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MATHEMATICAL NOTES

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THE TOEPLITZ–HAUSDORFF THEOREM EXPLAINED

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Paul Halmos expressed [3, p. 110] the general dissatisfaction with the usual proofs of this famous and important theorem. They all make it seem like an accidental product of a computation. A more conceptual proof was devised by N. P. Dekker(1). In spite of the elegance of his proof, the one offered below may have some claim to be regarded as "the reason the theorem is true". This proof was used to prove a stronger theorem in a previous paper [1, Theorem 10.1]; my reason for publishing it now separately is the wide interest in the Toeplitz–Hausdorff theorem. (The proof here was also found independently by W. M. Kahan.)

Let $\mathscr{H}$ be a complex Hilbert space, with elements $x, y, \ldots$. For $x \in \mathscr{H}$, let x^* denote the corresponding linear functional; thus x^*y means the inner product of x with y, but xy^* is an operator on $\mathscr{H}$, whose value at z is $xy^*z=(y^*z)x$. The numerical range of any operator A is the set of complex numbers

$$\{x^*Ax : x^*x = 1\}.$$

The Toeplitz–Hausdorff theorem asserts that *the numerical range is convex.*

Most proofs (but not Dekker's) begin by observing that the problem reduces to its two-dimensional special case. For if x^*Ax and y^*Ay are two of the numbers in the numerical range ($x^*x=y^*y=1$), and if λ is a number on the segment joining them, then we must be able to show that there is a unit vector z in the span of x and y such that $z^*Az=\lambda$, and to do so will complete the proof.

Accordingly, let us assume without loss of generality that $\dim H=2$.

The numbers x^*Ax under consideration may be written as $x^*Ax=\operatorname{tr}(Axx^*)$. This is the key to the proof. Consider the mapping Φ which takes the arbitrary hermitian operator H on $\mathscr{H}$ to

$$\Phi(H) = \operatorname{tr}(AH).$$

(1) I thank Professor Halmos for communicating Dekker's proof to me. I have not yet seen the original [2].

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It is plainly real-linear. Its domain is a space of four real dimensions: if you like, the space $\mathscr{M}$ of matrices

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (a = \bar{a},\, b = \bar{c},\, d = \bar{d}).$$

The range of Φ is a space of two real dimensions: namely, the complex numbers. The conclusion which is to be proved is that Φ takes the set of one-dimensional orthoprojectors xx^* onto a convex set.

In the matrix representation of $\mathscr{M}$, these orthoprojectors may be parametrized as follows. It is enough to consider

$$x = \begin{pmatrix} \cos\theta \\ e^{i\delta}\sin\theta \end{pmatrix} \qquad (\theta,\, \delta \text{ real})$$

because any other unit vector is a scalar multiple of one of these. For such x,

$$xx^* = \begin{pmatrix} \cos^2\theta & e^{-i\delta}\cos\theta\sin\theta \\ e^{i\delta}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

These matrices comprise a 2-sphere centred at $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and lying in a 3-flat in $\mathscr{M}$ (to wit, the set of H having trace 1).

But the image of a 2-sphere, under a linear map with range in real 2-space, is either an ellipse with interior, or a segment, or a point—in any case, convex. The proof is complete.

This argument also yields the following natural generalization of the Toeplitz–Hausdorff theorem:

THEOREM. *Let $A_1, \ldots, A_n$ be operators on $\mathscr{H}$. Then the subset*

$$\{(x^*A_1x, \ldots, x^*A_nx) : x^*x = 1\}$$

of $\mathbf{C}^n$ has the following property: with any two points, it contains an ellipsoid (perhaps degenerate) joining them.

(When $n=1$, the ellipsoid lies in $\mathbf{C}$, so it *must* degenerate and hence be convex: this gives the former result.)

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