

PERIODIC ORBITS OF ISOMETRIC FLOWS

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1. Introduction

Let M be a compact C^∞ Riemannian manifold, X a Killing vector field on M , and φ_t its 1-parameter group of isometries of M . In this paper, we obtain some basic properties of the set of periodic points of φ_t . We show that the set of least periods is always finite, and the set $P(X, t)$ of points of M having least period t for the vector field X is a totally geodesic submanifold, with possibly non-empty boundary. Moreover, we show there are at least m geometrically distinct closed geodesic orbits of φ_t , where m is the number of least periods which are not integral multiples of any other least period.

2. Finiteness of least periods

Let M be a complete Riemannian manifold of dimension n . Let $I^0(M)$ be the identity component of its isometry group, and $i(M)$ the Lie algebra of $I^0(M)$. $i(M)$ is naturally identified with the Lie algebra of Killing vector fields on M , and we will identify an element $X \in i(M)$ with the corresponding Killing vector field. If X is any vector field on M , let $\text{Zero}(X) = \{p \in M \mid X_p = 0\}$. We use $\text{Fix}(f)$ for the set of fixed points of a map $f: M \rightarrow M$.

LEMMA. *Let $X \in i(M)$ and φ_t its 1-parameter group. For each $p \in M$ there is a neighborhood U of p such that the set of least periods of periodic orbits of φ_t which intersect U is finite.*

Proof. (i) If $p \in \text{Zero}(X)$ then $(\varphi_t)_*: T_pM \rightarrow T_pM$ is a 1-parameter subgroup of the orthogonal subgroup of the orthogonal group of T_pM , so there is a basis of T_pM in which $(\varphi_t)_*|T_pM$ has the form:

$$(\varphi_t)_* = \text{diag} \{a_1(t), \dots, a_k(t), I_{n-2k}\}$$

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where

$$a_i(t) = \begin{pmatrix} \cos \alpha_i t & \sin \alpha_i t \\ -\sin \alpha_i t & \cos \alpha_i t \end{pmatrix}$$

$\alpha_i \neq 0$ for $i = 1, \dots, k$, and I_{n-2k} is the $(n - 2k) \times (n - 2k)$ identity matrix. The periodic points of φ_t in a spherical neighborhood of p correspond, by $\exp_p: T_p M \rightarrow M$, to periodic vectors in $T_p M$ under $(\varphi_t)_*$. The least periods of such vectors are the least common multiples of subsets of the set of numbers $\{2\pi/a_1, \dots, 2\pi/a_k\}$ and thus are finite in number.

(ii) If $p \notin \text{Zero}(X)$, let t_0 be the least period of p . (Let $t_0 = +\infty$ if p is not periodic). If $t_0 = \infty$, then either p lies in a neighborhood of non-periodic points, or there is a sequence of periodic points $p_i \rightarrow p$. In the first case we're done, and in the second we replace p by a p_i sufficiently close to p so p lies in a convex normal neighborhood of p_i . Thus we assume p is periodic of least period $0 < t_0 < +\infty$. For each $\varepsilon > 0$ let $N_\varepsilon = \{Y \in T_p M \mid Y \perp X_p \text{ and } |Y| < \varepsilon\}$. If ε is sufficiently small, then $\varphi_t(\exp_p N_\varepsilon) \cap \varphi_{t'}(\exp_p N_\varepsilon) \neq \emptyset$ only when $t - t' \equiv 0 \pmod{t_0}$, and $\bigcup_t \varphi_t(\exp_p N_\varepsilon)$ is a tubular neighborhood of the closed orbit $\{\varphi_t(p) \mid t \in \mathbb{R}\}$. Let $\delta \in (0, t_0/2)$ be so small that $U = \bigcup_t \{\varphi_t(\exp_p N_\varepsilon) \mid |t| < \delta\}$ is a normal neighborhood of p . We may assume the p_i we chose to replace p is close enough to p so they both lie in U . Now if we put $(\varphi_{t_0})_*|T_p M$ in the same normal form as in (i), the same argument shows there are only finitely many least periods of periodic points in U . q.e.d.

Remark. It follows from the proof that the number of least periods in each of the neighborhoods U is bounded above by the maximal number possible of least common multiples in a set of $[n/2]$ numbers; namely $2^{[n/2]}$. This is independent of the choice of X , but the neighborhood U does depend on X .

COROLLARY. *If M is compact, the set of least periods of φ_t is finite.*

This corollary thus follows from a simple geometrical argument. One can derive the same result using a theorem of Yang ([4]), which says that a compact Lie group acting differentiably on a compact manifold has only finitely many non-conjugate isotropy subgroups.

From now on, M will always be compact. The Lie algebra $\mathfrak{i}(M)$ has an $\text{ad}(I^0(M))$ -invariant positive definite symmetric bilinear form, and we let S be the unit sphere in $\mathfrak{i}(M)$ with respect to this form.

LEMMA. *There is a number $a > 0$ such that for every $X \in S$ and every isotropy subgroup G_p , if $0 < t_0 < a$ and $\exp(t_0 X)$ is in G_p , then $\exp(tX) \in G_p$ for all $t \in R$.*

Proof. $I^0(M)$ is a compact Lie group and therefore a compact symmetric space whose geodesics are the translates of 1-parameter subgroup $\exp(tX)$. If we assume $X \in S$ then t is arc-length. Let U_ε be the open ball of radius ε about 0 in $i(M)$, and assume $\varepsilon > 0$ is so small that $\exp(U_\varepsilon)$ lies in a normal neighborhood of e in $I^0(M)$. Suppose $0 < t_0 < \varepsilon$ and $\exp(t_0 X) \in G_p =$ identity component of G_p . There is a minimizing geodesic $\exp(tY)$ from e to $\exp(t_0 X)$ lying entirely in G_p^0 , whose length is less than ε (since $\exp(U_\varepsilon) \cap G_p^0$ is the ε -ball about e in G_p^0). Now $\exp|_{U_\varepsilon}$ is 1-1 so we must have $\exp(tY) = \exp(tX)$ for all t .

If G_p is not connected, it has finitely many components and we let $r_p =$ distance $(e, G_p - G_p^0)$. Clearly $r_p > 0$ since $G_p - G_p^0$ is compact; and r_p is constant on the conjugacy class of G_p since the metric in $I^0(M)$ is invariant by conjugation. There are only finitely many conjugacy classes so $r = \min r_p > 0$. Then any $0 < a < \min(\varepsilon, r)$ satisfies the requirements of the lemma. q.e.d.

From this lemma we can derive a "uniform" period bounding lemma for Killing vector fields:

COROLLARY. *The positive least periods of periodic orbits of 1-parameter groups of isometries are bounded away from zero uniformly if their generators X are taken from S .*

Proof. A point $p \in M$ is periodic of least period t_0 for the 1-parameter group $\exp(tX)$ if $\exp(t_0 X) \in G_p$ but $\exp(tX) \notin G_p$ if $0 < t < t_0$. The number a of the previous lemma is then the required lower bound. q.e.d.

3. Submanifolds of periodic points

For each $X \in i(M) - \{0\}$ and each $0 < t < \infty$, let $P(X, t) = \{p \in M \mid p$ has least period t for the 1-parameter group $\exp(tX)\}$. Let $P(X, 0) =$ Zero (X) , and $P(X, \infty) =$ set of non-periodic points of $\exp(tX)$. Then we know $P(X, t) = \emptyset$ except for a finite subset of $[0, \infty]$. Now assume $X \in i(M) - \{0\}$ is fixed, and φ_t is its 1-parameter group. It is well-known that for each $0 < t < \infty$, $\text{Fix}(\varphi_t)$ is a closed totally geodesic submanifold of M .

Let $0 < t_1 < t_2 < \dots < t_N$ be the positive least periods for $X \in \mathfrak{i}(M)$. Let β_1, \dots, β_m be the subset of least periods which are not integral multiples of any other least period. Use the notation $t_i | t_j$ if $t_j = kt_i$ for some integer k , and call the β_i the *basic periods* for X . It is easy to see that for each $i = 1, \dots, N$, $P(X, t_i) = \text{Fix}(\varphi_{t_i}) - \cup\{\text{Fix}(\varphi_{t_j}) | t_j | t_i\} - \text{Zero}(X)$, the union consisting of finitely many closed totally geodesic submanifolds of $\text{Fix}(\varphi_{t_i})$. Therefore we have:

PROPOSITION. *Each $P(X, t_i)$ is a totally geodesic submanifold of M (possibly with a finite number of closed submanifolds deleted).*

THEOREM. *If the Killing vector field X has m basic periods, there are at least m geometrically distinct smooth closed geodesics on M which are orbits of the 1-parameter group of X .*

Proof. For each $i = 1, \dots, m$, $P(X, \beta_i) = \text{Fix}(\varphi_{\beta_i}) - \text{Zero}(X)$. Now $\text{Fix}(\varphi_{\beta_i})$ is a closed totally geodesic submanifold of M , and X is tangent to it, so X is Killing vector field on $\text{Fix}(\varphi_{\beta_i})$. Let $p \in \text{Fix}(\varphi_{\beta_i})$ be a point at which $|X|$ achieves its maximum. Then $|X|^2$ has a critical point at p , so ([2]) the orbit of p is a geodesic. The orbit is non-trivial since $p \notin \text{Zero}(X)$. q.e.d.

Remarks. (1) In fact, we get a closed geodesic for each component of $\text{Fix}(\varphi_{\beta_i})$.

(2) The same argument as in Kobayashi ([1]), shows that the Euler numbers of the $\text{Fix}(\varphi_{t_i})$ all equal that of M .

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