acts in the same way as x we get a generalised retract semigroup $T = S \cup \{x, x'\}$ for which $J(x) \cap J(x') = S$ which is not a principal ideal. Also, this example gives a suggestion as to how to describe the structure of generalised retract semigroups which are not globally idempotent. The idea

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is to consider a semigroup whose elements are suitable subsets $\bar{x} = \{xf_{\alpha,\beta} \mid \beta \leqslant \alpha\} \subseteq S$ of a Tully semigroup S rather than the elements $x \in S$ themselves. First we need an auxiliary definition.

DEFINITION: Let S be a semigroup and $K \subseteq S$; then the equivalence δ_K is defined by

$$x\ \delta_K\ y \Leftrightarrow zx = zy \text{ and } xz = yz \text{ for all } z \in K.$$

DEFINITION: Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a Tully semigroup. A subset $F \subseteq S$ is a *fibre* if

(1) $F \neq \emptyset$;
(2) if $x, y \in F \cap I_\alpha^*$ for some $\alpha \in X$ then $x\ \delta_S\ y$;
(3) if $x \in F \cap I_\alpha^*$ then $xf_{\alpha,\beta} \in F$ for all $\beta \leqslant \alpha$.

By $\mathcal{F}(S) = \mathcal{F}(X; I_\alpha, f_{\alpha,\beta})$ we denote the set of all fibres of S.

The reason for condition (2) is that in general the retract homomorphisms $f_{J(x)}$ and thus the partial homomorphisms $f_{\alpha,\beta}$ need not be unique. This only is true if S is weakly reductive, that is, if $ax = bx$ and $xa = xb$ for all $x \in S$ imply $a = b$. In this case condition (2) becomes the more natural form: $|F \cap I_\alpha^*| \leqslant 1$ for all $\alpha \in X$.

PROPOSITION 1. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a Tully semigroup. If a multiplication on $\mathcal{F}(S)$ is defined as the usual complex product of subsets of S then $\mathcal{F}(S)$ is a semigroup in which S can be embedded in a natural way.*

PROOF: $\mathcal{F}(S)$ is not empty and the multiplication is associative. We thus have to show that the product $FG \in \mathcal{F}(S)$ for $F, G \in \mathcal{F}(S)$. Let $x \in FG \cap I_\gamma^*$; then $x = uv$ for some $u \in F \cap I_\alpha^*, v \in G \cap I_\beta^*$ for some $\alpha, \beta \geqslant \gamma$. Let $\delta \leqslant \gamma$; then $xf_{\gamma,\delta} = [(uf_{\alpha,\gamma})(vf_{\beta,\gamma})]f_{\gamma,\delta} = (uf_{\alpha,\delta})(vf_{\beta,\delta}) \in FG$ so that FG satisfies condition (3). Let $uv, xy \in FG \cap I_\gamma^*$ for some $u, x \in F, v, y \in G$; if $u \in I_\alpha^*, v \in I_\beta^*$ then $uv = (uf_{\alpha,\gamma})(vf_{\beta,\gamma})$ and $uf_{\alpha,\gamma} \in F, vf_{\beta,\gamma} \in G$. Therefore the elements u, x, v, y may be replaced by their respective images in I_γ^* under the appropriate functions $f_{.,\gamma}$. Hence we may assume that $u, x, v, y \in I_\gamma^*$. By condition (2) we have $u\ \delta_S\ x$ and $v\ \delta_S\ y$. Thus $uv = xv = xy$ which implies that $|FG \cap I_\gamma^*| \leqslant 1$ for all $\gamma \in X$. For the product of two fibres in fact a stronger condition than (2) holds. Let $x \in S, x \in I_\alpha^*$ and $\bar{x} = \{xf_{\alpha,\beta} \mid \beta \leqslant \alpha\}$; then $\bar{x} \in \mathcal{F}(S)$ and by the rules of multiplication in a Tully semigroup we have $\bar{x}\bar{y} = \overline{xy}$. Therefore, $S \cong \overline{S} = \{\bar{x} \mid x \in S\} \subseteq \mathcal{F}(S)$. ∎

REMARK. Defining a relation $\leqslant$ by $u \leqslant v$ if and only if $u = vf_{\alpha,\beta}$ for suitable $\alpha, \beta \in X$, each fibre $F \in \mathcal{F}(S)$ becomes an ordered set. The fibre $F \in \mathcal{F}(S)$ is of the form $F = \bar{x}$ for some $x \in S$ if and only if F possesses a greatest element (with respect to $\leqslant$).

DEFINITION: For $u \in \mathcal{F}(S)$ let $\operatorname{supp} u = \{\gamma \in X \mid u \cap I_\gamma^* \neq \emptyset\}$. Then $\operatorname{supp} u$ is an ideal in X for each $u \in \mathcal{F}(S)$.

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We now are ready to formulate

THEOREM 3. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a Tully semigroup and let $T \subseteq \mathcal{F}(S)$ be a semigroup of fibres of S subject to the following conditions:*

(1) $\overline{S} = \{\bar{x} \mid x \in S\} \subseteq T$;
(2) $t^2 \in \overline{S}$ *for all* $t \in T$;
(3) $\operatorname{supp} t \cap \operatorname{supp} u$ *has a greatest element for all* $t \neq u \in T$.

Let U be an inflation of T. Then U is a generalised retract semigroup. Conversely, every generalised retract semigroup can be so constructed.

PROOF: Direct part. Let T be a semigroup subject to the conditions (1)–(3) and let U be an inflation thereof with inflation function $\phi : U \to T$. Let $u \in U$ be a fixed element and let $f_u : U \to J(u)$ denote a mapping defined by

$$vf_u = \begin{cases} u & \text{if } v \in U \setminus S \text{ and } u\phi = v\phi \\ \overline{v_\gamma} & \text{for some } v_\gamma \in v\phi \cap I_\gamma^* \text{ where } \gamma = \max(\operatorname{supp} v\phi \cap \operatorname{supp} u\phi) \text{ else.} \end{cases}$$

Conditions (2) and (3) imply that $U^2 \subseteq \overline{S}$. From this we obtain that $J(u) = \{u\} \cup \bigcup_{\gamma \in \operatorname{supp} u\phi} \overline{I_\gamma^*}$. In particular $f_u : U \to J(u)$. Suppose that $u \in U \setminus S$; then $uf_u = u$. If $\bar{x} \in J(u), \bar{x} \neq u$ then $\bar{x} \in \overline{I_\gamma^*}$ for some $\gamma \in \operatorname{supp} u\phi$. Now x is the unique element in $\bar{x} \cap I_\gamma^*$ and $\gamma = \max \operatorname{supp} \bar{x} = \max(\operatorname{supp} \bar{x} \cap \operatorname{supp} u\phi)$. In particular, $\bar{x} f_u = \bar{x}$. If $u = \bar{z} \in \overline{S}, \bar{z} \in \overline{I_\gamma^*}$ then for each $\bar{x} \in J(\bar{z})$ we have $\bar{x} \in \overline{I_\beta^*}$ for some $\beta \leqslant \gamma$. In particular, x is the unique element in $\bar{x} \cap I_\beta^*$ and $\beta = \max(\operatorname{supp} \bar{x} \cap \operatorname{supp} \bar{z})$. Therefore, in this case we also have $f_u | J(u) = \mathrm{id}_{J(u)}$. It remains to show that f_u is a homomorphism. Let $v, t \in U$; then $vt = (v\phi)(t\phi) = \overline{v_\gamma t_\gamma}$ where $\gamma = \max\{\delta \in X \mid v_\delta t_\delta \in I_\delta^*, v_\delta \in v\phi \cap I_\delta^*, t_\delta \in t\phi \cap I_\delta^*\}$. Furthermore, $(vt)f_u = \overline{(v_\gamma t_\gamma) f_{\gamma,\delta}}$ where $\delta = \max\left(\operatorname{supp} u\phi \cap \operatorname{supp} \overline{v_\gamma t_\gamma}\right) = \max(\operatorname{supp} u\phi \cap (\gamma])$ where $(\gamma] = \{\alpha \in X \mid \alpha \leqslant \gamma\}$. Now consider the product $(vf_u)(tf_u)$ and suppose first that $vf_u = \overline{v'_\alpha}, v'_\alpha \in v\phi \cap I_\alpha^*, \alpha = \max(\operatorname{supp} v\phi \cap \operatorname{supp} u\phi)$ and $tf_u = \overline{t'_\beta}, t'_\beta \in t\phi \cap I_\beta^*, \beta = \max(\operatorname{supp} t\phi \cap \operatorname{supp} u\phi)$. Now $\overline{v'_\alpha}\,\overline{t'_\beta} = \overline{v'_\varepsilon t'_\varepsilon}$ where $v'_\varepsilon = v'_\alpha f_{\alpha,\varepsilon}, t'_\varepsilon = t'_\beta f_{\beta,\varepsilon}$ and

$$\begin{aligned} \varepsilon &= \max\{\nu \in X \mid (v'_\alpha f_{\alpha,\nu})(t'_\beta f_{\beta,\nu}) \in I_\nu^*, \nu \leqslant \alpha, \beta\} \\ &= \max\{\nu \in X \mid v_\nu t_\nu \in I_\nu^*, v_\nu \in v\phi \cap I_\nu^*, t_\nu \in t\phi \cap I_\nu^*, \\ &\qquad \nu \leqslant \max(\operatorname{supp} v\phi \cap \operatorname{supp} u\phi), \nu \leqslant \max(\operatorname{supp} t\phi \cap \operatorname{supp} u\phi)\} \\ &= \max\{\nu \in X \mid v_\nu t_\nu \in I_\nu^*, v_\nu \in v\phi \cap I_\nu^*, t_\nu \in t\phi \cap I_\nu^*, \nu \in \operatorname{supp} u\phi\} \\ &= \max(\{\nu \in X \mid v_\nu t_\nu \in I_\nu^*, v_\nu \in v\phi \cap I_\nu^*, t_\nu \in t\phi \cap I_\nu^*\} \cap \operatorname{supp} u\phi) \\ &= \max((\gamma] \cap \operatorname{supp} u\phi) = \delta. \end{aligned}$$

The elements $v_\gamma f_{\gamma,\delta}$ and $v'_\alpha f_{\alpha,\delta}$, respectively $t_\gamma f_{\gamma,\delta}$ and $t'_\beta f_{\beta,\delta}$ are not necessarily equal but are contained in the fibre $v\phi$, respectively $t\phi$, and hence are δ_S-equivalent. In par-

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ticular, $(v_\gamma t_\gamma)f_{\gamma,\delta} = (v_\gamma f_{\gamma,\delta})(t_\gamma f_{\gamma,\delta}) = (v'_\alpha f_{\alpha,\delta})\left(t'_\beta f_{\beta,\delta}\right)$ and thus $(vt)f_u = (vf_u)(tf_u)$. Similarly but more easily it can be seen that the latter is also true if $vf_u = u$ or/and $tf_u = u$.

Converse. Let U be a generalised retract semigroup. By Lemma 2 and Theorem 2, $U^2 = S$ is a Tully semigroup $(X; I_\alpha, f_{\alpha,\beta})$. For $\alpha \in X$ let $K(\alpha) = \bigcup_{\beta \leqslant \alpha} I^*_\beta$. Let $u \in U \setminus S$, $\gamma \in X$ and

$$F_\gamma(u) = \{y \in K(\gamma) \mid zu = zy \text{ and } uz = yz \text{ for all } z \in K(\gamma)\},$$

that is, $F_\gamma(u)$ is the set of all elements of $J(z)$ (for some $z \in I^*_\gamma$) whose action on $J(z)$ is the same as that of u. Let $F(u) = \bigcup_{\gamma \in X} F_\gamma(u)$; then $F(u) \in \mathcal{F}(S)$. Condition (1) holds since U is a generalised retract semigroup and thus $F_\gamma(u) \neq \emptyset$ for each $\gamma \in X$. Let $x, y \in F(u) \cap I^*_\gamma$; then $x \in F_\alpha(u), y \in F_\beta(u)$ for certain $\alpha, \beta \geqslant \gamma$. We have $uz = xz, zu = zx$ for all $z \in K(\alpha)$ and $uz = yz, zu = zy$ for all $z \in K(\beta)$. Since $K(\gamma) \subseteq K(\alpha) \cap K(\beta)$ it follows that $xz = uz = yz$ and $zx = zu = zy$ for all $z \in K(\gamma)$. By the rules of multiplication in a Tully semigroup it follows that $xz = yz, zx = zy$ for all $z \in S$ so that $x\ \delta_S\ y$. Further, let $x \in F(u), x \in I^*_\beta$; then $x \in F_\alpha(u)$ for some $\alpha \geqslant \beta$. If $\gamma \leqslant \beta$ then again by the multiplication in $S = (X; I_\alpha, f_{\alpha,\beta})$ it follows that $xf_{\beta,\gamma} \in F_\gamma(u)$ and thus $xf_{\beta,\gamma} \in F(u)$. Now consider the mapping $\phi : U \to \mathcal{F}(S)$ defined by

$$u\phi = F(u) \qquad \text{if } u \in U \setminus S,$$
$$u\phi = \bar{u} = \{uf_{\alpha,\beta} \mid \beta \leqslant \alpha\} \qquad \text{if } u \in I^*_\alpha \subseteq S.$$

First we show that ϕ is a homomorphism. Let $u, v \in U \setminus S$; then $uv = (uf)(vf) \in S$ where f denotes a retraction of U on $J(uv)$. In particular, $uf, vf, uv = (uf)(vf) \in I^*_\gamma$ and $J(uv) = K(\gamma)$ for some $\gamma \in X$. Now let $x \in u\phi, y \in v\phi, x \in I^*_\alpha, y \in I^*_\beta, \delta = \delta(x, y)$. There exists $z \in I^*_\delta$ such that $xyz \in I^*_\delta$. By definition of ϕ, $xs = us$ for all elements s of a principal ideal of S which contains $K(\delta)$ so that $xyz = uyz$ since $yz \in K(\delta)$. Similarly we obtain $yz = vz$ so that $xyz = uyz = uvz$. In particular, $\delta \leqslant \gamma$. Since $uf \in F_\gamma(u), vf \in F_\gamma(v)$ it follows that $uv = (uf)(vf) \in F(u)F(v) = (u\phi)(v\phi)$. Now $(u\phi)(v\phi)$ is a fibre for which $|(u\phi)(v\phi) \cap I^*_\delta| \leqslant 1$ for each $\delta \in X$ and which has the greatest element $uv = (uf)(vf)$ (with respect to the order relation defined in the Remark after Proposition 1.) Therefore, $(u\phi)(v\phi) = [(uf)(vf)]\phi = (uv)\phi$. If $u \in S$ or/and $v \in S$ then the same can be proved similarly. Hence ϕ is a homomorphism. Now consider the relation $\delta = \delta_U$. Let $\{u_i \mid i \in U/\delta\}$ be a system of representatives of the δ-classes of U such that $u_i \in S$ if $u_i\delta \cap S \neq \emptyset$. For $u \in U \setminus S$ let $u\psi = u_i$ if $u\ \delta\ u_i$, for $u \in S$ let $u\psi = u$. Let $T = U\psi\phi$; then T is a subsemigroup of $\mathcal{F}(S)$

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such that $T^2 = \overline{S}$ and in particular $t^2 \in \overline{S}$ for all $t \in T$. We have to show that $\psi : u \mapsto u\psi$ is an inflation function and that $\phi|U\psi$ is injective. The first follows immediately by definition: $u\psi\psi = u\psi$ and $uv = (u\psi)v = (u\psi)(v\psi)$ for all $u, v \in U$, and $U\psi$ is a subsemigroup of U. Let $u, v \in U$ such that $u\phi = v\phi$ and $z \in U$; then $u\phi z\phi = v\phi z\phi$ and $z\phi u\phi = z\phi v\phi$. Since ϕ is a homomorphism we obtain $(uz)\phi = (vz)\phi$ and $(zu)\phi = (zv)\phi$ and thus, since $\phi|S$ is injective, $uz = vz, zu = zv$ for arbitrary $z \in U$. Therefore, if $u, v \in U\psi$ then either $u = v$ or $u, v \in S$. In the latter case also $u = v$ since ϕ is injective on S. It remains to show that $\operatorname{supp} u\phi \cap \operatorname{supp} v\phi$ is a principal ideal in X if $u \neq v \in U\psi$. For convenience we shall identify the elements u and $u\phi$ in the following (in particular, $\operatorname{supp} u$ then stands for $\operatorname{supp} F(u)$ or $\operatorname{supp} \bar{u}$). Let $u \neq v \in U\psi$. If $u, v \in \overline{S}$ then clearly $\operatorname{supp} u \cap \operatorname{supp} v$ is a principal ideal since both ideals are principal ideals. Now let $u, v \notin \overline{S}$. The principal ideal in U which is generated by u is given by $J(u) = \{u\} \cup \bigcup_{\gamma \in \operatorname{supp} u} I^*_\gamma$. Let $f : U \to J(u)$ denote a retraction. We consider the element $vf \in J(u)$. First assume that $vf \neq u$; then $vf \in I^*_\gamma$ for some $\gamma \in \operatorname{supp} u$. We have $(vf)z = vz$ and $z(vf) = zv$ for all $z \in K(\gamma)$ since $K(\gamma) \subseteq J(u)$. In particular $vf \in F_\gamma(v)$ so that $\gamma \in \operatorname{supp} u \cap \operatorname{supp} v$. Let $\beta \in \operatorname{supp} u \cap \operatorname{supp} v$ and $x \in F(v) \cap I^*_\beta$. Then there exists $z \in I^*_\beta$ so that $xz = vz \in I^*_\beta$. Since $\beta \in \operatorname{supp} u$ we have $z \in J(u)$ and thus $xz = vz = (vf)z \in I^*_\beta$ so that $\beta \leqslant \gamma$ since $vf \in I^*_\gamma$. Therefore, $\operatorname{supp} u \cap \operatorname{supp} v$ possesses the greatest element γ and hence it is a principal ideal. Now consider the case $vf = u$. Then $vz = uz$ and $zv = zu$ for all $z \in J(u)$. Let $\gamma \in \operatorname{supp} u$ and $x \in F_\gamma(u)$. Then, since $K(\gamma) \subseteq J(u)$, $xz = uz = vz$ and $zx = zu = zv$ for all $z \in K(\gamma)$ and thus $x \in F_\gamma(v)$. In particular, $F(u) \subseteq F(v)$. Now consider the same procedure for $J(v) = \{v\} \cup \bigcup_{\gamma \in \operatorname{supp} v} I^*_\gamma$. If $uf \neq v$ for a retraction $f : U \to J(v)$ then in the same way as above we obtain that $\operatorname{supp} u \cap \operatorname{supp} v$ has a greatest element and thus it is a principal ideal. If $uf = v$ then by the same procedure as above we have $F(v) \subseteq F(u)$ and thus $F(u) = F(v)$. So $u\phi = v\phi$ and thus, similarly as above, $uz = vz, zu = zv$ for all $z \in U$, that is, $u\ \delta_U\ v$. Since $u, v \in U\psi \setminus S$ we get $u = v$. For the case when $u \in U\psi \setminus S$ and $v \in S, v \in I^*_\alpha$ we consider a retraction $f : U \to J(v)$ and the element $uf \in I^*_\beta$ for some $\beta \leqslant \alpha$. By the same reason as above β is the greatest element of $\operatorname{supp} u \cap \operatorname{supp} v$. ∎

REMARK. A similar example as above shows that $T^2 = \overline{S}$ in general is not sufficient in order that $\operatorname{supp} u \cap \operatorname{supp} v$ has a greatest element for all $u \neq v \in T$.

COROLLARY 1. *Let S be a generalised retract semigroup. If $S^2 = (X; I_\alpha, f_{\alpha,\beta})$ is a semilattice of simple semigroups, that is, if each I^*_α is closed under multiplication then S is an inflation of S^2.*

PROOF: Let $x \in S$; the action of x on S^2 is given by the fibre $F(x)$. Also, $x^2 = z \in I^*_\alpha$ for some $\alpha \in X$. Then $z = x_\alpha x_\alpha$ for some $x_\alpha \in F(x) \cap I^*_\alpha$ and

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$\alpha = \max\{\beta \mid x_\beta x_\beta \in I^*_\beta, x_\beta \in F(x) \cap I^*_\beta\}$. Since each I^*_β is closed under multiplication, $\alpha = \max \operatorname{supp} F(x)$. Then the mapping

$$x \mapsto \begin{cases} x_\alpha \text{ for some } x_\alpha \in F(x) \cap I^*_\alpha, \alpha = \max \operatorname{supp} F(x) & \text{if } x \in S \setminus S^2, \\ x & \text{if } x \in S^2. \end{cases}$$

is an inflation function. ∎

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