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MARK J. SCHERVISH
Department of Statistics and Data Science, Carnegie Mellon University

TEDDY SEIDENFELD
Department of Philosophy and Department of Statistics and Data Science, Carnegie Mellon University

JOSEPH B. KADANE
Department of Statistics and Data Science, Carnegie Mellon University

RUOBIN GONG
Department of Statistics, Rutgers University
and

RAFAEL B. STERN
Instituto de Matemática e Estatística Universidade de São Paulo

Abstract. In this paper, we show how to represent a non-Archimedean preference over a set of random quantities by a nonstandard utility function. Non-Archimedean preferences arise when some random quantities have no fair price. Two common situations give rise to non-Archimedean preferences: random quantities whose values must be greater than every real number, and strict preferences between random quantities that are deemed closer in value than every positive real number. We also show how to extend a non-Archimedean preference to a larger set of random quantities. The random quantities that we consider include real-valued random variables, horse lotteries, and acts in the theory of Savage. In addition, we weaken the state-independent utility assumptions made by the existing theories and give conditions under which the utility that represents preference is the expected value of a state-dependent utility with respect to a probability over states.

§1. Introduction.

1.1. Motivation. The primary goal of this paper is to extend three well-known theories of decision-making to allow for non-Archimedean (unbounded and/or discontinuous) preferences as defined in Definition 1.1 below. The theories that we extend are (i) coherent previsions of [11], (ii) lotteries and horse lotteries of [1, 41], and (iii) general acts of [35]. The objects that an agent compares differ amongst the three theories. In Section 3 below we show how to interpret each of the theories as a special case of the framework that we develop in this paper.

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DEFINITION 1.1. *Let $\mathcal{X}$ be a set.*

1. *A binary relation $\precsim$ on $\mathcal{X}$ is a* preorder *if it is both reflexive and transitive. A preorder $\precsim$ is* total *if, for all $X, Y \in \mathcal{X}$, $(X \precsim Y) \vee (Y \precsim X)$.*
2. *A binary relation $\ll$ on $\mathcal{X}$ is a* strict partial order *if it is transitive and asymmetric ($X \ll Y$ implies $\neg[Y \ll X]$).*
3. *If $\ll$ is a strict partial order on $\mathcal{X}$ we say that $\ll$ is* Archimedean[1] *if, for all $X, Y, Z \in \mathcal{X}$,*

$$(\alpha X + [1-\alpha]Y \ll Z \text{ for all real } 0 < \alpha \leq 1) \text{ implies } \neg(Z \ll Y). \tag{1}$$

4. *Let $\precsim$ be a preorder on $\mathcal{X}$ that expresses an agent's preferences. Let $\ll$ be a strict partial order on $\mathcal{X}$ such that $X \ll Y$ implies $X \precsim Y$. If $\ll$ is not Archimedean, we say that the agent's preferences are* non-Archimedean.

Archimedean preferences are continuous at $\alpha = 0$ for mixtures of the form in the first clause of (1). Also, if X is worth infinitely less than Z which in turn is just a little bit less valuable than Y, then (1) can fail.

EXAMPLE 1. *Let Ω be an infinite set with a σ-field Σ of measurable sets and a countably additive probability defined on it. Let X_0 be an unbounded real-valued function with infinite expected value, and let $\mathcal{X}$ be the linear span of $\{X_0\}$ and the set of bounded real-valued measurable functions defined on Ω. Each element of $\mathcal{X}$ has a unique representation as $X = aX_0 + Y$ where a is a real number and Y is bounded. The expected value of each such X is $E(Y)$ if $a = 0$ and $\mathrm{sign}(a)\infty$ if $a \neq 0$. Define $X \ll Z$ to mean $E(X) < E(Z)$. To see that $\ll$ is not Archimedean, let $X =- X_0$, $Y = 2$, and $Z = 1$ in (1).*

The need for non-Archimedean preferences arises in several ways. One way is when certain needs must be satisfied before others can be addressed as described by Rekola [32]. Another is when some options are deemed infinitely more valuable than others. While modelling ecological preferences, Gelso and Peterson [17] say (p. 36) "someone who regards biodiversity protection as a moral duty cannot be compensated for the extinction of a species." For preferences between random variables, larger is often preferred to smaller. It is common to express such preferences through expected utility maximization. When some random variables are unbounded (see Example 1 above) or when the domain space is uncountable (see Examples 4 and 5 below), certain "larger is better" preferences are inherently non-Archimedean.

All three Archimedean theories that we extend also assume that preferences are state-independent, are expressed via total preorders, and satisfy a linearity assumption. We relax the state-independence and total-preorder assumptions and drop altogether the assumption that preferences are Archimedean. Allowing utility to be state-dependent is particularly important in financial applications where different states of the world can entail different exchange rates between currencies as in [36] and/or different relative prices for commodities. We maintain a linearity assumption in order to achieve an expected-utility representation.

[1] There are many definitions of $\ll$ being *Archimedean*, each of which is combined with other assumptions to prove results about $\ll$. The particular definition here is taken from Chapter 5 of [16] and which he calls "the weakest Archimedean axiom that will suffice for the one-way representation." The one-way representation of a strict partial order $\ll$ corresponds to the existence of a function that *agrees with* $\ll$ as defined in Definition 1.3 in Section 1.3 below.

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A popular way to model non-Archimedean preference is through a lexicography of Archimedean preferences. Let $A = \{\precsim_\alpha\}_{\alpha \in \aleph}$ be a well-ordered set of total preorders that represent Archimedean preferences on a set $\mathcal{X}$. The resulting lexicographic preference $\precsim_A$ is defined by (i) $X \sim_A Y$ if $X \sim_\alpha Y$ for all α, and (ii) $X \prec_A Y$ if $X \prec_\alpha Y$ for the first α such that $\neg(X \sim_\alpha Y)$. A common example is to let each $\precsim_\alpha$ be a ranking by expected utility values $P_\alpha(U_\alpha(X))$. Previously mentioned authors [17, 32] use lexicographies to model their non-Archimedean preferences. Also, Blume, Brandenburger, and Dekel [7] use lexicographies in game theory to allow conditioning on events that would otherwise have zero probability and Miranda and Van Camp [25] use lexicographies to model sets of acceptable choices in a non-Archimedean setting. See also [6, 21, 31] for some theoretical considerations. Halpern [20] contains an extensive comparison of the uses of lexicographic preferences and nonstandard numbers for representing preferences. In particular, Examples 3.3 and 4.8 of that paper show that there are cases of non-Archimedean preferences that cannot be modeled via lexicographies of standard expected utilities while showing that all lexicographic preferences can be represented by a nonstandard utility. See also [33]. In this way, a nonstandard representation is a strict generalization of lexicographic preferences. In this paper we use nonstandard-valued functions to represent non-Archimedean preferences.

1.2. Standard versus nonstandard numbers. For the remainder of the paper, we refer to the familiar real numbers in the set $\mathbb{R}$ as *standard* numbers to distinguish them from the nonstandard numbers that we describe in Appendix A.1 and use liberally throughout the paper. We call a function *numerical* if it takes either standard or nonstandard values. In all cases, the calculations that are part of an agent's expressions of preference involve only standard numbers. We use nonstandard numbers to represent an agent's preferences after preferences are expressed and to infer a probability and utility to express that representation. Since we use multiple number systems, we need to be careful about what we mean by "linear" in various settings.

DEFINITION 1.2. *A space $\mathcal{W}$ of functions is a* standard-linear space *if $\alpha Y + \beta Z \in \mathcal{W}$ for all standard α, β and all $Y, Z \in \mathcal{W}$. A nonstandard-valued function U on a standard-linear space $\mathcal{W}$ is called a* standard-linear function *if $U(\alpha Y + \beta Z) = \alpha U(Y) + \beta U(Z)$, for all $Y, Z \in \mathcal{W}$ and all standard α, β. The* standard-linear span *of a set is the smallest standard-linear space containing the set.*

Notice that $U(0) = 0$ for every standard-linear function U. Definition 1.2 restricts the coefficients in linear combinations to be standard even though the values of U might be nonstandard. Readers desiring a more thorough understanding of nonstandards than we present in Appendix A could read one of the many treatments such as [27, 34].

Other treatments of probability and/or decision theory that make use of nonstandard numbers include [4, 13, 29, 42]. Section 3.2 of [29] has extensive references along with some details of some of the attempts to make use of nonstandard-valued probabilities. The same author, in [30], investigates representations of non-Archimedean coherent preference over unconditional real-valued gambles. Our representation incorporates coherent conditional preferences in Section 4. The approach of [4, 5] is primarily to define probabilities that take infinitesimal values. For a probability P on a set Ω to be a "non-Archimedean probability," in their terminology, they impose a condition that requires all singletons $\{\omega\} \in \Omega$ to have probabilities that are standard multiples of a common infinitesimal ε. That is, there is an infinitesimal ε such that for every $\omega \in \Omega$, there is a standard $a_\omega > 0$ such that $P(\{\omega\}) = a_\omega \varepsilon$. This assumption places restrictions on the forms of non-Archimedean preferences that can be expressed. For example, one cannot have a probability on the integers in which every even integer has the same positive infinitesimal probability ε and every odd integer has the same positive infinitesimal probability δ, but δ/ε is itself infinitesimal.

For those familiar with nonstandard models of the reals, all of our analysis is external rather than internal.[2] The main reason for an external analysis is that the nonstandards are non-Archimedean from an external perspective, but are Archimedean from an internal perspective. Furthermore, we want to insure that those aspects of the decision problem to which the agent must attend are all of a familiar nature, such as the concepts of "finite," "countable," "lottery," "set," "sum," and "linear." We use nonstandards solely to represent an agent's preferences when those preferences satisfy the assumptions we state in Section 1.4. None of those assumptions requires understanding of nonstandard concepts.[3]

Narens [26] develops a non-Archimedean theory of measurement. The theory leads to measurements whose values lie in nonstandard models of the reals. Narens' measurement systems have a number of features in common with probability and preference, so it is not surprising that nonstandard numbers are useful for representing non-Archimedean preference structures.

1.3. Some notation. Throughout this paper, Ω denotes a state space, $\mathcal{X}$ denotes a set of random quantities, which are functions from Ω to a set $\mathcal{O}$ of outcomes. Subsets of Ω are called events. When $\mathcal{X}$ is a set of random variables, the set $\mathcal{O}$ will be the standard numbers $\mathbb{R}$. For other cases, both $\mathcal{X}$ and $\mathcal{O}$ will be more complicated sets that are constructed later. We will make much use of the following concepts:

DEFINITION 1.3. *Let $\precsim$ be a preorder and let $\ll$ be a strict partial order on the same set $\mathcal{X}$, and let U be a numerical function defined on $\mathcal{X}$.*

[2] The distinction between internal and external analyses of nonstandards depends on some concepts of abstract set theory, such as what counts as a set. Nonstandard models of the reals, such as the ones in Appendix A.1, use standard objects, such as sequences and equivalence relations, to construct new objects which play the roles of nonstandard numbers. These constructed objects are numbers when looked at internally, i.e., as objects that satisfy the Peano postulates and to which the Zermelo–Fraenkel axioms of set theory can be applied. When the constructed objects are looked at externally, i.e., as functions of standard objects, what counts as a set is defined in terms of the sets of standard objects from which they are built. Some of the most useful of these external sets include the standard numbers, the standard natural numbers and the infinitesimal numbers. These are sets when viewed externally, i.e., from the point of view of the standard objects from which they are defined. However, they do not satisfy the definition of "set" according to the Zermelo–Fraenkel axioms applied to the nonstandard numbers when viewed internally. Both internal and external perspectives have their uses, but it is important to be clear on which perspective is being used. An internal perspective is used by [13] to solve a different decision theory problem than the one addressed in this paper.

[3] Bottazzi and Katz [8, 9] argue that internal analysis of nonstandard probabilities has advantages over external analysis. We believe that the simplicity of understanding nonstandard numbers as providing larger and smaller numbers than are available with the standard reals and the consequent non-Archimedean properties they accommodate are advantages to the approach we take in this paper.

1. *We say that* U represents $\precsim$ *if, for all* $X, Y \in \mathcal{X}$,

$$X \precsim Y \text{ if and only if } U(X) \leq U(Y). \tag{2}$$

2. *We say that* U agrees with $\precsim$ *if, for all* $X, Y \in \mathcal{X}$,

$$X \precsim Y \text{ implies } U(X) \leq U(Y). \tag{3}$$

3. *We say that* U agrees with $\ll$ *if, for all* $X, Y \in \mathcal{X}$,

$$X \ll Y \text{ implies } U(X) < U(Y). \tag{4}$$

The following results follow easily from Definition 1.3.

PROPOSITION 1.4. *Let U be a numerical function defined on a set $\mathcal{X}$.*

- *U represents a unique preorder $\precsim$ on $\mathcal{X}$, defined via (2) and $\precsim$ is total.*
- *U represents a preorder $\precsim$ if and only if $aU + b$ represents $\precsim$ for all positive a and all b.*

1.4. Expressed preference. In our approach, preference amongst random quantities is expressed by willingness to trade.

DEFINITION 1.5. *Let X and Y be elements of a standard-linear space $\mathcal{X}$. If an agent is willing to trade X to receive Y, we write $X \precsim Y$. If both $X \precsim Y$ and $Y \precsim X$, we say that the agent is* indifferent *between X and Y, which we express by $X \sim Y$. If $(X \precsim Y) \wedge [\neg(Y \precsim X)]$ we write $X \prec Y$.*

We deliberately give no name to the relation $\prec$ for reasons that will become apparent in Example 2 below. The first assumption that we make merely avoids the two extremes in which the agent either is willing to make no trades or is willing to make all trades.

ASSUMPTION 1. *For all $X \in \mathcal{X}$, $X \precsim X$, and there exist $X, Y \in \mathcal{X}$ such that $X \prec Y$.*

Our next assumption expresses the idea that willingness to trade depends only on the agent's net change in fortune, which we state formally as follows.

ASSUMPTION 2. *Suppose that $X, X', Y, Y' \in \mathcal{X}$ and $Y - X = Y' - X'$. The agent is willing to give X to get Y if and only if the agent is willing to give X' to get Y'.*

Our next assumption is the trading analog to de Finetti's assumption that an agent is willing to accept all finite sums of fair gambles.

ASSUMPTION 3. *Suppose that $X_j \precsim Y_j$ for $j = 1, 2$ and α_1, α_2 are positive standard numbers. Then*

$$\alpha_1 X_1 + \alpha_2 X_2 \precsim \alpha_1 Y_1 + \alpha_2 Y_2.$$

Proposition 1.6 states some straightforward properties of the first three assumptions.

PROPOSITION 1.6. *Suppose that $\precsim$ satisfies Assumptions 1–3. Then $\precsim$ is a preorder, $\prec$ is a strict partial order, and $\sim$ is an equivalence relation.*

A general preorder might not be total, and hence may leave some elements of $\mathcal{X}$ uncompared, i.e., neither $X \precsim Y$ nor $Y \precsim X$.

EXAMPLE 2 (Consensus). *Let $\aleph$ be a set, and let $\{\precsim_\alpha\}_{\alpha\in\aleph}$ be a collection, indexed by $\aleph$, of total preorders on a standard-linear space $\mathcal{X}$. Our agent might think of $\aleph$ as indexing a set of experts whose opinions the agent wants to adopt to the extent that they agree. Define the binary relation $\precsim$ on $\mathcal{X}$ by $X \precsim Y$ if, for all $\alpha \in \aleph$, $X \precsim_\alpha Y$. If each $\precsim_\alpha$ satisfies Assumptions 1–3, then so does $\precsim$, which will also be a preorder, but not necessarily total. In general, for each $X, Y \in \mathcal{X}$, $\aleph$ can be written as the union of three disjoint subsets*:

$$\aleph_{X\sim Y} = \{\alpha : X \sim_\alpha Y\},\ \aleph_{X\prec Y} = \{\alpha : X \prec_\alpha Y\},\ \textit{and}\ \aleph_{Y\prec X} = \{\alpha : Y \prec_\alpha X\}.$$

If $\aleph_{Y\prec X} = \emptyset$, then $X \precsim Y$. In that case, if either of the other two sets is empty, there is unanimity about how the experts would trade X and Y. For example, if $\aleph_{X\sim Y} = \emptyset$, the agent is willing to trade X to get Y and will refuse to trade Y to get X. If both are nonempty, the agent is willing to trade X to get Y but has expressed neither willingness nor refusal to trade Y to get X. For example the agent might want to look more closely at which experts lie in each of the sets $\aleph_{X\sim Y}$ and $\aleph_{X\prec Y}$ before deciding whether to trade Y to get X. If both $\aleph_{X\prec Y}$ and $\aleph_{Y\prec X}$ are nonempty, then X and Y are not compared by $\precsim$.

As other authors have done, e.g., [18, 19, 28], we find it useful to allow an agent to distinguish preferences like the two cases in which $\aleph_{Y\prec X} = \emptyset$ but $\aleph_{X\prec Y} \neq \emptyset$ that appear in Example 2.

DEFINITION 1.7. *Let $\precsim$ satisfy Assumptions 1–3. For each case of $X \prec Y$, the agent can express whether this is an* unambiguous *one-way preference, which we denote $X \ll Y$ or an* ambiguous *one-way preference, which we denote $X \lhd Y$.*

In Example 2, unambiguous one-way preference $X \ll Y$ corresponds to both $\aleph_{Y\prec X}$ and $\aleph_{X\sim Y}$ being empty, while $X \lhd Y$ corresponds to both $\aleph_{X\sim Y}$ and $\aleph_{X\prec Y}$ being nonempty while $\aleph_{Y\prec X}$ is empty. In order for an "unambiguous" one-way preference to mean what it sounds like, we impose the following assumption.

ASSUMPTION 4. *The relations $\ll$ and $\lhd$ satisfy the following*:

- $X \prec Y$ *if and only if* $(X \ll Y) \vee (X \lhd Y)$,
- $(X \ll Y) \wedge (Y \precsim Z)$ *implies* $X \ll Z$, *and*
- $(X \precsim Y) \wedge (Y \ll Z)$ *implies* $X \ll Z$.

If the second bullet in Assumption 4 were violated, the agent would be willing to trade Y to get Z and would be willing to contemplate trading Z to get X, which would violate the understanding of $X \ll Y$ as unambiguous willingness to trade only one way. A similar violation arises if the third bullet is violated. The first claim in Proposition 1.8 is a direct consequence of Theorem 3.4 of [19], and the second claim is straightforward.

PROPOSITION 1.8. *If $\precsim$ satisfies Assumptions 1–4, then $\precsim$ is a consensus as in Example 2. If $\precsim$ satisfies Assumptions 1–3 choosing $\ll$ to be $\prec$ satisfies Assumption 4, as does choosing $\ll$ to be empty.*

We alert the reader that, although $X \lhd Y$ expresses a willingness to consider trading Y for X, nothing in general can be inferred jointly from two or more such relations, as the next example illustrates.

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Example 3. *Let $\Omega = \{0, 1\}$ and let $\mathcal{X}$ be the set of all standard-valued functions defined on Ω. Define $X \precsim Y$ as $X(\omega) \leq Y(\omega)$ for $\omega = 0, 1$, and define $X \ll Y$ as $X(\omega) < Y(\omega)$ for $\omega = 0, 1$. It is straightforward that Assumptions 1–4 hold. Let $X = 0$, $Z = 1$, and $Y = I_{\{0\}}(\omega)$. Then $X \lhd Y$ and $Y \lhd Z$. The agent would be willing to contemplate trading Y to get X or trading Z to get Y. But both trades together cannot be contemplated because that would imply a willingness to contemplate trading Z to get X where $X \ll Z$. These same comparisions also illustrate that $\lhd$ is not transitive, whereas $\ll$ and $\prec$ are transitive.*

We are now ready to formalize our model for trading.

Definition 1.9. *Let Ω be a set, and for each $\omega \in \Omega$ let $\mathcal{O}_\omega$ be a standard-linear space. Let $\mathcal{O}_\Omega = \prod_{\omega\in\Omega} \mathcal{O}_\omega$ and $\mathcal{O} = \bigcup_{\omega\in\Omega} \mathcal{O}_\omega$. Let $\mathcal{X} \subseteq \mathcal{O}_\Omega$ be a standard-linear space of functions with domain Ω.*[4] *Let $\precsim$ and $\ll$ be binary relations on $\mathcal{X}$. If $\precsim$ and $\ll$ satisfy Assumptions 1–4, we call $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ a* trading system. *If $\precsim$ is a total preorder and $\ll$ is $\prec$, then $\mathcal{T}$ is a* total trading system. *The sum of finitely many terms of the form $\alpha(Y - X)$, where $X \sim Y$ and α is standard is called a* fair trade. *The sum of finitely many terms of the form $\alpha(Y - X)$, where $X \precsim Y$ and $\alpha > 0$ is standard is called an* acceptable trade. *Denote the set of acceptable trades as $\mathcal{V}_\mathcal{T}$.*

Proposition 1.10 states some straightforward properties of trading systems.

Proposition 1.10. *Suppose that $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ is a trading system. The set $\mathcal{V}_\mathcal{T}$ of all acceptable trades is a convex cone, and it is the set of all trades V such that $0 \precsim V$. The set of all fair trades is a standard-linear space, and it is the equivalence class (under $\sim$) that contains the trade 0. Finally, $V \in \mathcal{V}_\mathcal{T}$ if and only if for all $X \in \mathcal{X}$, $X \precsim X + V$.*

1.5. Dominance and coherence (part one). Suppose that a (possibly nonstandard-valued) function U on $\mathcal{X}$ represents a total trading system $(\mathcal{X}, \precsim, \prec)$. There is a necessary condition for $U(X)$ to be expressed as the expected value of the utility of $X(\omega)$ with respect to a probability over Ω. Loosely speaking, the condition is the following:

> Let $X, Y \in \mathcal{X}$. If for all ω, $Y(\omega)$ is at least as valuable as $X(\omega)$ when state ω occurs, then $U(X) \leq U(Y)$.

In the theory of [11], where $\mathcal{X}$ is a linear space of standard-valued random variables, we can be more precise about the above condition. For each standard number x and each $\omega \in \Omega$ and each random variable X such that $X(\omega) = x$, x is assumed to be the utility value to the agent, when the state ω occurs, of receiving the random variable X. The condition then becomes "$X(\omega) \leq Y(\omega)$ for all ω implies $U(X) \leq U(Y)$."

In more general theories, where each $X(\omega)$ may be some non-numerical object $x \in \mathcal{O}$ (the codomain of X) and the utility of each object in $\mathcal{O}$ might vary with ω, the utility to the agent of receiving $X(\omega) = x$ could depend on both ω and x. Later (Definition 4.3 in Section 4.1) we define what we mean by $X(\omega) \leq Y(\omega)$ and $X(\omega) < Y(\omega)$ when $\mathcal{O} \neq \mathbb{R}$. Regardless of what are the objects in $\mathcal{O}$, there are several ways in which $X \leq Y$ but $X \neq Y$.

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[4] The structure described has each $X \in \mathcal{X}$ being a function from Ω to $\mathcal{O}$ while $X(\omega) \in \mathcal{O}_\omega$ for each X and each ω.

DEFINITION 1.11. *Let $X, Y \in \mathcal{X}$.*

- *We say that* Y weakly dominates X *or* X is weakly dominated by Y *if $X(\omega) \le Y(\omega)$ for all $\omega \in \Omega$ and there is $\omega \in \Omega$ such that $X(\omega) < Y(\omega)$.*
- *We say that* Y strictly dominates X *or* X is strictly dominated by Y *if $X(\omega) < Y(\omega)$ for all $\omega \in \Omega$.*
- *We say that* Y uniformly dominates X *or* X is uniformly dominated by Y *if there exists a standard $\varepsilon > 0$ such that $X(\omega) \le Y(\omega) - \varepsilon$ for all $\omega \in \Omega$.*

It is trivial to see that weak dominance is an extension of strict dominance which, in turn, is an extension of uniform dominance. Many of our results do not depend on which version of dominance an agent chooses. For those results that depend on the form of dominance (primarily in Section 4), we are explicit about which form is needed. We use $X \prec_{\text{Dom}} Y$ to denote "Y dominates X" in whichever sense the agent chooses. In [11], dominance means uniform dominance. Our next assumption formalizes the idea that more is better.

ASSUMPTION 5. *The agent chooses one of the senses of dominance. Suppose that $X, Y \in \mathcal{X}$. If $X \le Y$, then $X \precsim Y$. If $X \prec_{Dom} Y$, then $X \ll Y$.*

DEFINITION 1.12. *A trading system $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ is called* coherent *if it satisfies Assumption 5.*

When $\mathcal{O} = \mathbb{R}$, note that dominance is defined on all of $\mathbb{R}^\Omega$, while Assumption 5 pertains only to elements of $\mathcal{X}$. Some of our results apply only to coherent trading systems, and these contain clauses such as "If $\mathcal{T}$ is coherent ...." Other results apply more generally, and do not contain such clauses.

EXAMPLE 4. *The smallest coherent trading system on a linear space $\mathcal{X}$ of standard-valued random variables must include all cases of dominance (whichever form we choose) amongst the elements of $\mathcal{X}$. Let Ω be an arbitrary set, and let $\mathcal{X}$ be the linear span of all constants and all indicators of singletons. Let $\prec_{Dom}$ stand for any of the three forms of dominance. Each element of $\mathcal{X}$ is constant except for at most finitely many points. Hence strict and uniform dominance are the same if Ω is infinite in cardinality. Let $X \precsim Y$ if and only if either $X = Y$ or $X \prec_{Dom} Y$. It is straightforward that $\precsim$ is a preorder, but not total. Note that, for all $X, Y \in \mathcal{X}$ and all standard $\alpha > 0$, $0 \precsim \alpha(Y - X)$ if and only if $X \precsim Y$. If $X_j \prec_{Dom} Y_j$ for $j = 1, 2$, then $\alpha_1 X_1 + \alpha_2 X_2 \prec_{Dom} \alpha_1 Y_1 + \alpha_2 Y_2$, so Assumptions 1–3 are satisfied. Define $\ll$ to be $\prec_{Dom}$ so that Assumptions 4 and 5 are satisfied. Then $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ is a coherent trading system. We will return to this example later (Example 5) to prove that its preferences are non-Archimedean when Ω is uncountable and $\prec_{Dom}$ means weak dominance.*

§2. Representing and extending a trading system.

2.1. Representations of total trading systems. In this section, we show how to represent a total trading system by a (possibly nonstandard-valued) numerical function.

DEFINITION 2.1. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a trading system. A numerical function U on $\mathcal{X}$* agrees with $\mathcal{T}$ *if U agrees with $\precsim$ and with $\ll$. (Recall Definition 1.3.) If $\mathcal{T}$ is a total trading system and U represents $\precsim$ then we say that* U represents $\mathcal{T}$.

The following result follows easily from Definition 2.1.

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PROPOSITION 2.2. *A numerical function U represents a total preorder $\precsim$ if and only if*

$$X \prec Y \text{ if and only if } U(X) < U(Y). \tag{5}$$

Next, we introduce a class of numerical functions that represent total trading systems.

DEFINITION 2.3. *A standard-linear function U (recall Definition 1.2) is called* monotone *if $X \leq Y$ implies $U(X) \leq U(Y)$. A monotone standard-linear function U is said to* respect dominance *if $X \prec_{Dom} Y$ implies $U(X) < U(Y)$.*

LEMMA 2.4. *Let U be a standard-linear function defined on a standard-linear space $\mathcal{X}$ of functions defined on a state space Ω. Then U represents a total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$. Also $\mathcal{T}$ is coherent if and only if U is monotone and respects dominance.*

Proof. Assume that U is a standard-linear function on a standard-linear space $\mathcal{X}$. Define the total preorder $\precsim$ on $\mathcal{X}$ by (2). For the first claim, we need to verify Assumptions 1–4. Assumption 1 follows because a preorder is reflexive. For Assumption 2, suppose that $Y - X = Y' - X'$. Since U is standard-linear,

$$U(Y) - U(X) = U(Y - X) = U(Y' - X') = U(Y') - U(X').$$

For Assumption 3, note that U being standard-linear implies that

$$U(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 U(X_1) + \alpha_2 U(X_2),$$

for all standard α_1, α_2 and $X_1, X_2 \in \mathcal{X}$. For Assumption 4, note that $\ll$ is $\prec$.

For the second claim, we need to prove that Assumption 5 holds if and only if U is monotone and respects dominance. For the "if" direction, assume that U is monotone and respects dominance. Since U is monotone, $X \leq Y$ implies $U(X) \leq U(Y)$ and $X \precsim Y$. Since U respects dominance, $X \prec_{\mathrm{Dom}} Y$ implies $U(X) < U(Y)$ and $X \prec Y$, so Assumption 5 holds. For the "only if" direction, assume that Assumption 5 holds. To see that U is monotone, assume that $X \leq Y$. The first requirement of Assumption 5 is that $X \precsim Y$, which implies that $U(X) \leq U(Y)$, and U is monotone. To see that U respects dominance, assume that $X \prec_{\mathrm{Dom}} Y$. The second requirement of Assumption 5 is that $X \prec Y$, which implies that $U(X) < U(Y)$, and U respects dominance. □

2.2. Agreement, representation and extension. If $\precsim$ is a not a total preorder, then there can be no numerical function U such that (2) holds. The problem is the "if" direction of (2) rather than the "only if" direction. In other words, representation as defined in Definition 1.3 is not achievable for preorders that are not total. On the other hand, agreement with a preorder and with a strict partial order are possible.

When it comes to extension of a trading system, there are two modes of extension that are important to our analysis. One mode corresponds to adding more comparisons (amongst elements of a single set $\mathcal{X}$) to the preorder, bringing it closer to being total. The other mode corresponds to expanding the domain of definition of the preorder (from one set $\mathcal{X}$ to a larger set $\mathcal{X}'$). Along with the second mode of extension comes a corresponding concept of restricting the domain of definition.

DEFINITION 2.5. *Let $\mathcal{X}$ and $\mathcal{X}'$ be standard-linear spaces with $\mathcal{X} \subseteq \mathcal{X}'$. Let ρ be a binary relation on $\mathcal{X}$, and let ρ' be a binary relation on $\mathcal{X}'$.*

- *If $\mathcal{X} = \mathcal{X}'$ and $X \rho Y$ implies $X \rho' Y$, we say that ρ' is an* extension$_1$ *of ρ.*
- *If $(X, Y \in \mathcal{X}) \wedge (X \rho Y)$ implies $X \rho' Y$, we say that ρ' is an* extension$_2$ *of ρ.*
- *If ρ' is an extension$_2$ of ρ, we say that ρ is the* restriction *of ρ' to $\mathcal{X}$.*

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- *Suppose that $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ and $\mathcal{T}' = (\mathcal{X}, \precsim', \ll')$ are trading systems. If $\precsim'$ and $\ll'$ are extensions$_1$ of $\precsim$ and $\ll$ respectively, we call $\mathcal{T}'$ an* extension$_1$ *of $\mathcal{T}$.*
- *Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ and $\mathcal{T}' = (\mathcal{X}', \precsim', \ll')$ be trading systems. If $\mathcal{X} \subseteq \mathcal{X}'$ and if $\precsim'$ and $\ll'$ are extensions$_2$ of $\precsim$ and $\ll$ respectively, we call $\mathcal{T}'$ an* extension$_2$ *of $\mathcal{T}$.*

To be clear, each binary relation and each trading system is both an extension$_1$ and an extension$_2$ of itself. The following result about extension$_2$ is key in our theorems on representation. Its proof appears in Appendix C.1.

LEMMA 2.6. *Let $\mathcal{X}$ and $\mathcal{X}'$ be standard-linear spaces of functions with domain Ω and such that $\mathcal{X} \subseteq \mathcal{X}'$. Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a trading system. There exists a trading system $\mathcal{T}' = (\mathcal{X}', \precsim', \ll')$ that is an extension$_2$ of $\mathcal{T}$. If it is not required that $\mathcal{T}'$ be coherent, $\mathcal{T}'$ can be chosen such that $\mathcal{V}_{\mathcal{T}'} = \mathcal{V}_{\mathcal{T}}$. If $\mathcal{T}$ is coherent and it is required that $\mathcal{T}'$ be coherent, assume that $\leq$ and $\prec_{Dom}$ are defined on $\mathcal{X}'$ and are extensions$_2$ of $\leq$ and $\prec_{Dom}$ on $\mathcal{X}$. Then a coherent $\mathcal{T}'$ can be chosen so that for every $V' \in \mathcal{V}_{\mathcal{T}'}$ there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \leq V'$.*

Note that Lemma 2.6 above (as well as Lemma 2.7 and Theorem 2.9 below) have language about $\leq$ and $\prec_{\text{Dom}}$ on a larger space being extensions$_2$ of $\leq$ and $\prec_{\text{Dom}}$ on a smaller space. When $\mathcal{O} = \mathbb{R}$, this condition is met trivially. The language is included to allow us to use these same results in other cases after Section 4.1 where $\leq$ and $\prec_{\text{Dom}}$ are defined in terms of each specific trading system.

2.3. Finding agreeing functions. Our main representation Theorem 2.8 states that a trading system $\mathcal{T}$ has a standard-linear function U that agrees with it and an extension$_1$ to a total trading system that is represented by U. Results from [18, 19, 28] give the extension$_1$ for a general preorder, but without the representing function and without attention to the properties of a trading system. The following result has both Theorems 2.8 and 2.9 as special cases, and its proof appears in Appendix C.3.

LEMMA 2.7. *Assume the following structure*:

- *$\mathcal{Y}$ and $\mathcal{W}$ are standard-linear spaces of functions defined on Ω with $\mathcal{Y}$ a proper subset of $\mathcal{W}$.*
- *$\mathcal{T}_{\mathcal{Y}} = (\mathcal{Y}, \precsim_{\mathcal{Y}}, \prec_{\mathcal{Y}})$ is a total trading system that is represented by the standard-linear function $U : \mathcal{Y} \to {}^*\mathbb{R}$, where ${}^*\mathbb{R}$ is a nonstandard model of the reals.*
- *$\mathcal{T}_{\mathcal{W}} = (\mathcal{W}, \precsim_{\mathcal{W}}, \ll_{\mathcal{W}})$ is the extension$_2$ of $\mathcal{T}_{\mathcal{Y}}$ obtained from Lemma 2.6.*

Then U can be extended to a standard-linear function $U' : \mathcal{W} \to {}^*\mathbb{R}'$, where ${}^*\mathbb{R}'$ contains ${}^*\mathbb{R}$ and such that U' represents a total trading system $\mathcal{T}' = (\mathcal{W}, \precsim', \prec')$ that is an extension$_2$ of $\mathcal{T}_{\mathcal{Y}}$. Also, if $\mathcal{T}_{\mathcal{Y}}$ is coherent and $\leq$ and $\prec_{Dom}$ are defined on $\mathcal{W}$ so as to be extensions$_2$ of $\leq$ and $\prec_{Dom}$ on $\mathcal{Y}$, then $\mathcal{T}'$ can be chosen to be coherent.

THEOREM 2.8. *Let $\mathcal{T}$ be a trading system. There exists a standard-linear function U that agrees with $\mathcal{T}$ and total trading system $\mathcal{T}'$ that is an extension$_1$ of $\mathcal{T}$ such that U represents $\mathcal{T}'$. If $\mathcal{T}$ is coherent, $\mathcal{T}'$ can be chosen to be coherent.*

Proof. Apply Lemma 2.7 with $\mathcal{Y} = \{0\}$ (the trivial standard-linear space containing only the additive identity in $\mathcal{X}$), $0 \precsim_{\mathcal{Y}} 0$, $\mathcal{W} = \mathcal{X}$, $\mathcal{T}_{\mathcal{W}} = \mathcal{T}$, $U(0) = 0$, and ${}^*\mathbb{R} = \mathbb{R}$. Let $\mathcal{T}'$ be the $\mathcal{T}'$ that results from Lemma 2.7, and let U be the corresponding U'. These satisfy the conclusion of Theorem 2.8. ☐

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EXAMPLE 5 (Continuation of Example 4). *Recall the smallest coherent trading system* $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ *from Example 4. This time, assume that* $\prec_{Dom}$ *means weak dominance and that* Ω *is uncountable. Then* $X \prec Y$ *and* $X \ll Y$ *both mean that* Y *weakly dominates* X*. Here we show that* $\precsim$ *is non-Archimedean. Theorem 2.8 says that* $\mathcal{T}$ *can be extended*$_1$ *to a coherent total trading system* $\mathcal{T}' = (\mathcal{X}, \precsim', \prec')$ *with a representing function* U *that agrees with* $\mathcal{T}$*. Without loss of generality, assume that* $U(0) = 0$ *and* $U(1) = 1$*. So* $X \prec Y$ *(i.e.,* $X \prec_{Dom} Y$*) implies* $U(X) < U(Y)$*. For every nonempty finite subset* $E = \{\omega_1, \dots, \omega_n\}$ *of* Ω*, dominance and standard-linearity imply that*

$$0 < U(I_E) = \sum_{j=1}^{n} U(I_{\{\omega_j\}}) < U(1) = 1. \tag{6}$$

For each $\omega \in \Omega$ *either* $U(I_{\{\omega\}})$ *is infinitesimal or it is greater than* $1/n$ *for some standard integer* n*. There can be no more than* $n-1$ *values of* $U(I_{\{\omega\}}) > 1/n$ *or (6) would be violated for a finite set* E *containing* n *such* ω *points. Hence, all but at most countably many* ω *have* $U(I_{\{\omega\}})$ *infinitesimal. Let* $\omega_1, \omega_2 \in \Omega$ *be such that* $U(I_{\{\omega_j\}})$ *is infinitesimal for* $j = 1, 2$*. Let* $X = -1$*,* $Y = I_{\{\omega_1, \omega_2\}}$*, and* $Z = I_{\{\omega_1\}}$*. Then,* $Z \prec_{Dom} Y$*,* $U(Z) > 0$*, and* $U(\alpha X + [1-\alpha]Y) < 0$ *if* $0 < \alpha \leq 1$*. It follows that*

$$\alpha X + (1-\alpha)Y \prec_{Dom} Z, \text{ for all } 0 < \alpha \leq 1,$$

but $Z \prec_{Dom} Y$*, which violates (1). Hence the preference is non-Archimedean. It also follows that for each coherent extension*$_1$ $\mathcal{T}'$*,* $\precsim'$ *is also non-Archimedean. For example, we might want all* $I_{\{\omega\}}$ *to be indifferent to each other. In this case, we can construct a representing function* U *as follows. For each constant function* $X \equiv c$*,* $U(X) = c$*, and for each nonconstant function* $X = h_0 + \sum_{j=1}^{n} h_j I_{\{\omega_j\}}$*,* $U(X) = h_0 + \varepsilon \sum_{j=1}^{n} h_j$*. Note that this* U *is standard-linear and monotone, and it respects weak dominance, so Lemma 2.4 says that it represents a coherent total trading system which extends*$_1$ $\mathcal{T}$*.*

Theorem 2 of [40] shows that every total preorder can be represented by a nonstandard-valued function, but the standard-linear property that we need is not proven in that paper.

2.4. Extending$_2$ a trading system. Let $\mathcal{T} = (\mathcal{X}, \precsim, \ll)$ be a (coherent) trading system. Theorem 2.8 says that there exists a standard-linear function U that agrees with $\mathcal{T}$ and such that U represents a (coherent) total trading system $\mathcal{T}'$ that is an extension$_1$ of $\mathcal{T}$. Extension$_2$ of $\mathcal{T}'$ is also possible if $\mathcal{X}$ is a subspace of a larger standard-linear space, as stated in Theorem 2.9.

THEOREM 2.9. *Let* $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ *be a total trading system on a standard-linear space* $\mathcal{X}$ *with standard-linear representing function* U*. Let* $\mathcal{X}'$ *be a standard-linear space of functions that includes* $\mathcal{X}$ *as a proper subset. Then there is a total trading system* $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ *that is an extension*$_2$ *of* $\mathcal{T}$ *and a standard-linear function* U' *on* $\mathcal{X}'$ *that represents* $\mathcal{T}'$ *and extends* U *to* $\mathcal{X}'$*. If* $\mathcal{T}$ *is coherent and* $\leq$ *and* $\prec_{Dom}$ *are defined on* $\mathcal{X}'$ *so that they are extensions*$_2$ *of* $\leq$ *and* $\prec_{Dom}$ *on* $\mathcal{X}$*, then* $\mathcal{T}'$ *can be chosen to be coherent.*

Proof. Apply Lemma 2.7 with $\mathcal{T}_{\mathcal{Y}} = \mathcal{T}$, $\mathcal{W} = \mathcal{X}'$, and U being the U in the statement of Theorem 2.9. The $\mathcal{T}'$ and U' that result from Lemma 2.7 satisfy the conclusion of Theorem 2.9. □

EXAMPLE 6 (Continuation of Example 5). *The linear space* $\mathcal{X}$ *used in Example 5 is the set of all standard-valued functions on an uncountable set* Ω *that are constant* *except at possibly finitely many points. Because weak dominance is defined on the set of all functions from Ω to $\mathbb{R}$, Theorem 2.9 says that we could extend$_2$ each of the non-Archimedean coherent total trading systems from Example 5 to non-Archimedean coherent total trading systems on larger linear spaces $\mathcal{X}'$ of standard-valued functions.*

For example, suppose that $\Omega = [0, 1]$, the closed unit interval and that we have chosen $\mathcal{T}$ to make $U(I_{\{\omega\}}) = \varepsilon$ for all ω. It is possible to extend$_2$ $\mathcal{T}$ to the set $\mathcal{X}'$ of all simple functions that are measurable with respect to the smallest field Σ of subsets of Ω that contains all intervals (closed, open, half-open, or degenerate). For example, the degenerate interval $[\omega, \omega]$ is the singleton $\{\omega\}$. (We do not allow $[\omega, \omega)$ to be called an interval.) The field Σ consists of all unions of finitely many disjoint intervals. The simple functions that are measurable with respect to Σ have the form

$$\sum_{j=1}^{n} h_j I_{J_j}, \tag{7}$$

for finite n, standard numbers $h_1, \dots, h_n$, and disjoint intervals $J_1, \dots, J_n$. For a nondegenerate interval J define $U'(I_J)$ to be the length of J minus ε for each missing endpoint, and $U'(I_{\{\omega\}}) = \varepsilon$. Although the given representation of functions in $\mathcal{X}$ is not unique, it is straightforward to show that, for X being the function in (7),

$$U'(X) = \sum_{j=1}^{n} h_j U'(I_{J_j}),$$

is well defined and extends the function U defined in Example 5. The function U' is standard-linear and monotone, and it respects weak dominance. Hence, by Lemma 2.4, U' represents a coherent total trading system which extends$_2$ $\mathcal{T}$ because U' extends U.

The following example is modified from [38]. It involves random variables with a familiar distribution in which externally infinite nonstandard numbers are needed to represent non-Archimedean preferences amongst them.

EXAMPLE 7. *Let $\mathbb{Z}^+$ stand for the set of positive integers, and let $\Omega = \mathbb{Z}^+ \times \{1, 2\}$. Let P be the following probability on Ω: $P(\{(n, j)\}) = 2^{-n-1}$ for $n \in \mathbb{Z}^+$ and $j \in \{1, 2\}$, which is countably-additive. Let $\mathcal{X}$ be the set of all bounded functions defined on Ω, each of which has a finite expected value $E(X)$ with respect to P. The function $E(X)$ represents a (weakly, strictly, and uniformly) coherent total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$.*

Define the following three unbounded functions on Ω:

$$X_1((n, j)) = \begin{cases} 1, & \text{if } j = 1, \\ 2^n, & \text{if } j = 2, \end{cases}$$

$$X_2((n, j)) = \begin{cases} 1, & \text{if } j = 2, \\ 2^n, & \text{if } j = 1, \end{cases}$$

$$W((n, j)) = 2^{n-1}.$$

It follows that all three of these functions have the same distribution, namely, each of them takes the value 2^k with probability 2^{-k} for $k = 1, 2, \dots$. Each of them also has infinite expected value. Let $\mathcal{X}'$ be the linear span of $\mathcal{X} \bigcup \{X_1, X_2, W\}$. All three forms of dominance are defined on $\mathcal{X}'$ and are extensions$_2$ of the same form of dominance on $\mathcal{X}$. We can use Theorem 2.9 to extend$_2$ $\mathcal{T}$ to a coherent total trading system $\mathcal{T}' = (\mathcal{X}, \precsim', \prec')$ with *representing function U'. Since each of X_1, X_2, and W has infinite expected value under P, it is necessary that all three of $U'(X_1)$, $U'(X_2)$, and $U'(W)$ must be externally infinite nonstandard numbers. However, we show next that, despite having identical distributions, X_1, X_2, and W cannot be indifferent in a (weakly, strictly, or uniformly) coherent trading system.*

First, note that

$$X_1(\omega) + X_2(\omega) = 2W(\omega) + 1, \tag{8}$$

for all ω. Hence there is a uniform dominance relation that cannot be respected by a preference in which all three are indifferent. We could, for example, have $W \prec' X_1 \sim' X_2$ by choosing an externally infinite x and setting $U'(X_1) = U'(X_2) = x$ and $U'(W) = x - 1/2$. There are other possible choices for $U'(X_1)$ and $U'(X_2)$, but they must satisfy $U'(X_1) + U'(X_2) = 2U'(W) + 1$ because of (8). Finally, U' is well defined on all of $\mathcal{X}'$ which includes all linear combinations of X_1, X_2, and W with both positive and negative coefficients.[5]

There are many other examples of standard-linear spaces with proper supersets. One can easily imagine an agent determining a set of preferences over a small set $\mathcal{X}$ of objects and then being offered additional options in a set $\mathcal{X}'$. Here is an example of a situation that might seem of a different nature, but which still fits the setup of Theorem 2.9. It is related to the concepts of awareness growth and reverse Bayesianism.[6]

EXAMPLE 8 (Awareness growth). *Suppose that an agent has a total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ where each element of $\mathcal{X}$ is a function from Ω to a space $\mathcal{O}$. At some point, the agent realizes that the "actual" state space is a different set Ω'. There are two common models for this realization.*

> ***Refinement:*** *The agent learns that the elements of Ω are not atomic. That is, each element ω appears to be a subset of Ω'.*
>
> ***Expansion:*** *The agent learns that Ω is merely a proper subset of the "actual" state space Ω'.*

In either case, one recommendation (called reverse Bayesianism) *is for the agent to preserve those preferences included in $\mathcal{T}$ when extending the preference relation to include any new objects that need to be considered. Theorem 2.9 can be applied in both the refinement and expansion cases to achieve that goal. Specifically, there is a one-to-one mapping T between $\mathcal{X}$ and a set $\mathcal{X}^* = \{T(X) : X \in \mathcal{X}\}$ of functions from Ω' to a standard-linear space $\mathcal{O}'$ such that $\mathcal{T}^* = (\mathcal{X}^*, \precsim^*, \prec^*)$ is a total trading system with $X \precsim Y$ if and only if $T(X) \precsim^* T(Y)$. If $\mathcal{T}$ is coherent and $\mathcal{O} = \mathbb{R}$, then $\leq$ and $\prec_{Dom}$ are defined on $\mathcal{X}'$ so as to be extensions$_2$ of $\leq$ and $\prec_{Dom}$ on $\mathcal{X}$.*

In the refinement case, for each $\omega \in \Omega$, there is a subset $C_\omega \subseteq \Omega'$ such that the distinct elements of $\mathcal{C} = \{C_\omega : \omega \in \Omega\}$ form a partition of Ω'. For each $X \in \mathcal{X}$ and $\omega' \in \Omega'$, define $T(X)(\omega') = X(\omega)$ where $\omega' \in C_\omega$. Each such $T(X)$ is constant on each element of $\mathcal{C}$. If $\mathcal{T}$ is coherent, then $X \prec_{Dom} Y$ in $\mathcal{T}$ if and only if $T(X) \prec_{Dom} T(Y)$ in $\mathcal{T}^*$ for all three forms of dominance.

[5] Definition 4.8 in Section 4.4 below says that U' acts as an expected value on $\mathcal{X}'$. No extended-real-valued (allowing $\pm\infty$ values) expectation exists on $\mathcal{X}'$.

[6] See [10, 22, 23] for an introduction and discussion of awareness growth and reverse Bayesianism. The purpose of this example is not to endorse or refute reverse Bayesianism, but rather to show how it can be achieved in the case of trading systems.

In the expansion case, the additional functions in $\mathcal{X}'$ are defined on the whole set Ω' and may take values that are not in the codomain $\mathcal{O}$ of the functions in $\mathcal{X}$. Let $\mathcal{O}'$ be the codomain of the functions defined on Ω' that the agent wants be able to trade. For each $X \in \mathcal{X}$ and $\omega \in \Omega$ define

$$T(X)(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in \Omega, \\ 0, & \text{if } \omega \in \Omega' \setminus \Omega. \end{cases}$$

All such $T(X)$ are identical on the part of Ω' of which the agent recently became aware. If $\mathcal{T}$ is weakly coherent, $X \prec_{Dom} Y$ if and only if $T(X) \prec_{Dom} T(Y)$. There are no cases of $X \prec_{Dom} Y$ in $\mathcal{T}^*$ for the other two forms of dominance, but if $\mathcal{T}$ is uniformly or strictly coherent, $\mathcal{T}^*$ is both uniformly and strictly coherent.

In both the refinement and expansion cases, the agent can determine what set $\mathcal{X}'$ (containing $\mathcal{X}^*$) of functions defined on Ω' are available for trading. Theorem 2.9 then allows extension$_2$ of $\mathcal{T}^*$ to a total trading system $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$. If $\mathcal{T}$ is coherent, the extension$_2$ of $\mathcal{T}^*$ to $\mathcal{T}'$ can be done so that $\mathcal{T}'$ retains the same form of coherence.

§3. Three decision theories. In this section, we show how the structure of Section 1 and the results of Section 2 extend three well-known theories of decision-making.

3.1. Previsions for random variables. The first Archimedean theory to which our results apply is the theory of previsions of [11], which begins with an arbitrary set of standard-valued random variables. For each X in that set, an agent chooses a standard value $P(X)$ (called the prevision of X) such that the agent is willing to trade away either X or $P(X)$ in order to receive the other one. Specifically, the change in fortune $\alpha[X - P(X)]$ is considered a fair gamble for all standard α. Although de Finetti's theory deals only in fair trades (indifference), there is an implicit assumption that "more is better," which is built into his notion of coherence (corresponding to uniform dominance in Definitions 1.11 and 1.12). In [11], the agent is willing to accept every finite sum of fair gambles. In particular, the agent is willing to accept

$$\alpha[X - P(X)] - \alpha[Y - P(Y)] = \alpha(X - Y) + \alpha[P(Y) - P(X)], \tag{9}$$

for all standard α. If $P(X) = P(Y)$, the right-hand side of (9) is $\alpha(X - Y)$, and the agent is implicitly willing to trade X to get Y or to trade Y to get X. If $P(X) \neq P(Y)$, there is an implicit strict preference in one direction, e.g., if $P(Y) > P(X)$ and $\alpha < 0$, the fair trade (9) is strictly smaller than $|\alpha|(Y - X)$, so the agent is willing to trade X to get Y, but not the other way. In addition, willingness to accept all finite sums of fair trades implies that a coherent prevision P on an arbitrary set $\mathcal{Y}$ of random variables extends uniquely to a coherent prevision on the linear span $\mathcal{X}$ of $\mathcal{Y}$. Define the total preorder $\precsim$ on $\mathcal{X}$ defined "$X \precsim Y$ if and only if $P(X) \leq P(Y)$." It follows that $(\mathcal{X}, \precsim, \prec)$ is a total trading system that is represented by the linear function P. Our theory extends that of [11] by dropping the requirement that every element of $\mathcal{X}$ be indifferent to some standard constant.

A simple example of a random variable that is not indifferent to a standard constant arises with an "almost-fair" coin. For an even-money bet (odds equal 1) the agent strictly prefers the bet that pays on heads over the bet that pays on tails. But, for every bet that is not at even money (i.e., odds are different from 1), the agent strictly prefers the side of the bet that pays the larger amount. Theorem 3.1 of [15] implies

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that there is no standard-valued prevision that ranks these bets in the order of the stated preferences. See also [12]. But a nonstandard-valued function can represent such preferences. Random variables with infinite previsions (such as Example 7) are also cases in which fair prices are not available. The nonstandard-valued functions that represent these preferences are not fair prices, but they provide a numerical representation of preference in the manner of expected utility.

3.2. Horse lotteries. The second theory to which our results apply is that of [1, 41] for decisions about horse lotteries, which are functions from Ω to the set of simple lotteries over a set of prizes.

3.2.1. Horse lotteries in general.

Definition 3.1. *For each $\omega \in \Omega$, let $\mathcal{P}_\omega$ be the set of prizes available in state ω. A* simple lottery *r is a probability on a finite subset $\mathcal{P}(r) \subseteq \mathcal{P}_\omega$. Let $\mathcal{R}_\omega$ be the convex set of simple lotteries available in state ω.*[7] *For ease of notation, let $\mathcal{P} = \bigcup_{\omega\in\Omega} \mathcal{P}_\omega$ and $\mathcal{R} = \bigcup_{\omega\in\Omega} \mathcal{R}_\omega$ be, respectively, the sets of all prizes available in at least one state and all lotteries available in at least one state. Let $\mathcal{R}_\Omega = \prod_{\omega\in\Omega} \mathcal{R}_\omega$, which is a subset of $\mathcal{R}^\Omega$. A* horse lottery *is a function $h \in \mathcal{R}_\Omega$, i.e., $h(\omega) \in \mathcal{R}_\omega$ for every $\omega \in \Omega$. Let $\mathcal{H}$ stand for the set of horse lotteries under consideration, which we assume to be a convex subset of $\mathcal{R}_\Omega$.*[8]

In each application, the set $\mathcal{H}$ of horse lotteries can be different, but each such $\mathcal{H}$ must be a convex subset of $\mathcal{R}_\Omega$. For $h_1, h_2 \in \mathcal{R}_\Omega$ and $\alpha \in [0,1]$, the meaning of $h_3 = \alpha h_1 + (1-\alpha)h_2$ is that $h_3(\omega) = \alpha h_1(\omega) + (1-\alpha)h_2(\omega) \in \mathcal{R}_\omega$, because $\mathcal{R}_\omega$ is convex. A set $\mathcal{H}$ of horse lotteries is not a linear space. Next, we show how to create a linear space that is equivalent to $\mathcal{H}$ in an appropriate sense.

3.2.2. A linear space for horse lotteries. The set $\mathcal{H}$ of horse lotteries is a convex subset of $\mathcal{R}_\Omega$, but is not a linear space. Hausner [21] (Sections 2–4) assumes that $\precsim'$ is a total preorder that satisfies the following axiom, which is part of the theory of [1, 41]:

Independence Axiom: Let $\precsim'$ be a preorder on a convex set $\mathcal{H}$ of horse lotteries. For all $h_1, h_2, g \in \mathcal{H}$ and standard $0 < \alpha < 1$, $h_1 \precsim' h_2$ if and only if $\alpha h_1 + (1-\alpha)g \precsim' \alpha h_2 + (1-\alpha)g$.

Hausner [21] shows how to create a standard-linear space $\mathcal{K}_0$ with a preorder $\precsim$ that satisfies our Assumptions 2 and 3 in Section 1.4 above. This is done as follows. For each $\omega \in \Omega$, let $\mathcal{O}_\omega$ be the set of all simple signed measures[9] on $\mathcal{P}_\omega$ that give measure 0 to the whole set $\mathcal{P}_\omega$. Let $\mathcal{O} = \bigcup_{\omega\in\Omega} \mathcal{O}_\omega$, and let $\mathcal{O}_\Omega = \prod_{\omega\in\Omega} \mathcal{O}_\omega$. Then

$$\mathcal{K}_0 = \{\alpha(h_1 - h_2) : h_1, h_2 \in \mathcal{H} \text{ and } \alpha \in \mathbb{R}\} \subseteq \mathcal{O}_\Omega$$

is a standard-linear space. Define $\precsim$ on $\mathcal{K}_0$ as follows. For each $k_1, k_2 \in \mathcal{K}_0$, express $k_2 - k_1 = \alpha(h_2 - h_1)$ with $\alpha > 0$ and $h_1, h_2 \in \mathcal{H}$. Then say that $k_1 \precsim k_2$ if $h_1 \precsim' h_2$. Hausner [21] (Section 4) shows that $\precsim$ is well defined and satisfies Assumptions 2 and 3. The theory of [1, 41] satisfies Assumption 4 vacuously since $\precsim$ is a total preorder.

[7] If $r_1, r_2 \in \mathcal{R}$ but $\mathcal{P}(r_1) \neq \mathcal{P}(r_2)$, $\mathcal{P}(\alpha r_1 + [1-\alpha]r_2) = \mathcal{P}(r_1) \bigcup \mathcal{P}(r_2)$ when $\alpha \in (0,1)$.
[8] Each $h \in \mathcal{H}$ is also a function from Ω to $\mathcal{R}$, but $\mathcal{R}$ may not be a convex set.
[9] A signed measure μ on a set $\mathcal{Y}$ is *simple* if there is a finite subset $\{y_1, \dots, y_n\} \subseteq \mathcal{Y}$ and numbers $\alpha_1, \dots, \alpha_n$ such that, for every $B \subseteq \mathcal{Y}$, $\mu(B) = \sum_{j=1}^{n} \alpha_j I_B(y_j)$.

Dominance and coherence are not issues that arise in the theory of [1, 41] as horse lotteries are not numerically comparable without further assumptions.

The state-independence assumption of [1, 41] implies that all $\mathcal{R}_\omega$ sets are the same. Our theory is general enough to include cases in which the $\mathcal{R}_\omega$ sets might all be the same or might be different. We also drop the Archimedean axiom and allow $\precsim'$ to not be total as do other authors such as [2, 3, 14]. Our weaker state-independence Assumption 7 is stated in Section 4.1.

3.2.3. Representing horse lotteries. For the remainder of this paper, when we refer to the *horse-lottery case*, we will assume that $\mathcal{X}$ is the standard-linear space $\mathcal{K}_0$ defined in Section 3.2.2. (The case in which $\mathcal{X}$ is a linear space of standard-valued random variables will be called the *random-variable case.*) In the horse-lottery case, it would be easier on the intuition if each representing function of a trading system had $\mathcal{H}$ as its domain rather than $\mathcal{K}_0$. This is easily arranged. Let $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ be a total trading system in a horse-lottery case with standard-linear representing function U. Let $\mathcal{H}$ be the set of horse lotteries that corresponds to $\mathcal{K}_0$ as in Section 3.2.2. For each $k \in \mathcal{K}_0$, we can write $k = \alpha(h_1 - h_2)$ with $\alpha > 0$ standard. Then $0 \precsim k$ is equivalent to $h_2 \precsim' h_1$ for a total preorder $\precsim'$ on $\mathcal{H}$. Let $h_0 \in \mathcal{H}$ be arbitrary, and define

$$V(h) = U(h - h_0). \tag{10}$$

It follows that $V(h_0) = 0$ and

$$U(\alpha[h_1 - h_2]) = \alpha[V(h_1) - V(h_2)]. \tag{11}$$

Also, V represents $\precsim'$ and satisfies

$$V(\beta h_1 + [1 - \beta]h_2) = \beta V(h_1) + (1 - \beta)V(h_2),$$

for all $h_1, h_2 \in \mathcal{H}$ and all standard $\beta \in [0, 1]$.

3.3. Savage-style acts. The third theory to which our results apply is that of Savage [35]. This theory makes some assumptions (including state-independence) about preferences amongst acts (functions) from a state space Ω to a set F of consequences (prizes) and then proves an expected-utility representation for those preferences. Lemma 3.2 below starts with those same assumptions and shows that there is a set of lotteries over the acts with an implied willingness to trade that satisfies the assumptions that appear in Section 1.4 of this paper. We then weaken the original assumptions of Savage [35] and show how to use our results for the horse-lottery case to represent non-Archimedean and state-dependent preferences over the acts of Savage [35]. Whenever we refer to "the horse-lottery case" in this paper, we implicitly include the theory of Savage in that case.

At this point, we can show how a non-Archimedean version of the theory of Savage [35] becomes a special case of trading systems in the horse-lottery case without any additional assumptions or choices by the agent. The proof of Lemma 3.2 is in Appendix C.4.

LEMMA 3.2. *Let $\mathcal{F}$ be a set of functions from Ω to F, and let $\precsim'$ be a total preorder on $\mathcal{F}$ that satisfies the seven postulates $(P1–P7)$ of Savage [35]. Let $\mathcal{H}$ be the set of finite mixtures of elements of $\mathcal{F}$. Then $\precsim'$ extends to a total preorder on $\mathcal{H}$, and the $\mathcal{K}_0$ and $\precsim$ constructed from $\mathcal{H}$ and $\precsim'$ in Section 3.2.2 form a total trading system $(\mathcal{K}_0, \precsim, \prec)$ that satisfies Assumptions 1–4.*

Despite the fact that Savage worked hard to avoid making the assumption that his set of acts contained the mixtures that we assume, his postulates are sufficient to show that his preorder extends to a total trading system that satisfies our assumptions without any further choices needed from the agent. For the purposes of this paper, instead of assuming a subset of P1–P7 or some weakened versions of them, assume only that there is a set $\mathcal{F}$ of Savage-style acts with a preorder (not necessarily total) $\precsim'$. Then embed $\mathcal{F}$ into the convex set $\mathcal{H}$ of Lemma 3.2 which is a special case of a set of horse lotteries. We then proceed with the same analysis and assumptions as in Sections 3.2.2 and 1.4. In particular, we make Assumptions 1–4. By so doing, we implicitly weaken some of Savage's postulates so as to allow non-Archimedean preferences, In addition, all of the extension and representation results in Section 2 above apply to the resulting trading system, as well as the results in Section 4 below. In the end, if the agent does not want to think about mixtures of Savage-style acts, we show (in Section 4.7) how to restrict the results of Section 4 to the original Savage-style acts.

§4. Probability and expected utility. In this section, we explore the relationship between finitely additive expectation and the representing function of a total trading system. Throughout the section, $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ denotes a total trading system on a standard-linear space $\mathcal{X}$ of functions from Ω to a set $\mathcal{O}$. Let U be a standard-linear function that represents $\mathcal{T}$. Let Σ be a field of subsets of Ω. In order to construct a probability from a trading system, the indicators of elements of Σ must play a role in the elements of $\mathcal{X}$. We make the following assumption about a total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ throughout this section.

Assumption 6. *For all $B \in \Sigma$ and all $X \in \mathcal{X}$, $XI_B \in \mathcal{X}$, where*

$$(XI_B)(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in B, \\ 0(\omega), & \text{if } \omega \in B^C. \end{cases}$$

If Assumption 6 is not met, one can apply Theorem 2.9 to find an extension$_2$ of $\mathcal{T}$ to $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$, where $\mathcal{X}'$ is the standard-linear span of $\mathcal{X} \bigcup \{XI_B : X \in \mathcal{X}, B \in \Sigma\}$. This extension needs to be done before attempting to infer a probability on (Ω, Σ) or attempting to interpret a representing function U as an expected utility. Once Assumption 6 is met, we can define conditional preference.

Definition 4.1. *Let $\mathcal{W}$ be a standard-linear space of functions defined on a set $\mathcal{Z}$. Let Γ be a field of subsets of $\mathcal{Z}$. Suppose that $XI_B \in \mathcal{W}$ for each $B \in \Gamma$ and $X \in \mathcal{W}$. Let $\precsim$ be a total preorder on $\mathcal{W}$. For $X, Y \in \mathcal{W}$, if $0 \prec (Y - X)I_B$, we write $X \prec Y|B$ and we say that Y* is conditionally preferred to *X given B. If $0 \precsim (Y - X)I_B$, we write $X \precsim Y|B$. If both $X \precsim Y|B$ and $Y \precsim X|B$, we write $X \sim Y|B$. An event $B \in \Gamma$ is* null *if $X \sim Y|B$ for all $X, Y \in \mathcal{W}$. An event is* non-null *if it is not null. If $z \in \mathcal{Z}$ and $\{z\}$ is a null event, we call z a* null state. *A state z is* non-null *if $\{z\}$ is non-null.*

Example 9 (Continuation of Example 6). *Consider the set $\mathcal{X}'$ of simple functions that are measurable with respect to the field Σ generated by the interval subsets of $\Omega = [0, 1]$ in Example 6. Every nonempty element of Σ is non-null. For example, if $\omega_0 \in \Omega$,*

$$(Y - X)I_{\{\omega_0\}}(\omega) = \begin{cases} Y(\omega_0) - X(\omega_0), & \text{if } \omega = \omega_0, \\ 0, & \text{otherwise}, \end{cases}$$

and $X \sim Y|\{\omega_0\}$ if and only if $X(\omega_0) = Y(\omega_0)$.

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4.1. Dominance and coherence (part two). In this section, we extend the concepts of dominance and coherence to certain horse-lottery cases. This extension is useful in Section 4.4 where we show how, in both the random-variable and horse-lottery cases, a standard-linear function that represents a coherent trading system can be interpreted as an expected value of a (possibly state-dependent) utility function defined on the codomain $\mathcal{O}$ of the elements of $\mathcal{X}$.

For the remainder of this section, we assume that $\mathcal{X}$ is the space $\mathcal{K}_0$ defined in Section 3.2.2 and $\mathcal{O}$ is a set of simple signed measures on subsets of the prize set $\mathcal{P}$ and that assign signed measure 0 to $\mathcal{P}$. As such, each element of $\mathcal{K}_0$ is a function from Ω to $\mathcal{O}$. Dominance for horse lotteries is defined in terms of conditional preference on a state-by-state basis. Suppose that there is a non-null event B that consists entirely of null states, i.e., every $\omega \in B$ is a null state. Then a state-by-state comparison of two elements X and Y of $\mathcal{X}$ given the elements of B tells us nothing about how X and Y compare (or should compare) given B. The reason is that $X \sim Y|\{\omega\}$ for every X and Y and every null state ω. To circumvent this problem, we make an assumption that is a generalization (weaker assumption) of the state-independent utility assumptions made by Anscombe and Aumann [1] (Assumption 1), Savage [35] (P3–P4), and Blume, Brandenburger, and Dekel [6] (Axioms 5 and 5′). Assumption 7 below allows varying degrees of state dependence for the utilities of prizes.

Assumption 7. *There exists a partition $\mathcal{B} \subseteq \Sigma$ of Ω into non-null events such that, for each $B \in \mathcal{B}$,*

- *for all $\omega \in B$, $\mathcal{O}_\omega$ is the same set $\mathcal{O}(B)$,*
- *for each $x \in \mathcal{O}(B)$, there is $X_x \in \mathcal{X}$ such that $X_x(\omega) = x$ for all $\omega \in B$, and*
- *for each $x \in \mathcal{O}(B)$, $0 \prec X_x|B$ if and only if $0 \prec X_x|C$ for every non-null $C \subseteq B$.*

For each ω, we will use B_ω to denote the element of $\mathcal{B}$ that contains ω.

The second bullet assumes that certain functions defined on $\mathcal{O}_\Omega$ are in $\mathcal{X}$. If these functions are not in $\mathcal{X}$ and the first bullet is satisfied, we can apply Theorem 2.9 to find an extension$_2$ of $\mathcal{T}$ to $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ where $\mathcal{X}'$ is the standard-linear span of $\mathcal{X} \bigcup \{X_x I_B I_A : B \in \mathcal{B}, x \in \mathcal{O}(B), A \in \Sigma\}$. Such an extension$_2$ will continue to satisfy Assumption 6 and the first bullet of Assumption 7. We will assume that such an extension$_2$ has been done for the remainder of the paper. Note that there is no guarantee that such a $\mathcal{T}'$ satisfies the third bullet of Assumption 7. But we cannot even check whether the third bullet is satisfied until the second bullet is satisfied.

The state-independence assumptions made by Anscombe and Aumann; Savage; and Blume, Brandenburger, and Dekel [1, 6, 35] correspond to the extreme case of Assumption 7 in which $\mathcal{B} = \{\Omega\}$, i.e., utility is independent of state for the whole state space. The opposite extreme case of Assumption 7 has $\mathcal{B} = \{\{\omega\} : \omega \in \Omega\}$, i.e., every state is non-null, in which case all three bullets of Assumption 7 are tautological because each B is a singleton. There are cases between these two extremes, such as Example 13 in Section 4.2. A more concrete example would be the following.

- Each state consists of a specification of exchange rates between a set of currencies and a specification of a set of meteorological conditions.
- The values of the prizes depend only on the exchange rates and not on the meteorological conditions.
- Each set in the partition $\mathcal{B}$ is the set of states with a fixed specification of exchange rates.

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Definition 4.3 below defines $\leq$ and dominance in those horse-lottery cases that satisfy Assumption 7. It allows us to talk about coherent trading systems in the horse-lottery case with no modifications to Assumption 5 or the definition of coherence (Definition 1.12). As in the random-variable case, whichever (if any) sense of dominance is reflected in the agent's preferences, we express "Y dominates X" by $X \prec_{\text{Dom}} Y$.

As in Definition 1.11 for the random-variable case, Definition 4.3 defines dominance on a larger set of objects than just $\mathcal{X}$. The reason is that we may need elements of that larger set of objects in order to infer the existence of a probability on the field Σ. (See Lemma 4.5 in Section 4.2.)

LEMMA 4.2. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a total trading system that satisfies Assumptions 6 and 7. Let $X \in \mathcal{O}_\Omega$, and let $\omega_0 \in \Omega$. There exists $G_{X,\omega_0} \in \mathcal{X}$ such that $G_{X,\omega_0}(\omega) = X(\omega_0)$ for all $\omega \in B_{\omega_0}$.*

Proof. Because $x = X(\omega_0) \in \mathcal{O}(B_{\omega_0})$, Assumption 7 says that there is $X_x \in \mathcal{X}$ such that $X_x(\omega) = x = X(\omega_0)$ for all $\omega \in B_{\omega_0}$. Rename X_x to be G_{X,ω_0}. □

Lemma 4.2 gives us what we need to define dominance on all of $\mathcal{O}_\Omega$.

DEFINITION 4.3. *Suppose that a trading system satisfies Assumptions 6 and 7. For each $\omega_0 \in \Omega$ and each $X \in \mathcal{O}_\Omega$, let $G_{X,\omega_0} \in \mathcal{X}$ be as defined in Lemma 4.2.*

- *If $G_{X,\omega_0} \precsim G_{Y,\omega_0} | B_{\omega_0}$, we write $X(\omega_0) \leq Y(\omega_0)$.*
- *If $G_{X,\omega_0} \prec G_{Y,\omega_0} | B_{\omega_0}$, we write $X(\omega_0) < Y(\omega_0)$.*
- *If, for all $B \in \mathcal{B}$ and all $\omega_0 \in B$, $X(\omega_0) \leq Y(\omega_0)$, we write $X \leq Y$.*
- *If, for all ω_0, $X(\omega_0) < Y(\omega_0)$, we say that* Y strictly dominates X.
- *If $X \leq Y$ and there exists ω_0 such that $X(\omega_0) < Y(\omega_0)$, we say that* Y weakly dominates X.

For trading systems in the horse-lottery case that satisfy Assumptions 6 and 7, the definition of "coherent" (Definition 1.12) applies. We did not define "uniform dominance" because the $<$ symbol in Definition 4.3 is not a relation between numerical values, but rather a non-numerical preference relation. Once we define a state-dependent utility function (Definition 4.7 in Section 4.3) we can define uniform dominance in the horse-lottery case.

4.2. Numeraires and probability.

DEFINITION 4.4. *Assume the conditions stated in Definition 4.1. Let $Z \in \mathcal{W}$. If $0 \prec Z$ and $0 \precsim Z|B$ for all B, we call Z a* numeraire for Γ. *If, in addition, $0 \prec Z|B$ for every non-null B, we call Z a* strong numeraire for Γ.

In the random-variable case with a coherent trading system, every non-negative function in $\mathcal{X}$ that is strictly preferred to 0 is a numeraire.

EXAMPLE 10 (Continuation of Example 9). *Consider the set $\mathcal{X}'$ of simple functions that are measurable with respect to the field Σ generated by the interval subsets of $\Omega = [0, 1]$ in Examples 6 and 9. Every strictly positive function $X \in \mathcal{X}'$ is a strong numeraire. The probability associated with the constant function 1 is $P_1(E) = U'(I_E)$.*

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For an arbitrary positive function X of the form (7),

$$P_X(E) = \frac{\sum_{j=1}^{n} h_j U'(I_{E\cap J_j})}{\sum_{j=1}^{n} h_j U'(I_{J_j})}$$

is another probability that corresponds to X being the numeraire.

Sometimes, even positive constants might not be strong numeraires.

EXAMPLE 11. *Let $\Omega = \mathbb{Z}^+$, the positive integers, and let Σ be the finite/cofinite field, i.e., the collection of all finite subsets of Σ and their complements. Let $G : \Omega \to \mathbb{R}$ be $G(\omega) = \omega$. Let $\mathcal{X}$ be the standard-linear span of all standard-valued bounded functions and the functions $\{GI_B : B \in \Sigma\}$. Each $X \in \mathcal{X}$ can be written uniquely as*

$$X = X_b + \alpha_X GI_E, \tag{12}$$

where X_b is bounded, α_X is standard, and $E = \{11, 12, \ldots\}$.[10] *Define*

$$U(X) = \frac{1}{10}\sum_{\omega=1}^{10} X(\omega) + 2\alpha_X. \tag{13}$$

Also, $U : \mathcal{X} \to \mathbb{R}$ is standard-linear (and standard-valued). If $X \prec_{Dom} Y$ (uniform or strict, but not *weak dominance) then $U(X) < U(Y)$, so U represents a coherent (with uniform or strict dominance) total trading system $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ with $X \precsim Y$ meaning $U(X) \le U(Y)$. Since $U(1) = 1$, $0 \prec 1$. Since $0 \le 1$, $0 \precsim 1|B$ for all $B \in \Sigma$, and 1 is numeraire for Σ. Since $0 \prec G|E$, E is a non-null event. On the other hand, $0 \sim 1|E$, so 1 is not a strong numeraire for Σ. The non-null events are all cofinite sets and all elements of Σ that are supersets of the singletons $\{1\}, \ldots, \{10\}$. For each $\omega = 1, \ldots, 10$, $0 \prec G|\{\omega\}$, and for each cofinite set B, $0 \prec G|B$, so G is a strong numeraire for Σ.*

In Section 4.5, we show how different numeraires relate to each other within the same trading system. In general, we need a numeraire in order to derive a probability on (Ω, Σ) from a trading system. Since 1 is a numeraire in the random-variable case, the following result is needed only in the horse-lottery case.

LEMMA 4.5. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a coherent total trading system in the horse-lottery case that satisfies Assumptions 6 and 7. There is a coherent extension$_2$ $\mathcal{T}' = (\mathcal{X}', \precsim', \prec')$ of $\mathcal{T}$ such that:*

- *there is a numeraire Z for Σ,*
- *Z is constant on each element of $\mathcal{B}$ (from Assumption 7), and*
- *$0 \prec Z|B$ for every $B \in \mathcal{B}$.*

Proof. First, we show that, for each $B \in \mathcal{B}$, there is $x_B \in \mathcal{O}(B)$ such that $0 \prec X_{x_B}|B$, where X_x is defined in the second bullet of Assumption 7. Let $B \in \mathcal{B}$, which, by definition, is non-null. So there is $Y \in \mathcal{X}$ such that $0 \prec Y|B$. For each $\omega \in B$, let $y_\omega = Y(\omega)$. If $X_{y_\omega} \precsim 0|B$ for every $\omega \in B$, then $YI_B \le 0$ and $Y \precsim 0|B$ by coherence. This contradicts $0 \prec Y|B$, so there must be $\omega_0 \in B$ such that $0 \prec X_{x_B}|B$ for $x_B = y_{\omega_0}$.

Define $Z : \Omega \to^* \mathbb{R}$ as follows: For all $B \in \mathcal{B}$ and all $\omega \in B$, let $Z(\omega) = x_B$. By construction $0 \prec Z|B$ for all $B \in \mathcal{B}$ and $0 \prec_{\text{Dom}} Z$ by either definition of dominance. If $Z \in \mathcal{X}$, let $\mathcal{T}' = \mathcal{T}$. If not, let $\mathcal{X}'$ be the standard-linear span of $\mathcal{X} \bigcup \{ZI_D : D \in \Sigma\}$.

[10] Note that $\alpha_X = \lim_{\omega\to\infty} X(\omega)/G(\omega)$, and $X_b = X - \alpha_X GI_E$.

Use Theorem 2.9 to extend$_2$ $\mathcal{T}$ to a coherent $\mathcal{T}'$. In $\mathcal{T}'$, $0 \precsim' Z|D$ for every $D \in \Sigma$ by construction. Since $0 \prec' Z$, Z is a numeraire for Σ. $\square$

For the remainder of the paper, assume that each trading system $\mathcal{T}$ in the horse-lottery case is an extension$_2$ from Lemma 4.5.

LEMMA 4.6. *Let $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ be a coherent total trading system that satisfies Assumption 6 (and Assumption 7 in the horse-lottery case). Let Z be a numeraire for Σ, and let U be a standard-linear function that represents $\mathcal{T}$. Define $P(B) = U(ZI_B)/U(Z)$ for each $B \in \Sigma$. Then P is a finitely additive (possibly nonstandard-valued) probability on Σ.*

Proof. First, note that $U(Z) > 0$ and $P(\Omega) = U(ZI_\Omega)/U(Z) = 1$. For each $B \in \Sigma$, $0 \precsim Z|B$ and $P(B) = U(ZI_B)/U(Z) \geq 0$. Suppose that B_1 and B_2 are disjoint events. Then $ZI_{B_1} + ZI_{B_2} = ZI_{B_1 \cup B_2}$. Since U is standard-linear, we have $U(ZI_{B_1}) + U(ZI_{B_2}) = U(ZI_{B_1 \cup B_2})$ and $P(B_1) + P(B_2) = P(B_1 \cup B_2)$. $\square$

We take probability to be finitely additive in Lemma 4.6 and elsewhere due to the lack of countably additive probabilities in some relatively simple situations. (See Example 18 in Section A.3.)

EXAMPLE 12 (Continuation of Example 11). *The two numeraires, 1 and G for Σ lead to two different probabilities P_1 and P_G on (Ω, Σ), respectively. From (13), it is clear that $P_1(\{\omega\}) = 1/10$ for $\omega = 1, \ldots, 10$, and $P_1(B) = 0$ for each B that is a subset of E. In particular, $P_1(E) = 0$ despite E being non-null. For each ω, $U(GI_{\{\omega\}}) = \omega/10$ for $\omega \in E^C$ and 0 for $\omega \in E$. For each cofinite set $B \subseteq E$, $U(GI_B) = 2$, since $\alpha_G = 1$. Then $P_G(\{\omega\}) = \omega/75$ for $\omega \in E^C$ and 0 for $\omega \in E$. Also, $P_G(B) = 4/15$ for every cofinite subset B of E, including E itself.*

EXAMPLE 13. *Let $\Omega = (0, 1)$, the unit interval with Σ the Borel σ-field. Let $\mathcal{P} = \{a, b\}$. Let $\mathcal{R}$ be the set of all lotteries over $\mathcal{P}$. Each element $r \in \mathcal{R}$ is characterized by $r(\{b\})$. Let $\mathcal{H}$ be that subset of $\mathcal{R}^\Omega$ for which $g_h(\omega) = h(\omega)(\{b\})$ is a Borel-measurable function of ω. In this example, for states $\omega < 1/2$, prize a is better than b, and the situation is the reverse for all $\omega > 1/2$. We include the null event $\{1/2\}$ with $\omega > 1/2$ for the rest of the example. Define*

$$V(h) = \int_0^{1/2} [1 - g_h(\omega)]d\omega + \int_{1/2}^1 g_h(\omega)d\omega. \tag{14}$$

Each element of $\mathcal{K}_0$ has the form $\alpha(h_1 - h_2)$ for standard $\alpha \geq 0$ and $h_1, h_2 \in \mathcal{H}$. Define

$$U(\alpha[h_1 - h_2]) = \alpha[V(h_1) - V(h_2)].$$

The trading system represented by U satisfies Assumption 7 with $\mathcal{B} = \{(0, 1/2), [1/2, 1)\}$. There are many numeraires for Σ. Let $g_{h_0}(\omega) = I_{(0,1/2)}(\omega)$, so that $V(h_0) = 0$. Each h *with $g_h(\omega) < 1$ for $\omega < 1/2$ and $g_h(\omega) > 0$ for $\omega \geq 1/2$ has $V(h) > 0$ and can be combined with h_0 to define a numeraire for Σ as follows. Define $Z_h = (h - h_0)/V(h)$. Since $U(Z_h I_A) \geq 0$ for each Borel set A, $0 \precsim Z_h|A$. Also, $0 \prec Z_h$, so each such Z_h is a numeraire for Σ. As (14) would suggest, the probability derived from Z_h has the following* *density with respect to Lebesgue measure*:

$$f_h(\omega) = \frac{1}{V(h)}\left([1 - g_h(\omega)]I_{(0,1/2)}(\omega) + g_h(\omega)I_{[1/2,1)}(\omega)\right).$$

Since $f_h > 0$, Z_h *is a strong numeraire for* Σ.

4.3. State-dependent utility. In the random-variable case, one can think about a numeraire as the units that correspond to numerical values of the random variables. For example, suppose that the random variables in a trading system $\mathcal{T}$ are in units of dollars. A European foreign-exchange trader might be more comfortable comparing units of euros rather than dollars. If $G(\omega)$ is the exchange rate (in dollars per euro) in state ω, $X(\omega)/G(\omega)$ is the utility of X in state ω to the European trader measured in the preferred currency units of euros. We need something analogous to X/G for the horse-lottery case.

DEFINITION 4.7. *Let* Σ *be a field of subsets of* Ω, *and let* $\mathcal{T}$ *be a coherent total trading system that satisfies Assumption 6 (and Assumption 7 in the horse-lottery case). Let* $U : \mathcal{X} \to^* \mathbb{R}$ *be a standard-linear function that represents* $\mathcal{T}$. *In the horse-lottery case, let* Z *be a numeraire of the sort obtained through Lemma 4.5. In the random-variable case, let* Z *be a strictly positive numeraire. Define, for each* $X \in \mathcal{X}$, *each* $\omega \in \Omega$, *and each* $x \in \mathcal{O}_\omega$:

$$U_Z^*(\omega, x) = \begin{cases} \dfrac{x}{Z(\omega)}, & \textit{in the random-variable case,} \\ \dfrac{U(xI_{B_\omega})}{U(ZI_{B_\omega})}, & \textit{in the horse-lottery case.} \end{cases}$$

We call U_Z^* *the* state-dependent utility function (relative to the numeraire Z). *Let* $X, Y \in \mathcal{X}$. *If there exists a standard* $\varepsilon > 0$ *such that* $U_Z^*(\omega, X(\omega)) \le U_Z^*(\omega, Y(\omega)) - \varepsilon$ *for all* ω, *we say that* X is uniformly dominated by Y *relative to* Z *or* Y uniformly dominates X *relative to* Z.

As Definition 4.7 makes explicit, which elements of $\mathcal{X}$ uniformly dominate each other depends on which numeraire Z is used. Example 14 illustrates this fact.

EXAMPLE 14 (Continuation of Example 13). *The state-dependent utility that corresponds to the numeraire* Z_h *is, for each* $X = \alpha(h_1 - h_2) \in \mathcal{K}_0$,

$$U_h^*(\omega, X(\omega)) = \frac{\alpha}{f_h(\omega)} \begin{cases} g_{h_2}(\omega) - g_{h_1}(\omega), & \textit{for } \omega < 1/2, \\ g_{h_1}(\omega) - g_{h_2}(\omega), & \textit{for } \omega \ge 1/2, \end{cases}$$

so that $U_h^*(\omega, Z_h(\omega)) = 1$ *for all* ω, *and* Z_h *uniformly dominates* 0 *relative to* Z_h. *The same could be said for each numeraire constructed in the same fashion. However, what "uniformly dominates" 0 depends on which numeraire is used to construct the probability. To be specific, suppose that* $g_h(\omega) = 1/2$ *for all* ω *so that* f_h *is constant. Let* h' *be another horse lottery with* $g_{h'}(\omega) = 1 - \omega$ *for all* ω, *so that* $f_{h'}$ *is* $\wedge$*-shaped and* $U_{h'}^*(\omega, Z_{h'}(\omega)) = 1$ *for all* ω. *Then* h' *uniformly dominate* 0 *relative to* $Z_{h'}$, *but*

$$U_h^*(\omega, Z_{h'}(\omega)) = \begin{cases} \omega, & \textit{for } \omega < 1/2, \\ 1 - \omega, & \textit{for } \omega \ge 1/2, \end{cases}$$

so h' *does not uniformly dominate* 0 *relative to* Z_h.

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To see that U_Z^* is well defined in the horse-lottery case, note that for each $B \in \mathcal{B}$ of Assumption 7, $U(ZI_B) > 0$ because $0 \prec Z|B$. In general U_Z^* is a nonstandard-valued function defined on the set $\bigcup_{\omega\in\Omega}(\{\omega\} \times \mathcal{O}_\omega)$. The interpretation of $U_Z^*(\omega, x)$ is the utility to the agent of X in state ω measured in units of numeraire Z when $X(\omega) = x$. In Section 4.4, Theorem 4.9 shows that $U(X)$ can be interpreted as an expected value of the state-dependent utility of X with respect to the probability P on (Ω, Σ) that corresponds to the chosen numeraire Z via Lemma 4.6.

4.4. Expected utility. The following definition is a generalization of the concept of Daniell integral to the finitely additive nonstandard-valued case. See [37] for discussion of the finitely additive standard-valued case. We give additional motivation for this definition in Appendix B.

Definition 4.8. *Let ${}^*\mathbb{R}$ be a nonstandard model of the reals, and let $\mathcal{Z}$ be a set. Let $\mathcal{W}$ be a standard-linear space that is a subset of $({}^*\mathbb{R})^{\mathcal{Z}}$, and that contains all standard constants. Let $W : \mathcal{W} \to {}^*\mathbb{R}$ be a standard-linear function that satisfies*

- $W(1) = 1$, *and*
- *for* $w_1, w_2 \in \mathcal{W}$, $w_1 \le w_2$ *implies* $W(w_1) \le W(w_2)$ (*i.e.*, W *is* monotone).

Then W acts as an expected value on $\mathcal{W}$. *Suppose, in addition, that* Γ *is a field of subsets of* $\mathcal{Z}$ *and for every* $B \in \Gamma$ *and* $w \in \mathcal{W}$, $wI_B \in \mathcal{W}$. *Define* $P(B) = W(I_B)$ *for* $B \in \Gamma$ *so that* P *is a finitely additive probability on* $(\mathcal{Z}, \Gamma)$. *Then* W acts as an expected value on $\mathcal{W}$ *with respect to* P. *For each* $w \in \mathcal{W}$, *we also say that* $W(w)$is an expected value of w *with respect to* P. *We also use the notation* $P(w)$ *to denote* $W(w)$.

It should be apparent that Definition 4.8 agrees with the familiar countably additive definition of expected value for simple probabilities when ${}^*\mathbb{R} = \mathbb{R}$. Furthermore, all finite standard-valued countably additive expected values satisfy Definition 4.8, but they also have an additional continuity property that does not carry over to the nonstandard-valued case.

Theorem 4.9. *Assume the conditions from and terms defined in Definition 4.7. Let* $U_X(\omega) = U_Z^*(\omega, X(\omega))$ *for each* $X \in \mathcal{X}$ *and* $\omega \in \Omega$, *and let* P *be the probability on* (Ω, Σ) *from Lemma 4.6 using* Z *as the numeraire for* Σ. *Let* $\mathcal{W}_Z = \{U_X : X \in \mathcal{X}\}$. *Then*:

- $\mathcal{W}_Z$ *is a standard-linear space of functions from* Ω *to* ${}^*\mathbb{R}$,
- $W_Z(U_X) = U(X)/U(Z)$ *defines an expected value of* U_X *with respect to* P, *and*
- *if dominance means weak dominance, then every non-empty element of* Σ *is non-null and has positive probability.*

Proof. First, we show that $U_X \le U_Y$ implies $X \le Y$. This is immediate in the random-variable case. In the horse-lottery case, if $U_X \le U_Y$, then for each $B \in \mathcal{B}$ and each $\omega_0 \in B$, $G_{X,\omega_0} \precsim G_{Y,\omega_0}|B$ in the notation of Lemma 4.2 and Definition 4.3. It follows that $X \le Y$.

Next, we show that W_Z is well defined. Let $X, Y \in \mathcal{X}$ with $U_X = U_Y$. Then $U_X \le U_Y$ and $U_Y \le U_X$. We just proved that $X \le Y$ and $Y \le X$. By coherence of $\mathcal{T}$, $U(X) \le U(Y)$ and $U(Y) \le U(X)$, so $U(X) = U(Y)$ and W_Z is well defined.

Next, we prove the three bullets in the theorem. Since U is standard-linear, so is $U_Z^*(\omega, \cdot)$ for each ω. Since $U_{\alpha X+\beta Y}(\omega) = \alpha U_X(\omega) + \beta U_Y(\omega)$ for all $X, Y \in \mathcal{X}$, $\omega \in \Omega$, and standard α, β, we see that $\mathcal{W}_Z$ is a standard-linear space and that W_Z is standard-linear. Since $U_Z(\omega) = 1$ for all ω, we see that $\mathcal{W}_Z$ contains all constants and $W_Z(1) = U(Z)/U(Z) = 1$. Since $U_{XI_B}(\omega) = U_X(\omega)I_B(\omega)$ for all $X \in \mathcal{X}$, $B \in \Sigma$, and $\omega \in \Omega$, we see that $\mathcal{W}_Z$ contains wI_B for all $w \in \mathcal{W}_Z$ and $B \in \Sigma$. Thus, the first bullet and part of the second bullet are proven.

For the rest of the second bullet we must show that (i) $W_Z(I_B) = P(B)$ for all $B \in \Sigma$, and (ii) W_Z is monotone. For (i), note that for each $B \in \Sigma$, $U_{ZI_B} = I_B$, and $W_Z(I_B) = U(ZI_B) = P(B)$ (including $B = \Omega$). For (ii) $U_X \leq U_Y$ implies $X \leq Y$ by what we proved earlier. Then, $W_Z(U_X) = U(X)/U(Z) \leq U(Y)/U(Z) = W_Z(U_Y)$, where the inequality follows from coherence of $\mathcal{T}$.

For the third bullet, assume that dominance means weak dominance and that $E \in \Sigma$ is non-empty. The construction of Z implies that $0 \prec_{\text{Dom}} ZI_E$, so $0 \prec Z|E$, and E is non-null. Also, $0 = U(0)/U(Z) < U(ZI_E)/U(Z) = P(E)$. □

Corollary 1. *Assume the conditions of Theorem 4.9 in the random-variable case with numeraire $Z = 1$. Then $\mathcal{W}_Z = \mathcal{X}$, $U_X = X$ for all $X \in \mathcal{X}$, and U acts as an expected value on $\mathcal{X}$ with respect to P.*

4.5. Changes of numeraire. Assume the conditions stated in Definition 4.7. Let Z_1, Z_2 be two numeraires of the sort described there with corresponding probabilities P_{Z_1}, P_{Z_2}. Then

$$\mathcal{W}_{Z_j} = \{U^*_{Z_j}(\cdot, X(\cdot)) : X \in \mathcal{X}\},$$

for $j = 1, 2$. And $W_{Z_j} : \mathcal{W}_{Z_j} \to^* \mathbb{R}$, defined by

$$W_{Z_j}(U^*_{Z_j}(\cdot, X(\cdot))) = U(X)/U(Z_j),$$

acts as an expected value on $\mathcal{W}_{Z_j}$ with respect to P_{Z_j}. Note that

$$U^*_{Z_2}(\cdot, Z_1(\cdot)) = \frac{1}{U^*_{Z_1}(\cdot, Z_2(\cdot))}, \text{ and}$$

$$\mathcal{W}_{Z_2} = \left\{wU^*_{Z_2}(\cdot, Z_1(\cdot)) : w \in \mathcal{W}_{Z_1}\right\}.$$

For each $w \in \mathcal{W}_{Z_2}$,

$$W_{Z_2}(w) = W_{Z_1}\left[wU^*_{Z_2}(\cdot, Z_1(\cdot))\right]\frac{U(Z_1)}{U(Z_2)}.$$

In other words, $U^*_{Z_2}(\cdot, Z_1(\cdot))U(Z_1)/U(Z_2)$ has the defining feature of a Radon–Nikodym derivative of W_{Z_2} with respect to W_{Z_1} (when restricted to indicators of events), without referring to absolute continuity.

Example 15 (Continuation of Example 12). *In the notation of Theorem 4.9, the two numeraires, 1 and G, along with their probabilities P_1 and P_G, correspond to linear spaces $\mathcal{W}_1 = \mathcal{X}$ and $\mathcal{W}_G = \{X/G : X \in \mathcal{X}\}$ with the standard-linear functions $W_1(X) = U(X)$ (for $X \in \mathcal{W}_1$) and $W_G(Y) = U(YG)/U(G)$ (for $Y \in \mathcal{W}_G$). So, $U^*_1(\omega, x) = x$ while $U^*_G(\omega, x) = x/\omega$. It then appears as if $dP_G/dP_1(\omega) = 1/[\omega U(G)]$ acts as a Radon–Nikodym derivative despite the fact that no existing definition of absolutely continuous has P_G absolutely continuous with respect to P_1. Nevertheless, we can still express $P_G(B) = P_1(BG)/P_1(G)$ for each event $B \in \Sigma$, including $B = E = \{11, 12, \ldots\}$.*

4.6. Another layer of expected utility. As in Section 3.2.3, in the horse-lottery case, it would be easier on the intuition if state-dependent utility could be expressed as a function of lotteries in state ω rather than elements of $\mathcal{O}_\omega$. Let $\mathcal{W}$ and U_Z^* be as in Theorem 4.9. Let $h_0 \in \mathcal{H}$ be arbitrary, and define $V_Z^* : \bigcup_{\omega\in\Omega}(\{\omega\} \times \mathcal{R}_\omega) \to^* \mathbb{R}$ by

$$V_Z^*(\omega, r) = U_Z^*(\omega, r - h_0(\omega)). \tag{15}$$

This has the effect of shifting the utility in state ω so that $h_0(\omega)$ has value 0. It follows that, for $r_1, r_2 \in \mathcal{R}_\omega$,

$$U_Z^*(\omega, \alpha[r_1 - r_2]) = \alpha[V_Z^*(\omega, r_1) - V_Z^*(\omega, r_2)], \tag{16}$$

which gives us equivalent ways to express state-dependent utility in both $\mathcal{H}$ and $\mathcal{K}_0$. Equivalent ways to express marginal utilities were given in (10) and (11) in Section 3.2.3. Finally, for each $\omega \in \Omega$ and $p \in \mathcal{P}_\omega$, let r^p stand for the simple lottery that assigns probability 1 to the prize p, and define

$$V_{0,Z}^*(\omega, p) = V_Z^*(\omega, r^p). \tag{17}$$

A useful consequence of the above notation is the following simple corollary of Theorem 4.9.

COROLLARY 2. *Assume the conditions of Theorem 4.9. Then $V(h)$ is an expected value of $V_Z^*(\cdot, h(\cdot))$ with respect to P.*

Our final goal, in this section, is to show that $V_Z^*(\omega, r)$ can be interpreted as an expected value of $V_{0,Z}^*(\omega, \cdot)$ with respect to the simple probability r.

LEMMA 4.10. *For each $\omega \in \Omega$ and $r \in \mathcal{R}_\omega$, $V_Z^*(\omega, r)$ is an expected value of $V_{0,Z}^*(\omega, \cdot)$ with respect to* r.

Proof. Let $\omega \in \Omega$, and let $r \in \mathcal{R}_\omega$ be a simple lottery with $\mathcal{P}(r) = \{p_1, \dots, p_n\}$ and $r(\{p_j\}) = \alpha_j$ for $j = 1, \dots, n$. For $j = 1, \dots, n$, let $x_j = r^{p_j} - h_0(\omega) \in \mathcal{O}_\omega$, where r^{p_j} stands for the simple lottery that assigns prize p_j with probability 1. Then $r - h_0(\omega) = \sum_{j=1}^n \alpha_j x_j$. Hence

$$\begin{aligned}
V_Z^*(\omega, r) &= U^*(\omega, r - h_0(\omega)) \\
&= \sum_{j=1}^n \alpha_j U_Z^*(\omega, x_j) \\
&= \sum_{j=1}^n \alpha_j [V_Z^*(\omega, r^{p_j}) - V_Z^*(\omega, h_0(\omega))] \\
&= \sum_{j=1}^n \alpha_j V_{0,Z}^*(\omega, p_j),
\end{aligned}$$

where the first equality follows from (15), the second follows from standard-linearity of $U_Z^*(\omega, \cdot)$, the third follows from (16), and the last follows from the definition of $V_{0,Z}^*$ and the fact that $V_Z^*(\omega, h_0(\omega)) = 0$. Let $Z = \mathcal{P}(r)$, $\Gamma = 2^{\mathcal{P}(r)}$ and $\mathcal{W}$ equal to the standard-linear span of $\{V_{0,Z}^*(\omega, \cdot)I_A : A \in \Gamma\} \bigcup \{I_A : A \in \Gamma\}$ in Definition 4.8. The two bullets in the definition are clearly satisfied. □

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Corollary 2 and Lemma 4.10 combine to say that $V(h)$ is an iterated expected value of $V^*_{0,Z}(\omega, p)$ where, for each $\omega \in \Omega$, the inner expected value is with respect to the probability $h(\omega)$ on $\mathcal{P}(h(\omega))$, and the outer expected value is with respect the probability P on Ω. To be precise, $Q(\omega) = h(\omega)[V^*_{0,Z}(\omega, \cdot)]$ is the inner expected value, and $V(h) = P[Q(\cdot)]$ is the outer expected value.

4.7. The original Savage-style acts. In this section, we assume that a coherent total trading system $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ that satisfies Assumptions 1–7 was generated from a set $\mathcal{F}$ of Savage-style acts by creating the set $\mathcal{H}$ of horse lotteries that are simple mixtures of elements of $\mathcal{F}$. The probability P that results from Lemma 4.6 is a function from Σ to ${}^*\mathbb{R}$, and as such can be associated with the set $\mathcal{F}$, regardless of how its existence was proved. Similarly, the function V of (10), when restricted to the elements of $\mathcal{F}$, represents the preorder $\precsim'$ on $\mathcal{F}$ that is the restriction of $\precsim$. Also, the state-dependent utility of act $f \in \mathcal{F}$ in state ω is $V^*_{0,Z}(\omega, f(\omega))$ from (17). Hence the expected-utility interpretation of Corollary 2, when restricted to elements of $\mathcal{F}$ gives an interpretation of V as an expected value of the state-dependent utility $V^*_{0,Z}(\omega, f(\omega))$ with respect to P.

4.8. Existence of coherent trading systems in the horse-lottery case. In the random-variable case, all forms of dominance are defined independently of an agent's willingness to trade. Furthermore, a respect for a chosen form of dominance can be enforced while the agent is stating preferences. The same is not true for the horse-lottery case, where dominance is defined based on an existing total trading system. If that trading system does not respect dominance, the agent either has to start over or can try to modify the stated preferences.

There is a way for the agent to enforce respect for a chosen form of dominance, but it places restrictions on the order in which preferences can be stated. To mimic the random-variable case, for each $X, Y \in \mathcal{X}$, the agent needs to be able to determine, prior to saying which trades between X and Y are acceptable, whether $X \prec_{\text{Dom}} Y$ or $Y \prec_{\text{Dom}} X$ or neither. To do this, one needs a partition $\mathcal{B}$ of Ω into non-null events that satisfy the first two bullets of Assumption 7. The agent also needs to be willing to require preferences to satisfy the third bullet.

If the agent wishes to respect weak dominance, then all states will be non-null, and $\mathcal{B}$ can be taken to be $\{\{\omega\} : \omega \in \Omega\}$. The agent would first determine whether $X \prec Y|\{\omega\}$ or $Y \prec X|\{\omega\}$ or neither. Such comparisons reduce to checking, for each $x, y \in \mathcal{O}_\omega$, how $xI_{\{\omega\}}$ and $yI_{\{\omega\}}$ compare. Once all such determinations are made, weak dominance is defined on all of $\mathcal{O}_\Omega$, and all instances of $X \prec_{\text{Dom}} Y$ can be labelled as $X \ll Y$ before the rest of the trading system is determined.

If the agent wishes to respect only strict dominance, the agent needs to determine, for each $B \in \mathcal{B}$ and each $(x, y) \in \mathcal{O}(B)^2$, whether $x \prec y|B$. If so, the agent then declares that $x \prec y|C$ for every non-null subset C of B. As above, strict dominance is now defined on all of $\mathcal{O}_\Omega$, and all instances of $X \prec_{\text{Dom}} Y$ can be labelled as $X \ll Y$ before the rest of the trading system is determined.

4.9. Conditional trading. Conditional preference (see Definition 4.1) can be interpreted as a willingness to trade given that some event occurs. Throughout this section, assume the conditions of Definition 4.7 and Theorem 4.9. In particular:

- $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$ is a coherent total trading system with standard-linear representing function U.
- Σ is a field of events for which Assumption 6 holds.

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- A numeraire Z for Σ exists with corresponding probability P as in Lemma 4.6.
- There exists a state-dependent utility $U_Z : \mathcal{O}_\Omega \to {}^*\mathbb{R}$ such that $U(X) = U(Z)P[U_Z(\cdot, X(\cdot))]$.

The following result is straightforward from standard-linearity of U.

Proposition 4.11. *Let $E_1, \dots, E_n$ be a finite partition of Ω. If $X \precsim Y|E_j$ for $j = 1, \dots, n$, then $X \precsim Y$. If, in addition, $X \prec Y|E_j$ for at least one j, then $X \prec Y$.*

In the spirit of conditioning on an event, the following result shows how to restrict a trading system on a state space Ω to a smaller state space consisting of an element E of Σ with $P(E) > 0$. The restriction $X|_E$ of a function to a subset E of its domain is defined to be the function

$$X|_E(\omega) = X(\omega) \text{ for all } \omega \in E,$$

which maps E into the codomain of X. The corresponding restriction of a field Σ is $\Sigma_E = \{A \cap E : A \in \Sigma\}$.

Lemma 4.12. *Let E be a non-null element of Σ, and define*

- *$\mathcal{X}_E = \{X|_E : X \in \mathcal{X}\}$, and*
- *$\precsim_E$ to mean $X|_E \precsim_E Y|_E$ if and only if $X \precsim Y|E$.*

Then, $\mathcal{T}_E = (\mathcal{X}_E, \precsim_E, \prec_E)$ is a total trading system with representing function $U_E(X|_E) = U(XI_E)$, and which satisfies Assumption 6. If $P(E) > 0$, then (a) $Z|_E$ is a numeraire for Σ_E with associated probability $P_E(B) = P(B \cap E)/P(E)$, and (b) there is an expected state-dependent utility representation for U_E with respect to P_E. Finally, if dominance in $\mathcal{T}$ means weak dominance, then $\mathcal{T}_E$ is coherent with weak dominance.

Proof. It is straightforward that $\precsim_E$ is a well-defined total preorder on $\mathcal{X}_E$ as is the fact that Assumptions 1–4 hold in $\mathcal{T}_E$. Since U is standard-linear and the operation of restriction to E commutes with linear combinations, it follows that U_E is also standard-linear. Since $X \precsim Y|E$ means $XI_E \precsim YI_E$, it is clear that U_E represents $\mathcal{T}_E$. For Assumption 6, the appropriate field is Σ_E, and the assumption holds in $\mathcal{T}_E$.

Next, assume that $P(E) > 0$ so that $0 \precsim Z|E$ and $Z|_E$ is a numeraire for Σ_E. Since $U(ZI_E) = P(E)$ and $U(ZI_{B\cap E}) = P(B \cap E)$, the probability associated with $Z|_E$ is P_E as stated. Also, for each $X \in \mathcal{X}$,

$$\begin{aligned} U_E(X|_E) &= U(XI_E) = P[U_Z^*(\cdot, X(\cdot))I_E] \\ &= P_E[U_Z^*(\cdot, X(\cdot))]P(E) \\ &= P_E[U'_{Z,E}(\cdot, X|_E)], \end{aligned}$$

where $U'_{Z,E}(\omega, x) = U_Z^*(\omega, x)P(E)$ for $\omega \in E$ and $x \in \mathcal{O}_\omega$.

Finally, assume that dominance in $\mathcal{T}$ means weak dominance. If $X|_E \prec_{\mathrm{Dom}} Y|_E$ in $\mathcal{T}_E$, then $X|_E \le Y|_E$, and there is $\omega \in E$ such that $X|_E(\omega) < Y|_E(\omega)$ so it follows that $XI_E \prec_{\mathrm{Dom}} YI_E$ and $XI_E \prec YI_E$ in $\mathcal{T}$. Hence $X|_E \prec_E Y|_E$, and Assumption 5 holds in $\mathcal{T}_E$. □

As a corollary, we have a version of the law of total probability/expectation for conditional trading systems. Note that, if E is a non-empty null event then $P(E) = 0$ and $U_E(X|_E) = 0$ for all $X \in \mathcal{X}$.

COROLLARY 3. *Let n be a standard finite integer, and let $E_1, \dots, E_n$ be a partition of Ω into non-empty events. Then, for each $X \in \mathcal{X}$,*

$$U(X) = \sum_{j=1}^{n} U_{E_j}(X|_{E_j})P(E_j).$$

The following example illustrates why coherence of $\mathcal{T}_E$ in Lemma 4.12 is proven only for weak dominance.

EXAMPLE 16 (Continuation of Examples 11 and 12). *Recall that the trading system $\mathcal{T}$ is coherent using either uniform or strict dominance, and that the set $E = \{11, 12, \dots\}$ is non-null. The two probabilities P_1 and P_G computed in Example 12 differ most notably by the fact that $P_1(E) = 0$ while $P_G(E) > 0$. Suppose that we try to restrict the trading system to the set E as is done in Lemma 4.12. With $Z = 1$ as numeraire, $P_1(E) = 0$, so Lemma 4.12 doesn't apply. In particular, $0 \sim 1|E$ which violates all forms of dominance. With $Z = G$ as numeraire, $P_G(E) > 0$, but $U_G^*(\omega, x) = x/\omega$. In order for X to uniformly dominate 0 there must be a standard $\varepsilon > 0$ such that $X(\omega)/\omega > \varepsilon$ for all ω. A necessary condition for this is $\alpha_X > 0$, so only some unbounded functions uniformly dominate 0, and all $X \in \mathcal{X}$ with $\alpha_X > 0$ satisfy $0 \prec X|E$. So $\mathcal{T}'$ is coherent using uniform dominance with numeraire G. Finally, recall that $\mathcal{T}'$ is not coherent using strict dominance because $0 \sim 1|E$, despite the fact that $\mathcal{T}$ is coherent using strict dominance.*

§5. Discussion. The major contributions of this paper are:

- a systematic representation of coherent preferences amongst random variables or horse lotteries regardless of how strong is the form of dominance that one wishes to respect,
- the use of nonstandard models of the reals to represent non-Archimedean preferences,
- an extension theorem from one standard-linear space of random variables to a larger space,
- an expected utility interpretation for the nonstandard representation in special cases (including those of existing theories) and
- a derivation of conditional preferences.

Some, but not all, of the examples of non-Archimedean preferences arise from the use of weak dominance in the definition of coherence. Weak dominance is the weakest of the three dominance concepts in Definition 1.11. Weak dominance is the same as the form of dominance used to define inadmissibility in statistical decision theory. The strongest of the three dominance concepts is the one used in de Finetti's theory, namely uniform dominance. Strict dominance is intermediate to the other two. Since dominance is used to prevent calling a trading system coherent, the stronger the dominance condition, the weaker the sense of coherence, i.e., the easier it is to call a trading system coherent. Since some of our results use the weakest form of dominance, those results use the strongest form of coherence.

There is room for future work. Assumption 7, in the presence of the other assumptions, is sufficient to prove the existence of an expected-utility representation of preference, but it is not necessary. If we define the finitely additive signed measure $\mu_X(B) = U(XI_B)$ for $B \in \Sigma$, then $U_Z^*(\cdot, X(\cdot))$ behaves like a Radon–Nikodym derivative of μ_X with respect to P in Theorem 4.9. The missing necessary condition would be equivalent to a Radon–Nikodym theorem for nonstandard-valued finitely additive signed measures. A standard-valued finitely additive Radon–Nikodym theorem for bounded measures was proved by Maynard [24], but it is heavily dependent on standard real numbers. Here is an example of a total trading system with an expected-utility representation that fails Assumption 7.

EXAMPLE 17. *Let $\Omega = (0, 1)$ with $\mathcal{P} = \{a, b, c\}$ and Σ being the field generated by the intervals (the unions of finitely many disjoint intervals, including singletons). Each $r \in \mathcal{R}$, the set of all lotteries, is a simple probability $(r(\{a\}), r(\{b\}), r(\{c\}))$. Let $V^*(\omega, r) = \omega r(\{b\}) + r(\{c\})$. This corresponds to a having utility 0 in every state, b having utility ω in state ω, and c having utility 1 in every state. Let $\mathcal{H}$ be the set of functions $h : \Omega \to \mathcal{R}$ such that $V^*(\omega, h(\omega))$ is a Borel-measurable function of ω. We will work in the space $\mathcal{H}$ rather than $\mathcal{K}_0$ where possible. Define*

$$V(h) = \int_0^1 V^*(\omega, h(\omega))d\omega.$$

Let $h \precsim' g$ mean $V(h) \le V(g)$. Let $Z = h_c - h_a$, where $h_c(\omega) = (0, 0, 1)$ for all ω and $h_a(\omega) = (1, 0, 0)$ for all ω. Then $V^*(\omega, h_a) = 0$ and $V^*(\omega, h_c) = 1$ for all ω. This makes $Z \in \mathcal{K}_0$ a numeraire, and V^* is actually V_Z^* in the notation of Section 4.6. The probability corresponding to Z is Lebesgue measure on (Ω, Σ). Every non-null event B contains an interval C. Let α be the midpoint of C, and partition $C = C_1 \bigcup C_2$ by splitting at the midpoint, which makes C_1 and C_2 non-null. Notice that $(0, 1, 0) \prec' (1 - \alpha, 0, \alpha)|C_1$ and $(1 - \alpha, 0, \alpha) \prec' (0, 1, 0)|C_2$. Hence no partition of the kind required by Assumption 7 exists.

§Appendix A. Overview of nonstandard models. This appendix is intended only to give examples and an intuitive overview of the concept of nonstandard models of the reals. Those needing a more thorough understanding should read one of the many treatments such as [27, 34].

A.1. Nonstandard models of the reals. A nonstandard model of the reals is an embedding of the real numbers $\mathbb{R}$ into a superset ${}^*\mathbb{R}$ that preserves many of the familiar properties of the reals (e.g., being a linearly ordered algebraic field) while introducing others that are convenient for certain analyses (e.g., "infinite" numbers that obey the usual rules of arithmetic). (For convenience, we take $\mathbb{R}$ to be a nonstandard model of the reals, despite its being standard.)

DEFINITION A.1. *A linearly ordered algebraic field that contains the standard reals $\mathbb{R}$ as a subfield is called a* nonstandard model of the reals. *A nonstandard z such that $|z| < y$ for every positive standard y is called* infinitesimal. *A nonstandard z such that $|z| > y$ for every positive standard y is called* externally infinite. *A nonstandard that is not externally infinite is called* externally finite.

The infinitesimals and standard reals are externally finite, as are hybrid nonstandards such as $1 + x$, where x is infinitesimal.

One common class of nonstandard models of the reals are known as ultraproduct models. They are constructed as follows. Let $\mathcal{Z}$ be an infinite set, let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathcal{Z}$, and let $\mathcal{F}$ be a nonstandard model of the reals (possibly $\mathbb{R}$ itself). The

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elements of $\mathcal{F}^{\mathcal{Z}}$ are functions from $\mathcal{Z}$ to $\mathcal{F}$. Define a binary relation of $\mathcal{F}^{\mathcal{Z}}$ by

$$f \sim_{\mathcal{U}} g \text{ if and only if } \{z \in \mathcal{Z} : f(z) = g(z)\} \in \mathcal{U}.$$

It is easy to see that $\sim_{\mathcal{U}}$ is an equivalence relation because the intersection of finitely many elements of $\mathcal{U}$ is in $\mathcal{U}$. Call ${}^*\mathcal{F}_{\mathcal{U}} = \mathcal{F}^{\mathcal{Z}}/\sim_{\mathcal{U}}$ (the set of equivalence classes corresponding to $\sim_{\mathcal{U}}$) the ultraproduct corresponding to $\mathcal{F}$, $\mathcal{Z}$ and $\mathcal{U}$. The *natural embedding* of $\mathcal{F}$ into ${}^*\mathcal{F}_{\mathcal{U}}$ is $x \mapsto [d_x]_{\mathcal{U}}$, where (for $x \in \mathcal{F}$) $d_x \in \mathcal{F}^{\mathcal{Z}}$ is the constant function $d_x(z) = x$ for all $z \in \mathcal{Z}$. We take the liberty of using the symbol x to stand for $[d_x]_{\mathcal{U}}$ when $x \in \mathcal{F}$. The classic ultraproducts are those that start with $\mathcal{F} = \mathbb{R}$ and $\mathcal{Z} = \mathbb{Z}^+$ (the positive integers). Such models are often denoted ${}^*\mathbb{R}$.

In this paper, each nonstandard model $\mathcal{F}$ of the reals (other than $\mathbb{R}$) will be non-Archimedean in one of the many equivalent senses, such as the following: There exist $x, y \in \mathcal{F}$ with $x < y$ such that $nx < y$ for every standard integer n. For those with more knowledge of nonstandard models, our analysis is entirely external. The formal meaning of “external” is not important here, but it includes the ability to refer to subsets of the standard reals (the familiar $\mathbb{R}$) as subsets of $\mathcal{F}$. The cost of an external analysis includes, among other things, the inability to carry theorems and proofs back and forth between the standard and nonstandard models. The external approach also requires us to distinguish between standard and nonstandard notions of finite, infinite, and countable. The main thing that we gain from the external approach is the non-Archimedean nature of $\mathcal{F}$ as opposed to $\mathbb{R}$. One manifestation of a non-Archimedean property is the non-existence of suprema and/or infima for certain bounded external subsets of $\mathcal{F}$.

In the theorems of Section 2, we assign numerical values to objects in a trading system sequentially. There are two different situations when the number system $\mathcal{F}$ we are using does not have a value that is appropriate for the next object. One situation arises when the next object to be assigned a value is strictly preferred (dispreferred) to every element of a set B and each number in $\mathcal{F}$ is already less (greater) than or equal to the value assigned to an element of B. In this case, we need to expand $\mathcal{F}$ to include values that are larger (smaller) than everything already in $\mathcal{F}$. The second situation arises when the next object needs to be assigned a value strictly between two non-empty sets B_1 and B_2 that already partition $\mathcal{F}$. The following lemma shows how to extend a number system $\mathcal{F}$ in each of those situations.

Lemma A.2. *Let $\mathcal{F}$ be a nonstandard model of the reals.*

1. *There exists a nonstandard model ${}^*\mathcal{F}$ of the reals such that (i) $\mathcal{F}$ is naturally embedded in ${}^*\mathcal{F}$ and (ii) there exist $z_-, z_+ \in {}^*\mathcal{F}$ such that $z_- < x < z_+$ for all $x \in \mathcal{F}$.*
2. *Let $B_1 \subseteq \mathcal{F}$ and $B_2 = B_1^C$. Suppose that, for all $b_1 \in B_1$ and all $b_2 \in B_2$, $b_1 < b_2$. There exists a nonstandard model ${}^*\mathcal{F}$ of the reals such that (i) $\mathcal{F}$ is naturally embedded in ${}^*\mathcal{F}$ and (ii) there exists $z \in {}^*\mathcal{F}$ such that $b_1 < z < b_2$ for all $b_1 \in B_1$ and $b_2 \in B_2$.*

Proof. If $\mathcal{F} = \mathbb{R}$, let $\mathcal{Z} = \mathbb{Z}^+$, the positive integers. If $\mathcal{F}$ is already nonstandard, let $\mathcal{Z} = \mathcal{F}$. Let $\mathcal{U}$ be a nonprincipal ultrafilter that contains all subsets of $\mathcal{Z}$ of the form $\{z : z > z_0\}$ for some $z_0 \in \mathcal{Z}$, and let ${}^*\mathcal{F}$ be the ultraproduct corresponding to $\mathcal{F}$, $\mathcal{Z}$ and $\mathcal{U}$ as defined above. Let $f(z) = z$ define a function in $\mathcal{F}^{\mathcal{Z}}$. The equivalence class $[f]_{\mathcal{U}}$ is greater than every element of $\mathcal{F}$ by construction and $[-f]_{\mathcal{U}}$ is smaller than every element of $\mathcal{F}$. For claim 2, if B_1 has a least upper bound b, (for example, if $\mathcal{F} = \mathbb{R}$) then b is also a greatest lower bound for B_2. In this case $x = b + [1/f]_{\mathcal{U}}$ satisfies claim 2 if $b \in B_1$, and $x = b - [1/f]_{\mathcal{U}}$ satisfies claim 2 if $b \in B_2$.

The only case remaining is claim 2 when B_1 has no least upper bound, which might occur if $\mathcal{F} \neq \mathbb{R}$. In this case, let $\mathcal{Z} = B_2$ and let $\mathcal{U}$ be a nonprincipal ultrafilter that contains all of the sets of the from $\{z \in \mathcal{Z} : z < z_0\}$ for some $z_0 \in \mathcal{Z}$. Let $f(z) = z$ for all $z \in \mathcal{Z}$. The equivalence class $[f]_{\mathcal{U}} > b_1$ for all $b_1 \in B_1$ because every $f(z) > b_1$. Also, $[f]_{\mathcal{U}} < b_2$ for all $b_2 \in B_2$ because $\{z : f(z) < b_2\} \in \mathcal{U}$. $\square$

Some of our results rely on the possibility of applying Lemma A.2 infinitely many times, in a well-ordered manner. To be specific, let Γ be an ordinal. Let $\mathcal{F}_0 = \mathbb{R}$. For each successor $\gamma \leq \Gamma$, let $\mathcal{F}_\gamma$ be either $\mathcal{F}_{\gamma-1}$ or the result of applying Lemma A.2 to $\mathcal{F}_{\gamma-1}$. For each limit $\gamma \leq \Gamma$ (if any), let $\mathcal{F}_{\gamma-} = \bigcup_{\delta<\gamma} \mathcal{F}_\delta$. It is straightforward to show that $\mathcal{F}_{\gamma-}$ is a nonstandard model of the reals when γ is a limit ordinal.

A.2. A note about infinity. The use of the symbol ∞ to stand for "larger than every standard number" has a long history, and rarely causes trouble when discussing standard reals. Certain conventions allow some arithmetic with ∞. For example,

- for all finite x, $\infty + x = \infty$, and
- for all finite, non-zero x, $x\infty$ equals $\pm\infty$, with the sign matching that of x.

However, there is no place for standard infinity in a nonstandard model of the reals. Externally infinite, but internally finite, nonstandards replace standard infinity, and they require no special conventions to allow externally finitary arithmetic. Whenever we need to represent something in a nonstandard model that is larger than every number in the model, we appeal to Lemma A.2 which essentially iterates the ultraproduct construction to produce a larger nonstandard model that contains internally finite numbers to represent what we need.

A.3. Countable additivity. Shimony [39, pp. 19–20] calls the nonexistence of a standard-valued probability on a countable set with equal positive probabilities for each singleton "a serious difficulty" for decision theory. He also notes that allowing probabilities to take "infinitesimal" values might be appropriate. Following that suggestion, Example 18 shows that there is no nonstandard-valued countably additive probability on a countable state space that assigns the same probability to each singleton, but there are finitely-additive probabilities that do so.

EXAMPLE 18. *Let $\Omega = \{\omega_{i,j} : i, j \in \mathbb{Z}^+\}$ be an arbitrary countably infinite set with its elements indexed by the pairs of positive integers in the first quadrant for purposes of this example. Define the following subsets of* Ω:

$$\begin{aligned} A &= \{\omega_{i,j} : j > i\}, \\ V_i &= \{\omega_{i,j} : j \in \mathbb{Z}^+\}, \textit{ for each } i \in \mathbb{Z}^+, \\ H_j &= \{\omega_{i,j} : i \in \mathbb{Z}^+\}, \textit{ for each } j \in \mathbb{Z}^+. \end{aligned}$$

Each V_i is the set of points whose indices form a vertical strip in the first quadrant, and the indices of the points in each H_j form a horizontal strip. Also A is the set of points with indices above the main diagonal in the first quadrant while A^C is the set of points with indices on or below the main diagonal.

Suppose that we want each singleton $\{\omega_{i,j}\} = V_i \cap H_j$ *to have the same probability* ε. *It follows that* ε *must be smaller than every positive standard number and hence must be an infinitesimal. It does not matter from which nonstandard model of the reals the infinitesimal* ε *is extracted. The following analysis applies equally well in all nonstandard models of the reals. What we show next is that no countably additive probabilities can be assigned to all of the sets described above while preserving equal positive probability for all singletons. The example concludes by showing that finitely additive probabilities can be assigned to all of the sets described above while preserving equal positive probability for all singletons.*

Let P *be a probability whose domain contains all of the sets listed above and for which* $P(\{\omega_{i,j}\}) = \varepsilon$ *for all* i, j. *For each* i, j, *let* $v_i = P(V_i)$ *and let* $h_j = P(H_j)$. *We know that for all* i, j, ε/v_i *and* ε/h_j *are strictly less than every positive standard number, e.g.,* 0.1. *Hence* $1 - i\varepsilon/v_i$ *and* $1 - (j-1)\varepsilon/h_j$ *are both greater than* 0.9 *for all* i, j. *Also,*

$$P(A \cap V_i) = v_i - i\varepsilon = v_i(1 - i\varepsilon/v_i) > 0.9v_i = 0.9P(V_i),$$
$$P(A^C \cap H_j) = h_j - (j-1)\varepsilon = h_j[1 - (j-1)\varepsilon/h_j] > 0.9P(H_j),$$

for every $i, j \in \mathbb{Z}^+$. *If* P *were countably additive, then*

$$P(A) = \sum_{i=1}^{\infty} P(A \cap V_i) > 0.9 \sum_{i=1}^{\infty} P(V_i) = 0.9,$$
$$P(A^C) = \sum_{j=1}^{\infty} P(A^C \cap H_j) > 0.9 \sum_{i=1}^{\infty} P(H_j) = 0.9.$$

But $P(A) + P(A^C) = 1$, *hence* P *is* not *countably additive.*

Finally, let $\mathcal{X}$ *be the standard-linear span of all constant functions and the indicators of all of the singletons. Each element of* $\mathcal{X}$ *is constant except at possibly finitely many points. For each constant element* X *of* $\mathcal{X}$, *define* $U(X)$ *to be that constant. For each nonconstant element* X *of* $\mathcal{X}$, *define* $U(X) = c + \sum_{j=1}^{n} \alpha_j \varepsilon$, *where* n *is the (at most finite) number of points where* $X \neq c$, $X(\omega) = \alpha_j - c$ *at the jth such point, and* ε *is the infinitesimal mentioned earlier. Then* U *represents a coherent total trading system,* $\mathcal{T} = (\mathcal{X}, \precsim, \prec)$. *Theorem* 2 *allows us to extend* $\mathcal{T}$ *to a coherent total trading system on whatever larger standard-linear space* $\mathcal{X}'$ *that we wish, including the space of all bounded standard-valued functions defined on* Ω. *The constant* 1 *is a strong numeraire for every field of subsets of* Ω. *Lemma* 7 *shows that, so long as* $\mathcal{X}'$ *is large enough to contain the sets required by Assumption* 6, $P(B) = U'(I_B)$ *is a finitely additive probability on each such field.*

In light of the lack of countably additive probabilities in examples such as Example 18, we do not require that probabilities be countably additive. There are additional reasons as well. It is not even clear how to define the sum of countably many positive nonstandard numbers even if the sequence of partial sums is bounded. For example, in Example 18, the finite partial sums of the numbers $P(\{\omega_{1,j}\})$ for $j = 1, 2, \ldots$ is bounded above by every positive standard but it would be difficult to justify giving that sum a specific value x because $x + \varepsilon$ is also the sum of the same number of ε values. Furthermore, for every upper bound on the partial sums, there is a smaller upper bound, so none of them qualifies to be called the countable sum. See Section 4.3 of [20] for more discussion of this point.

§Appendix B. Expected values for nonstandard-valued functions. In the theory of countably additive probability and expected value for standard-valued functions, each probability P on a measurable space $(\mathcal{Z}, \Gamma)$ has a unique extension from indicators of elements of Γ to an expected value for all bounded measurable functions and for all non-negative measurable functions (where ∞ may be the resulting expected value). Finally, there is a unique extension to all functions whose positive and negative parts don't both have ∞ as their expected value. The first step is the trivial extension to the simple functions, those that assume only finitely many values. This trivial extension applies equally well in the nonstandard-valued and/or the finitely additive cases, namely

$$P\left(\sum_{j=1}^{n} a_j I_{A_j}\right) = \sum_{j=1}^{n} a_j P(A_j).$$

The extension of expected value to a bounded measurable function f is done by a sequence of uniform approximations of f by a sequence of pairs of simple functions $\{(f_{n,<}, f_{n,>}\}_{n=1}^{\infty}$ where $f_{n,<}(z) \leq f_{n,>}(z)$ and $f_{n,>}(z) - 1/n \leq f(z) \leq f_{n,<}(z) + 1/n$ for all $z \in \mathcal{Z}$ and all $n \in \mathbb{Z}^+$. The sequences of expected values $\{P(f_{n,>})\}_{n=1}^{\infty}$ and $\{P(f_{n,>)}\}_{n=1}^{\infty}$ both converge to the same number, and that number is $P(f)$. This feature applies in the finitely additive standard-valued case, but not so much to nonstandard-valued cases. In particular, we can get uniform approximations of bounded functions by simple functions to within each positive standard value, which allows us to pin down the standard part of $P(f)$, but we cannot uniquely determine the value of $P(f)$ from these simple functions alone. To get a uniform approximation to within an infinitesimal amount requires a "simple" function with an externally infinite nonstandard integer number of terms. The sum of countably many nonstandard values is generally not possible to define, and countably additive probabilities are not generally additive over an externally infinite nonstandard number of values.

The standard-valued countably additive extension to a non-negative measurable function f is done by

$$P(f) = \sup_{\text{simple } g \leq f} P(g), \tag{18}$$

which is still possible in the standard-valued finitely additive case. In the nonstandard-valued cases, there are many sets of finite numbers, even bounded sets of finite numbers, for which no supremum (least upper bound) exits, as noted in Appendix A. So, one cannot use (18) to define the expected value of a non-negative nonstandard-valued function. Instead, $P(f)$ must be at least as large as $P(g)$ for every simple $g \leq f$. This makes expected values of unbounded standard-valued functions and general nonstandard-valued functions non-unique extensions of the underlying probabilities, be they countably additive or merely finitely additive. Example 16 in Section 4.9 is a case of a countably additive probability with a finitely additive extension to unbounded functions.

The implications of the non-uniqueness of extensions are handled as follows. In the finitely additive standard-valued case, it is coherent, in the sense of [11] and for a single unbounded non-negative function f, to assign a value to $P(f)$ that equals the right-hand side of (18) plus c for $c > 0$. However, doing so has implications for the expected values of other unbounded non-negative functions. Our extension Theorems 2.8 and 2.9 are set up to take into account all of those implications if and when they arise. The reader should also note that (18) often forces $P(f) = \infty$ because there can be simple functions $g \leq f$ with arbitrarily large $P(g)$. In our nonstandard approach, we would assign a nonstandard externally infinite expected value as $P(f)$. In fact, the very idea of what counts as "bounded" or "unbounded" changes when nonstandard externally infinite numbers are being used.

For the above reasons, we use Definition 4.8 to define expected values of nonstandard-valued functions with respect to a finitely additive (or even a countably-additive) probability P. Essentially, an expected-value functional W with respect to P is a standard-linear mapping $W : \mathcal{W} \to {}^*\mathbb{R}$, where $\mathcal{W}$ is a standard-linear space of (possibly nonstandard-valued) functions that includes the indicators of the sets on which P is defined as well as other functions for which one desires expected values. The functional W needs to have two additional properties: (i) $g \leq f$ implies $W(g) \leq W(f)$ (monotonicity) and (ii) $W(1) = 1$ (normalized). Each finitely additive probability P has multiple extensions to each $\mathcal{W}$ that includes non-simple (even unbounded) functions. Each extension involves the space of functions whose expected values need to be computed as well as the specific expected values assigned to the functions. For each finitely additive probability P and each standard-linear space $\mathcal{W}$ of functions whose expected values we want, there is a convex set $\mathcal{E}_{P,\mathcal{W}}$ of possible extensions of P to $\mathcal{W}$.

Our assumptions refer to an agent's willingness to engage in various trades amongst elements of a set $\mathcal{X}$. If the agent's willingness to trade satisfies our assumptions, then there is a (possibly nonstandard-valued) function U on $\mathcal{X}$ that represents the trades that the agent is willing to make. We then show that $U(X)$ can be interpreted as an expected value of a state-dependent utility of the value of X in state ω with respect to a probability over (Ω, Σ). Unless we impose more restrictions on which trades an agent should be willing to make, i.e., make more restrictive assumptions, the inferred expected-value functionals could be arbitrary elements of $\mathcal{E}_{P,\mathcal{W}}$.

§Appendix C. Lengthy proofs. This appendix contains the lengthier proofs of the results in the main paper.

C.1. Proof of Lemma 2.6. If $\mathcal{X}' = \mathcal{X}$, then $\mathcal{T}' = \mathcal{T}$ satisfies the conclusions of the lemma. For the remainder of the proof, assume that $\mathcal{X}$ is a proper subset of $\mathcal{X}'$.

We start with the case in which it is not required that $\mathcal{T}'$ be coherent. Define $\precsim'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \precsim' Y$ if $Y - X \in \mathcal{V}_{\mathcal{T}}$. Then $\precsim'$ is an extension$_2$ of $\precsim$. If $\ll$ on $\mathcal{X}$ is nonempty, define $\ll'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \ll' Y$ if $0 \ll Y - X$. Since $0 \in \mathcal{V}_{\mathcal{T}}$, Assumption 1 holds. For Assumption 2, suppose that $X, Y, X', Y' \in \mathcal{X}'$ with $Y - X = Y' - X'$. Then $X \precsim' Y$ if and only if $Y' - X' = Y - X \in \mathcal{V}_{\mathcal{T}}$ if and only if $X' \precsim' Y'$. For Assumption 3, suppose that $X_j \precsim' Y_j$ for and $\alpha_j > 0$ is standard for $j = 1, 2$. Then

$$\alpha_1 Y_1 + \alpha_2 Y_2 - (\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 (Y_1 - X_1) + \alpha_2 (Y_2 - X_2) \in \mathcal{V}_{\mathcal{T}},$$

hence $\alpha_1 X_1 + \alpha_2 X_2 \precsim' \alpha_1 Y_1 + \alpha_2 Y_2$. For Assumption 4 on $\mathcal{X}'$, only the final two bullets need to be proven. To that end, $(X \ll' Y) \wedge (Y \precsim' Z)$ implies $[0 \ll (Y - X)] \wedge [0 \precsim (Z - Y)]$, and $(X \precsim' Y) \wedge (Y \ll' Z)$ implies $[0 \precsim (Y - X)] \wedge [0 \ll (Z - Y)]$. Each of the last two implies $0 \ll (Z - X)$, hence $X \ll' Z$.

If $\mathcal{T}$ is coherent and it is required that $\mathcal{T}'$ be coherent, define $\precsim'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \precsim' Y$ if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \leq Y - X$. Then $\precsim'$ extends$_2$ $\precsim$ and

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$\mathcal{V}_{\mathcal{T}'}$ satisfies the final claim of the lemma. Define $\ll'$ on $\mathcal{X}'$ as follows: For $X, Y \in \mathcal{X}'$, $X \ll' Y$ if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that either $V \prec_{\text{Dom}} Y - X$ or $0 \ll V \leq Y - X$. This makes $\mathcal{T}'$ satisfy Assumption 5. Since $0 \in \mathcal{V}_{\mathcal{T}}$ and $0 \leq X - X$, Assumption 1 holds. For Assumption 2, suppose that $X, Y, X', Y' \in \mathcal{X}'$ with $Y - X = Y' - X'$. Then $X \precsim' Y$ if and only if there is $V \in \mathcal{V}_{\mathcal{T}}$ such that $V \leq Y - X = Y' - X'$ if and only if $X' \precsim' Y'$. For Assumption 3, suppose that $X_j \precsim' Y_j$ and $\alpha_j > 0$ is standard for $j = 1, 2$. For $j = 1, 2$, let $V_j \in \mathcal{V}_{\mathcal{T}}$ be such that $V_j \leq Y_j - X_j$. Then $V = \alpha_1 V_1 + \alpha_2 V_2 \in \mathcal{V}_{\mathcal{T}}$, and

$$V \leq \alpha_1 Y_1 + \alpha_2 Y_2 - (\alpha_1 X_1 + \alpha_2 X_2).$$

hence $\alpha_1 X_1 + \alpha_2 X_2 \precsim' \alpha_1 Y_1 + \alpha_2 Y_2$. For Assumption 4 on $\mathcal{X}'$, the first bullet is immediate from the definition of $\lhd'$. For the last two bullets,

$$(X \ll' Y) \wedge (Y \precsim' Z) \text{ implies } X \ll' Z, \text{ and}$$
$$(X \precsim' Y) \wedge (Y \ll' Z) \text{ implies } X \ll' Z,$$

there are several things that could lead to the left-hand clauses:

(i) there is $V_1 \in \mathcal{V}_{\mathcal{T}}$ such that $V_1 \prec_{\text{Dom}} Y - X$,
(ii) there is $V_2 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_2 \leq Y - X$,
(iii) there is $V_3 \in \mathcal{V}_{\mathcal{T}}$ such that $V_3 \leq Z - Y$,
(iv) there is $V_4 \in \mathcal{V}_{\mathcal{T}}$ such that $V_4 \leq Y - X$,
(v) there is $V_5 \in \mathcal{V}_{\mathcal{T}}$ such that $V_5 \prec_{\text{Dom}} Z - Y$,
(vi) there is $V_6 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_6 \leq Z - Y$.

Similarly, there are two ways to achieve the right-hand clause(s):

(vii) there is $V_7 \in \mathcal{V}_{\mathcal{T}}$ such that $V_7 \prec_{\text{Dom}} Z - X$,
(viii) there is $V_8 \in \mathcal{V}_{\mathcal{T}}$ such that $0 \ll V_8 \leq Z - X$.

We need to prove two implications based on the above possibilities:

1 [{(i) or (ii)} and (iii)] implies [(vii) or (viii)], and
2 [(iv) and {(v) or (vi)}] implies [(vii) or (viii)].

For 1, [(i) and (iii)] implies $V_1 + V_3 \prec_{\text{Dom}} Z - Y + Y - X = Z - X$, which implies (vii) with $V_7 = V_1 + V_3$. Alternatively, [(ii) and (iii)] implies $0 \ll V_2 + V_3 \leq Y - X + Z - Y = Z - X$, which implies (vii) with $V_8 = V_2 + V_3$. For 2, (iv) and (v) implies $V_4 + V_5 \prec_{\text{Dom}} Y - X + Z - Y = Z - X$, which implies (vii) with $V_7 = V_4 + V_5$. Alternatively (iv) and (vi) implies $0 \ll V_4 + V_6 \leq Z - Y + Y - X = Z - X$, which implies (viii) with $V_8 = V_4 + V_6$.

C.2. Lemma C.1 and its proof. The proofs of Theorems 2.8 and 2.9 are transfinite inductions. Lemma C.1 is a template for the successor ordinal steps in the transfinite inductions. Lemma 2.7 (whose proof is in Appendix C.3) is the remainder of the transfinite induction, including the limit ordinal steps.

The proof of Lemma C.1 uses an argument that resembles the proof of de Finetti's fundamental theorem of prevision. The main step is constructing bounds for the possible values of the agreeing function (prevision in de Finetti's case, U in Theorem 2.8) at a new object Z given previously chosen values of the agreeing function. In de Finetti's theorem, one uses existing previsions of random variables X for which either $X \leq Z$ or $Z \leq X$. In Theorem 2.8, we replace prevision by an agreeing function U, and

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we replace $X \leq Z$ by a combination of $X \precsim Z$, $X \ll Z$, and/or $X \prec_{\text{Dom}} Z$. Additional steps are needed to deal with strict preferences of a non-Archimedean nature and with sets of nonstandards that don't have suprema and/or infima.

LEMMA C.1. *Assume the following structure*:

- $\mathcal{Y}$ *and* $\mathcal{W}$ *are linear spaces of functions from* Ω *to* $\mathcal{O}$ *with* $\mathcal{Y}$ *a proper subset of* $\mathcal{W}$.
- $\mathcal{T}_Y = (\mathcal{Y}, \precsim_{\mathcal{Y}}, \prec_{\mathcal{Y}})$ *is a total trading system that is represented by the standard-linear function* $U : \mathcal{Y} \to^{*} \mathbb{R}$, *where* $^{*}\mathbb{R}$ *is a nonstandard model of the reals.*
- $\mathcal{T}_{\mathcal{W}} = (\mathcal{W}, \precsim_{\mathcal{W}}, \ll_{\mathcal{W}})$ *is the extension*$_2$ *of* $\mathcal{T}_Y$ *obtained from Lemma 2.6.*

Let $Z \in \mathcal{W}$. *Let* $\mathcal{Z}$ *be the standard-linear span of* $\mathcal{Y} \bigcup \{Z\}$. *Then* U *can be extended to a standard-linear function* $U' : \mathcal{Z} \to^{*} \mathbb{R}'$, *where* $^{*}\mathbb{R}'$ *contains* $^{*}\mathbb{R}$ *and such that* U' *represents a total trading system* $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ *that is an extension*$_2$ *of* $\mathcal{T}_Y$. *Also, if* $\mathcal{T}_{\mathcal{Y}}$ *is coherent, then* $\mathcal{T}'$ *can be chosen to be coherent.*

Proof. If coherence is an issue, note that dominance has the same type (uniform, strict, or weak) in both $\mathcal{Y}$ and $\mathcal{W}$, so we will use the same notation $X \prec_{\text{Dom}} Y$ to mean that Y dominates X regardless of whether X, Y are both in $\mathcal{Y}$, both in $\mathcal{W}$ or one in each. We have

$$\mathcal{Z} \setminus \mathcal{Y} = \{\alpha Z + X : X \in \mathcal{Y}, \alpha \in \mathbb{R} \setminus \{0\}\}. \tag{19}$$

It is straightforward to show that the representation of elements of $\mathcal{Z} \setminus \mathcal{Y}$ in (19) is unique. Define $U'(X) = U(X)$ for $X \in \mathcal{Y}$.

Start with the case in which there is $Y \in \mathcal{Y}$ such that $Y \sim_{\mathcal{W}} Z$. In this case, set $U'(Z) = U(Y)$, $^{*}\mathbb{R}' =^{*} \mathbb{R}$, and

$$U'(\alpha Z + X) = \alpha U'(Z) + U(X), \tag{20}$$

for all other elements of $\mathcal{Z}$. Set $\precsim'$ to be the total preorder on $\mathcal{Z}$ that U' represents. The only thing that remains to show, in this case, is that $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ is coherent if $\mathcal{T}_{\mathcal{W}}$ is coherent. Suppose that $\alpha Z + W \prec_{\text{Dom}} \alpha' Z + Y$, for α, α' standard and $W, Y \in \mathcal{Y}$. If $\alpha = \alpha'$, then $W \prec_{\text{Dom}} Y$ and $W \ll_{\mathcal{Y}} Y$, so $W \prec_{\mathcal{Y}}$, $W \prec' Y$, and $\alpha Z + W \prec' \alpha Z + Y$. If $\alpha > \alpha'$, then $Z \prec_{\text{Dom}} (Y - W)/(\alpha - \alpha')$ and $Z \prec_{\mathcal{W}} (Y - W)/(\alpha - \alpha')$. Let $X \sim_{\mathcal{W}} Z$. Then $X \prec_{\mathcal{W}} (Y - W)/(\alpha - \alpha')$, $X \prec_{\mathcal{Y}} (Y - W)/(\alpha - \alpha')$, $X \prec' (Y - W)/(\alpha - \alpha')$, and $Z \prec' (Y - W)/(\alpha - \alpha')$. Hence $\alpha Z + W \prec' \alpha Z + Y$. A similar argument works of $\alpha < \alpha'$.

For the remainder of the proof, assume that for all $X \in \mathcal{Y}$, $\neg(X \sim Z)$. We start by choosing a value for $U'(Z)$. After that, we make U' standard-linear by defining it through (20). Then, we show that setting $\ll'$ to $\prec'$ satisfies Assumption 4. Finally, we prove that the trading system $\mathcal{T}'$ that U' represents (recall Lemma 2.4) is coherent if $\mathcal{T}_{\mathcal{Y}}$ is coherent. Since U' extends U, $\mathcal{T}'$ extends $\mathcal{T}_Y$.

When we attempt to choose a value for $U'(Z)$, we need to attend to instances of $\ll_{\mathcal{W}}$, if any.

$$\mathcal{L}_1 = \{U(X) : X \in \mathcal{Y}, X \precsim_{\mathcal{W}} Z\},$$
$$\mathcal{U}_1 = \{U(X) : X \in \mathcal{Y}, Z \precsim_{\mathcal{W}} X\},$$
$$\mathcal{L}_2 = \{U(X) : X \in \mathcal{Y}, X \ll_{\mathcal{W}} Z\},$$
$$\mathcal{U}_2 = \{U(X) : X \in \mathcal{Y}, Z \ll_{\mathcal{W}} X\}.$$

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The definition of $\ll_{\mathcal{W}}$ and the fact that U represents $\mathcal{T}_Y$ guarantee that, for $j = 1, 2$, $\ell < u$ for all $\ell \in \mathcal{L}_j$ and $u \in \mathcal{U}_j$. Also, $\mathcal{U}_2 \subseteq \mathcal{U}_1$ and $\mathcal{L}_2 \subseteq \mathcal{L}_1$. (If $\ll_{\mathcal{W}}$ is empty, then $\mathcal{L}_2 = \mathcal{U}_2 = \emptyset$.)

There are several cases (and subcases) to handle:

- (a) Both $\mathcal{L}_1$ and $\mathcal{U}_1$ are nonempty, and
  - (a)(i) there is $x \in {}^*\mathbb{R}$ such that $\ell \leq x \leq u$ for all $\ell \in \mathcal{L}_1$ and $u \in \mathcal{U}_1$, and at least one such x satisfies $x \notin \mathcal{L}_2 \bigcup \mathcal{U}_2$, or
  - (a)(ii) there is no x as described in case (a)(i).
- (b) $\mathcal{L}_1$ is empty, $\mathcal{U}_1$ is nonempty, and
  - (b)(i) there is $x \in {}^*\mathbb{R}$ such that $x \leq u$ for all $u \in \mathcal{U}_1$, and at least one such x satisfies $x \notin \mathcal{U}_2$, or
  - (b)(ii) there is no x as described in case (b)(i).
- (c) $\mathcal{U}_1$ is empty, $\mathcal{L}_1$ is nonempty, and
  - (c)(i) there is $x \in {}^*\mathbb{R}$ such that $\ell \leq x$ for all $\ell \in \mathcal{L}_1$, and at least one such x satisfies $x \notin \mathcal{L}_2$, or
  - (c)(ii) there is no x as described in case (c)(i).
- (d) Both $\mathcal{L}_1$ and $\mathcal{U}_1$ are empty.

In cases (a)(i), (b)(i), and (c)(i) set $U'(Z) = x$ and ${}^*\mathbb{R}' = {}^*\mathbb{R}$. In case (a)(ii), apply claim 2 of Lemma A.2 (in Appendix A) to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $\ell < x < u$ for all $\ell \in \mathcal{L}_1$ and all $u \in \mathcal{U}_1$, and set $U'(Z) = x$.

In case (b)(ii), apply claim 1 of Lemma A.2 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $x < u$ for all $u \in \mathcal{U}_1$, and set $U'(Z) = x$.

In case (c)(ii), apply claim 1 of Lemma 1.2 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and $x \in {}^*\mathbb{R}'$ such that $x > \ell$ for all $\ell \in \mathcal{L}_1$, and set $U'(Z) = x$.

In case (d), we have two choices. One choice is to let $U'(Z) \in {}^*\mathbb{R}$ and set ${}^*\mathbb{R}' = {}^*\mathbb{R}$. The other choice is to apply either claim of Lemma A.2 to find an extension ${}^*\mathbb{R}'$ of ${}^*\mathbb{R}$ and let $x \in {}^*\mathbb{R}'$.

By construction, we have that U' extends U from $\mathcal{Y}$ to $\mathcal{Z}$.

We define $\mathcal{T}' = (\mathcal{Z}, \precsim', \prec')$ by saying that

$$X \precsim' Y \text{ if and only if } U'(X) \leq U'(Y).$$

It follows that $\precsim'$ extends$_2$ $\precsim_{\mathcal{Y}}$ from $\mathcal{Y}$ to $\mathcal{Z}$. If $\ll_{\mathcal{W}}$ is empty, the proof is over.

For the remainder of the proof, assume that $\ll_{\mathcal{W}}$ is not empty. We must show that, if $X, Y \in \mathcal{Z}$ and $X \ll_{\mathcal{W}} Y$, then $X \prec' Y$. This involves comparing $U'(X)$ to $U'(Y)$ for various $X, Y \in \mathcal{Z}$. Represent such X and Y as in (19) by

$$X = q(X)Z + X',$$
$$Y = q(Y)Z + Y',$$

with $X', Y' \in \mathcal{Y}$ and $q(X), q(Y) \in \mathbb{R}$. Then

$$Y - X = [q(Y) - q(X)]Z + Y' - X', \tag{21}$$

$$U'(Y) - U'(X) = [q(Y) - q(X)]x + U(Y' - X'), \tag{22}$$

where $x = U'(Z)$. If $q(X) = q(Y)$, then $Y - X = Y' - X'$, $X' \prec_{\mathcal{Y}} Y'$, and $U'(Y) - U'(X) = U(Y' - X') > 0$, so that $X \prec' Y$. If $q(X) < q(Y)$, then

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$$\frac{Y' - X'}{q(X) - q(Y)} \ll_{\mathcal{W}} Z,$$

and $x = U'(Z) > [U(X' - Y')]/[q(X) - q(Y)]$ by construction. It follows from (22) that $U'(X) < U'(Y)$, as needed. If $q(X) > q(Y)$, a similar argument shows that $U'(X) < U'(Y)$, so $\mathcal{T}'$ preserves instances of $X \ll_{\mathcal{W}} Y$.

Finally, assume that $\mathcal{T}_Y$ is coherent. It follows from the previous paragraph, that U' respects dominance. We complete the proof by showing that U' is monotone. Suppose that $X, Y \in \mathcal{Z}$ with $X \leq Y$. If $q(X) = q(Y)$, then (21) yields $0 \leq Y' - X' = Y - X$ and

$$U'(Y) - U'(X) = U(Y' - X') \geq 0.$$

If $q(X) > q(Y)$, then

$$\frac{X' - Y'}{q(Y) - q(X)} \leq Z,$$

and $x = U'(Z) \geq [U(X' - Y')]/[q(Y) - q(X)]$ by construction. It follows from (22) that

$$U'(Y) - U'(X) \geq U(X' - Y') + U(Y' - X') = 0.$$

If $q(X) < q(Y)$, a similar argument shows that $U'(Y) \geq U'(X)$, so U' is monotone. □

C.3. Proof of Lemma 2.7. The proof proceeds by transfinite induction on $\mathcal{W} \setminus \mathcal{Y}$. Let Λ be an ordinal, and let $\{X_\lambda\}_{0<\lambda<\Lambda}$ be a well-ordering of the elements of $\mathcal{W} \setminus \mathcal{Y}$. Let $\mathcal{X}_0 = \mathcal{Y}$, $\mathcal{T}_0 = \mathcal{T}_\mathcal{Y}$, ${}^*\mathbb{R}_0 = {}^*\mathbb{R}$, and $U_0 = U$. Then the following induction hypothesis holds for $\lambda = 0$:

Induction hypothesis: Let $\lambda < \Lambda$ be an ordinal. There is a total trading system $\mathcal{T}_\lambda = (\mathcal{X}_\lambda, \precsim_\lambda)$ such that:

- $\mathcal{X}_\lambda$ contains $\{X_\gamma\}_{\gamma \leq \lambda}$,
- $\precsim_\lambda$ is a total preorder and is an extension$_2$ of $\precsim_\mathcal{Y}$ to $\mathcal{X}_\lambda$,
- $\mathcal{T}_\lambda$ is represented by a standard-linear function $U_\lambda : \mathcal{X}_\lambda \to {}^*\mathbb{R}_\lambda$, where ${}^*\mathbb{R}_\lambda$ is a nonstandard model of the reals that contains ${}^*\mathbb{R}_\gamma$ for each $\gamma < \lambda$, and
- $\mathcal{T}_\lambda$ is coherent if $\mathcal{T}_\mathcal{Y}$ is coherent.

Next, we deal with an arbitrary successor ordinal γ. Assume that the induction hypothesis holds for $\lambda = \gamma - 1$. We must prove that the induction hypothesis holds for $\lambda = \gamma$. Apply Lemma C.1 with $\mathcal{Y} = \mathcal{X}_{\gamma-1}$, $Z = X_\gamma$, $U = U_{\gamma-1}$, $\precsim_\mathcal{Y} = \precsim_{\gamma-1}$, and ${}^*\mathbb{R} = {}^*\mathbb{R}_{\gamma-1}$. Then $\mathcal{X}_\gamma$ is the $\mathcal{Z}$ in Lemma C.1. Let U_γ and ${}^*\mathbb{R}_\gamma$ be, respectively, the U' and ${}^*\mathbb{R}'$ that result from Lemma C.1. Then, the induction hypothesis holds for $\lambda = \gamma$.

Finally, we prove that the induction hypothesis holds for each limit ordinal λ. We start by creating objects to play the roles of $\mathcal{Y}$, U, $\precsim_\mathcal{Y}$, and ${}^*\mathbb{R}$ in the statement of Lemma C.1. Define

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$$^*\mathbb{R}_{<\lambda} = \bigcup_{\gamma<\lambda} {}^*\mathbb{R}_\gamma,$$
$$\mathcal{X}_{<\lambda} = \bigcup_{\gamma<\lambda} \mathcal{X}_\gamma.$$

Clearly, $^*\mathbb{R}_{<\lambda}$ is a nonstandard model of the reals that contains $^*\mathbb{R}_\gamma$ for all $\gamma < \lambda$. For $X \in \mathcal{X}_{<\lambda}$, let $U_{<\lambda}(X) = U_\gamma(X)$, where γ is the first ordinal such that $X \in \mathcal{X}_\gamma$. Each such γ is strictly less than λ. This makes $U_{<\lambda} : \mathcal{X}_{<\lambda} \to {}^*\mathbb{R}_{<\lambda}$. Define $\precsim_{<\lambda}$ on $\mathcal{X}_{<\lambda}$ by $X \precsim_{<\lambda} Y$ if $X \precsim_\gamma Y$ for γ being the first ordinal such that both $X, Y \in \mathcal{X}_\gamma$. Then $\gamma < \lambda$ and $U_{<\lambda}$ represents $\precsim_{<\gamma}$ on $\mathcal{X}_{<\lambda}$. To see that $U_{<\lambda}$ is standard-linear, let $X^1, X^2 \in \mathcal{X}_{<\lambda}$. Let γ be the first ordinal for which both X^1 and X^2 are in $\mathcal{X}_\gamma$. Then $\gamma < \lambda$ and $U_{<\lambda}(X^j) = U_\gamma(X^j)$ for $j = 1, 2$. Since U_γ is standard-linear,

$$\begin{aligned} U_{<\lambda}(\alpha X^1 + \beta X^2) &= U_\gamma(\alpha X^1 + \beta X^2) \\ &= \alpha U_\gamma(X^1) + \beta U_\gamma(X^2) \\ &= \alpha U_{<\lambda}(X^1) + \beta U_{<\lambda}(X^2), \end{aligned}$$

so $U_{<\lambda}$ is standard-linear.

To complete the proof, apply Lemma C.1 with $\mathcal{Y} = \mathcal{X}_{<\lambda}$, $^*\mathbb{R} = {}^*\mathbb{R}_{<\lambda}$, $Z = X_\lambda$, and $U = U_{<\lambda}$.

C.4. Proof of Lemma 3.2. Note that each function $f \in \mathcal{F}$ is a special case of a horse-lottery h for which each lottery $h(\omega)$ puts probability 1 on a single prize (consequence) $f(\omega)$. In this way, we can think of $\mathcal{F}$ as a subset of a set of horse lotteries. Savage [35] proves that there is a probability P on Ω and a utility $U : \mathcal{P} \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$, $f \precsim' g$ if and only if $P[U(f(\cdot))] \leq P[U(g(\cdot))]$. Let $\mathcal{H}$ be the set of finite mixtures of elements of $\mathcal{F}$, and let $\mathcal{P}'$ be the set of finite mixtures of elements of $\mathcal{P}$. Define U' on $\mathcal{P}'$ by $U'\left(\sum_{j=1}^n \alpha_j p_j\right) = \sum_{j=1}^n \alpha_j U(p_j)$.

Next, we show that U' is well defined. Suppose that

$$\sum_{j=1}^n \alpha_j p_j = \sum_{k=1}^m \alpha'_k p'_k, \tag{23}$$

with all α_j and all α'_k strictly positive. A necessary condition for (23) is that the set of distinct p_j be the same as the set of distinct p'_k. Another necessary condition is that, if $p_j = p'_k$, the sums of the α_j and/or α'_k corresponding to repeated values of p_j and/or p'_k must be equal. This makes U' well defined. Hence, $U'[h(\omega)]$ is well defined for every ω and every $h \in \mathcal{H}$. It follows that $V(h) = P(U'[h(\cdot)])$ is well defined, and can be used to represent a preorder $\precsim^*$ on $\mathcal{H}$ by "$h \precsim^* g$ if and only if $V(h) \leq V(g)$." It is straightforward that $V(\alpha h + [1-\alpha]g) = \alpha V(h) + (1-\alpha)V(g)$ for all $\alpha \in [0,1]$ and all $h, g \in \mathcal{H}$. There is a corresponding $\mathcal{K}_0 = \{\alpha(h-g) : h, g \in \mathcal{H}\}$, and $U^\dagger(\alpha[h-g]) = \alpha[V(h) - V(g)]$ is also well defined on $\mathcal{K}_0$.

Next, we show that $U^\dagger$ is standard-linear. Let $k_j = \alpha_j(h_j - g_j)$ for $j = 1, 2$. Then

$$\begin{aligned} \beta_1 k_1 + \beta_2 k_2 &= \beta_1\alpha_1(h_1 - g_1) + \beta_2\alpha_2(h_2 - g_2) \\ &= \beta_1\alpha_1 h_1 + \beta_2\alpha_2 h_2 - \beta_1\alpha_1 g_1 - \beta_2\alpha_2 g_2 \\ &= \gamma(J_1 - J_2), \end{aligned}$$

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where γ, J_1, J_2 depend on the signs of $\alpha_1, \beta_1, \alpha_2, \beta_2$. For example, if both $\alpha_j > 0$ and both $\beta_j < 0$,

$$\gamma = \beta_1\alpha_1 + \beta_2\alpha_2,$$
$$J_1 = \frac{\beta_1\alpha_1}{\gamma}h_1 + \frac{\beta_2\alpha_2}{\gamma}h_2,$$
$$J_2 = \frac{\beta_1\alpha_1}{\gamma}g_1 + \frac{\beta_2\alpha_2}{\gamma}g_2.$$

Each of $J_1, J_2 \in \mathcal{K}_0$, so

$$\begin{aligned} U^\dagger(\beta_1 k_1 + \beta_2 k_2) &= \gamma[V(J_1) - V(J_2)] \\ &= \beta_1\alpha_1 V(h_1) + \beta_2\alpha_2 V(h_2) - \beta_1\alpha_1 V(g_1) - \beta_2\alpha_2 V(g_2) \\ &= \beta_1 U^\dagger(k_1) + \beta_2 U^\dagger(k_2). \end{aligned}$$

There are 15 other combinations of signs that produce various formulae for γ, J_1, J_2, but all of them lead to the same conclusion. Lemma 2.4 says that $U^\dagger$ represents a total trading system $\mathcal{T}$ that satisfies Assumptions 1–4.

Finally, we show that $\mathcal{T} = (\mathcal{K}_0, \precsim, \prec)$ as described in Section 3.2.2. According to the discussion in Section 3.2.2, $\precsim$ corresponds to $\precsim^*$ as follows. Let $k_1, k_2 \in \mathcal{K}_0$ be expressed as $k_j = \alpha_j(h_j - g_j)$ with $\alpha_j > 0$ and $h_j, g_j \in \mathcal{H}$ for $j = 1, 2$. First, we need to express $k_2 - k_1 = \gamma(s_2 - s_1)$ with $\gamma > 0$ and $s_1, s_2 \in \mathcal{H}$. Then, we need to show that $k_1 \precsim k_2$ if and only if $s_1 \precsim^* s_2$. First, note that

$$\begin{aligned} k_2 - k_1 &= \alpha_1 h_1 + \alpha_2 g_2 - [\alpha_2 g_1 + \alpha_1 h_2] \\ &= \gamma(s_2 - s_1), \end{aligned}$$

where $\gamma = \alpha_1 + \alpha_2$, and

$$s_1 = \beta h_1 + (1-\beta)g_2,$$
$$s_2 = \beta h_2 + (1-\beta)g_1,$$

where $\beta = \alpha_1/\gamma$. Next, note that $k_1 \precsim k_2$ if and only if $U^\dagger(k_1) \le U^\dagger(k_2)$, which is true if and only if

$$\alpha_1[V(h_1) - V(g_1)] \le \alpha_2[V(h_2) - V(g_2)],$$

which is true if and only if

$$V(\beta h_1 + [1-\beta]g_2) \le V([1-\beta]h_2 + \beta g_1),$$

which is true if and only if $s_1 \precsim^* s_2$.

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DEPARTMENT OF STATISTICS AND DATA SCIENCE
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PA 15213, USA
E-mail: mark@cmu.edu

DEPARTMENT OF PHILOSOPHY AND DEPARTMENT OF STATISTICS AND DATA SCIENCE
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PA 15213, USA
E-mail: teddy@stat.cmu.edu

DEPARTMENT OF STATISTICS AND DATA SCIENCE
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PA 15213, USA
E-mail: kadane@stat.cmu.edu

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
PISCATAWAY, NJ 08854, USA
E-mail: ruobin.gong@rutgers.edu

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO
SÃO PAULO, SP, BRAZIL
E-mail: rbstern@gmail.com