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CENTRALIZING AUTOMORPHISMS OF LIE IDEALS IN PRIME RINGS

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ABSTRACT. Let R be a prime ring of characteristic not equal to two and let T be an automorphism of R. If U is a Lie ideal of R such that T is nontrivial on U and $xx^T - x^T x$ is in the center of R for every x in U, then U is contained in the center of R.

A linear mapping T from a ring to itself is called *centralizing* on a subset S of the ring if $xx^T - x^T x$ is in the center of the ring for every x in S. In [7] Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [5] and [8] the same result is proved for a prime ring with a nontrivial centralizing automorphism. A number of authors have generalized these results by considering mappings which are only assumed to be centralizing on an appropriate ideal of the ring.

In [1] Awtar considered centralizing derivations on Lie and Jordan ideals. In the Jordan case, he proved that if a prime ring of characteristic not two has a nontrivial derivation which is centralizing on a Jordan ideal, then the ideal must be contained in the center of the ring. This result is extended in [6] where it is shown that if R is any prime ring with a nontrivial centralizing automorphism or derivation on a nonzero ideal or (quadratic) Jordan ideal, then R is commutative. Recently Bell and Martindale [2] have proved similar results assuming that the ring is only semi-prime.

For prime rings Awtar also showed that a nontrivial derivation which is centralizing on a Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two or three. In [4] Lee and Lee obtained the same result while removing the characteristic not three restriction. In this paper the corresponding result for automorphisms on Lie ideals is proved.

THEOREM. *If R is a prime ring of characteristic not equal to two and T is an automorphism of R which is centralizing and nontrivial on a Lie ideal U of R, then U is contained in the center of R.*

From now on assume that R is a prime ring of characteristic not equal to two with center Z. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. Let $[x, y] = xy - yx$ and note the following basic identities valid in any associative ring :

(a) $[x, yz] = y[x, z] + [x, y]z$

(b) $[xy, z] = x[y, z] + [x, z]y$

(c) $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$

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The first two identities show that the commutator $[x,y]$ acts as a derivation on R if either of its arguments is fixed. The third identity is the Jacobi identity. Note also that whenever $a[x,r]=0$ for all r in R, then $0=a[x,rs]=ar[x,s]+a[x,r]s=ar[x,s]$ for all r and s in R. Since R is prime, either $a=0$ or x is the center of R.

Now assume that U is a Lie ideal of R and T is a homomorphism of R to R such that $[x,x^T]$ is in Z for every x in U. Linearizing this gives $[x,y^T]+[y,x^T]$ is in Z for every x and y in U. Since U is a Lie ideal, y can be replaced by $[x,r]$ with r in R resulting in $\big[x,[x,r]^T\big]+\big[[x,r],x^T\big]$ is in Z. Using the Jacobi identity and the fact that $[x,x^T]$ is in Z, $\big[x,[x,r]^T\big]+\big[[x,r],x^T\big]=\big[x,[x^T,r^T]\big]+\big[[x,r],x^T\big]=\big[x,[x^T,r^T]\big]-\big[x,[x^T,r]\big]$. Thus

(1) $$\big[x,[x^T,r-r^T]\big] \text{ is in } Z \text{ for all } x \text{ in } U \text{ and } r \text{ in } R.$$

If a mapping T satisfies $[x,x^T]=0$ for all x in some subset S of R, then T is called *commuting* on S.

LEMMA 1. *If T is an automorphism of R which is centralizing on U, then T is commuting on U.*

PROOF. Let r be replaced by xx^Tx in (1). Then using the fact that $[x,x^T]$ is in Z and thus $[x^T,x^{TT}]$ is also in Z, $\big[x,[x^T,xx^Tx]\big]-\big[x,[x^T,x^Tx^{TT}x^T]\big]=\big[x,x[x^T,x^Tx]+[x^T,x]x^Tx\big]-\big[x,x^T[x^T,x^{TT}x^T]\big]=\big[x,xx^T[x^T,x]+[x^T,x]x^Tx\big]-\big[x,x^Tx^T[x^T,x^{TT}]\big]=[x,xx^T][x^T,x]+[x^T,x][x,x^Tx]-[x,x^Tx^T][x^T,x^{TT}]=x[x,x^T][x^T,x]+[x^T,x][x,x^T]x-x^T[x,x^T][x^T,x^{TT}]-[x,x^T]x^T[x^T,x^{TT}]=2x[x,x^T][x^T,x]-2x^T[x,x^T][x^T,x^{TT}]$ is in Z for all x in U. Commuting this last expression with x^T gives $2[x,x^T][x,x^T][x^T,x]=0$. Since R is prime and all commutators in the product are in Z, $[x,x^T]=0$. Hence T is commuting on U.

From now on assume that T is an automorphism centralizing on the Lie ideal U. By Lemma 1, this means $[x,x^T]=0$ for all x in U.

LEMMA 2. $(x-x^T)\big[x^T,[x,r]\big]=0$ *for all x in U and r in R.*

PROOF. Linearizing $[x,x^T]=0$ gives $[x,y^T]+[y,x^T]=0$ for all x and y in U. As in the derivation of equation (1) replace y by $[x,r]$ to obtain

(2) $$\big[x,[x^T,r-r^T]\big]=0 \text{ for all } x \text{ in } U \text{ and } r\in R.$$

Replacing r by xr in (2), $\big[x,[x^T,xr-x^Tr^T]\big]=\big[x,x[x^T,r]-x^T[x^T,r^T]\big]=x\big[x,[x^T,r]\big]-x^T\big[x,[x^T,r^T]\big]=0$ for all x in U and r in R. Multiplying (2) by x^T on the left and subtracting from this last equation gives $(x-x^T)\big[x,[x^T,r]\big]=0$. By the Jacobi identity $(x-x^T)\big[x^T,[x,r]\big]=0$. A similar argument shows that $\big[x^T,[x,r]\big](x-x^T)=0$.

LEMMA 3. $(x-x^T)[x,r](x-x^T)=0$ *and* $(x-x^T)[x^T,r](x-x^T)=0$ *for all x in U and r in R.*

PROOF. By Lemma 2, $(x-x^T)\big[x^T,[x,r]\big]=0$. Replacing r by rs gives

$$\begin{aligned}(x-x^T)\big[x^T,[x,rs]\big]&=(x-x^T)\big[x^T,r[x,s]+[x,r]s\big]\\&=(x-x^T)\big\{r\big[x^T,[x,s]\big]+[x^T,r][x,s]+\big[x^T,[x,r]\big]s+[x,r][x^T,s]\big\}\\&=0.\end{aligned}$$

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Hence by Lemma 2,

(3) $(x-x^T)\{r[x^T,[x,s]]+[x^T,r][x,s]+[x,r][x^T,s]\}=0$ for all x in U, r and s in R.

Replacing s by $(x-x^T)s$ in (3) gives $0+(x-x^T)[x^T,r](x-x^T)[x,s]+(x-x^T)[x,r]$ $(x-x^T)[x^T,s]=0$. If s is replaced by $[x,s]$, then again by Lemma 2,

(4) $(x-x^T)[x^T,r](x-x^T)[x,[x,s]]=0$ for all x in U and r and s in R.

Let r be replaced by rt, then $(x-x^T)r[x^T,t](x-x^T)[x,[x,s]]+(x-x^T)[x^T,r]t$ $(x-x^T)[x,[x,s]]=0$. Let r be replaced by $r(x-x^T)$. Then $(x-x^T)r(x-x^T)[x^T,t]$ $(x-x^T)[x,[x,s]]+(x-x^T)[x^T,r](x-x^T)t(x-x^T)[x,[x,s]]=0$. But by (4) the first term is zero and so $(x-x^T)[x^T,r](x-x^T)t(x-x^T)[x,[x,s]]=0$ for all x in U and all r, s and t in R. Since R is prime either $(x-x^T)[x^T,r](x-x^T)=0$ or $(x-x^T)[x,[x,s]]=0$. Now equation (3) with r replaced by $r(x-x^T)$ results in $(x-x^T)[x^T,r](x-x^T)[x,s]+(x-x^T)[x,r]$ $(x-x^T)[x^T,s]=0$, so $(x-x^T)[x^T,r](x-x^T)=0$ if and only if $(x-x^T)[x,r](x-x^T)=0$. Thus, if $(x-x^T)[x^T,r](x-x^T)=0$, the Lemma is proved. If $(x-x^T)[x,[x,s]]=0$, then by replacing s by rs, $(x-x^T)[x,[x,rs]]=(x-x^T)[x,r[x,s]+[x,r]s]=(x-x^T)$ $\{r[x,[x,s]]+2[x,r][x,s]\}=0$. If r is replaced by $r(x-x^T)$, then $(x-x^T)[x,r]$ $(x-x^T)[x,s]=0$ which implies $(x-x^T)[x,r](x-x^T)=0$ and the Lemma is true in this case also.

LEMMA 4. *If x is in U and $(x-x^T)^2\neq 0$, then x is in Z.*

PROOF. By Lemma 3, $(x-x^T)[x,r](x-x^T)=0$. Letting r be rs gives $(x-x^T)$ $(r[x,s]+[x,r]s)(x-x^T)=0$. Replacing r by $[x^T,r]$ and using Lemma 2 results in

(5) $(x-x^T)[x^T,r][x,s](x-x^T)=0$ for x in U and all r,s in R.

Replacing s by $[x^T,s]$ would have given

(5′) $(x-x^T)[x,r][x^T,s](x-x^T)=0$ for x in U and all r,s in R.

Now replacing r by rt in (5) to obtain $(x-x^T)(r[x^T,t][x,s]+[x^T,r]t[x,s])(x-x^T)=0$ and then replacing r by $[x,r]$ gives

(6) $(x-x^T)[x,r][x^T,t][x,s](x-x^T)=0$ for x in U and all r,s,t in R.

Now $(x-x^T)[x,r][x-x^T,t][x,s](x-x^T)=0$ since $(x-x^T)[x,r](x-x^T)=0$ and adding this to (6) results in

(7) $(x-x^T)[x,r][x,t][x,s](x-x^T)=0$ for x in U and all r,s,t in R.

Now in (5) if s is replaced by ts and then s by $[x^T,s]$, $(x-x^T)[x^T,r][x,t][x^T,s](x-x^T)=0$. Subtracting this from (7) gives $(x-x^T)[x-x^T,r][x,t][x-x^T,s](x-x^T)=0$. Thus $\{(x-x^T)^2r-(x-x^T)r(x-x^T)\}[x,t]\{(x-x^T)s(x-x^T)-s(x-x^T)^2\}=0$. Replacing r by $[x^T,r]$ reduces this to $(x-x^T)^2[x^T,r][x,t]s(x-x^T)^2=0$ by (5) and Lemma 3. So if

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$(x-x^T)^2 \neq 0$, $(x-x^T)^2[x^T,r][x,t] = 0$ and thus x is in Z or x^T is in Z which implies x is in Z.

LEMMA 5. *If x is in U and $x - x^T \neq 0$, then x is in Z.*

PROOF. If $(x-x^T)^2 \neq 0$, then by Lemma 4, x is in Z, so assume that $(x-x^T)^2 = 0$. By the Jacobi identity, (2) is equivalent to $\big[x^T,[x,r-r^T]\big] = 0$ and linearizing this gives $\big[x^T,[y,r-r^T]\big]+\big[y^T,[x,r-r^T]\big] = 0$. Letting r be x in this results in $\big[x^T,[y,x-x^T]\big]+0 = [x^T, y(x-x^T)-(x-x^T)y] = 0$ or

(8) $$(x-x^T)[x^T,y] = [x^T,y](x-x^T) \text{ for all } x \text{ and } y \text{ in } U.$$

Now by Lemma 3 and using (8), $0 = (x-x^T)[x^T,yz](x-x^T) = (x-x^T)[x^T,y]z(x-x^T) + (x-x^T)y[x^T,z](x-x^T) = [x^T,y](x-x^T)z(x-x^T) + (x-x^T)y(x-x^T)[x^T,z]$ for y and z in U. Letting y be $[y,r]$ gives

(9)
$$\big[x^T,[y,r]\big](x-x^T)z(x-x^T) + (x-x^T)[y,r](x-x^T)[x^T,z] = 0 \text{ for all } r \text{ in } R \text{ and } y,z \text{ in } U.$$

Now by expanding and using $\big[x^T,[y,x-x^T]\big] = 0$,

(10) $$\big[x^T,[y,r(x-x^T)]\big] = [x^T,r][y,x-x^T] + \big[x^T,[y,r]\big](x-x^T).$$

So letting r be $r(x-x^T)$ in (9) and using $(x-x^T)^2 = 0$ and (10) implies $[x^T,r][y,x-x^T](x-x^T)z(x-x^T) + (x-x^T)r[y,x-x^T](x-x^T)[x^T,z] = 0$ or $[x^T,r](x-x^T)y(x-x^T)z(x-x^T) + (x-x^T)r(x-x^T)y(x-x^T)[x^T,z] = 0$. Let r be $[y,r]$ which is of course in U since y is in U, then using (8) on the first term,

(11) $$(x-x^T)\big[x^T,[y,r]\big]y(x-x^T)z(x-x^T) + (x-x^T)[y,r](x-x^T)y(x-x^T)[x^T,z] = 0.$$

Now again by Lemma 3, $(x-x^T)\big[x^T,[y,r]y\big](x-x^T) = 0$ and so $(x-x^T)[y,r][x^T,y](x-x^T) + (x-x^T)\big[x^T,[y,r]\big]y(x-x^T) = 0$. Thus using this in the first term of (11) results in $-(x-x^T)[y,r][x^T,y](x-x^T)z(x-x^T) + (x-x^T)[y,r](x-x^T)y(x-x^T)[x^T,z] = 0$ and by (8) $(x-x^T)[y,r](x-x^T)([x^T,y]z - y[x^T,z])(x-x^T) = 0$. But this implies that $(x-x^T)[y,r](x-x^T)y[x^T,z](x-x^T) = 0$. Linearizing by replacing y by $y+w$ results in $(x-x^T)[w,r](x-x^T)y[x^T,z](x-x^T) + (x-x^T)[y,r](x-x^T)w[x^T,z](x-x^T) = 0$ and now replacing w by $[x,s]$ so that the second term is 0 by (5′),

(12) $$(x-x^T)\big[[x,s],r\big](x-x^T)y[x^T,z](x-x^T) = 0 \text{ for } y,z \text{ in } U \text{ and } r,s \text{ in } R.$$

Now Bergen, Herstein and Kerr [3, Lemma 4] have shown that if a nonzero Lie ideal U is not in the center of a prime ring of characteristic not equal to two, then $aUb = 0$ implies $a = 0$ or $b = 0$. So if U is in the center, then so is x and the Lemma is proved. If U is not in the center, then since (12) is true for all y, either

(13) $$(x-x^T)\big[[x,s]r\big](x-x^T) = 0 \text{ for all } r \text{ and } s \text{ in } R$$

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or

(14) $$[x^T, z](x - x^T) = 0 \text{ for all } z \text{ in } U.$$

If (14) holds, then replacing z in it by $[y, r(x - x^T)]$ and using (10) results in $[x^T, r][y, x - x^T](x - x^T) = -[x^T, r](x - x^T)y(x - x^T) = 0$. So x is in Z or $(x - x^T)y(x - x^T) = 0$ which by Lemma 4 of [3] then forces $x - x^T = 0$ if x is not in Z. So the Lemma is true in this case. If (13) holds, replacing s by st gives $(x - x^T)\big[[x, st], r\big](x - x^T) = (x - x^T)\big[s[x, t] + [x, s]t, r\big](x - x^T) = (x - x^T)\big\{[s, r][x, t] + s\big[[x, t]r\big] + [x, s][t, r] + \big[[x, s], r\big]t\big\}(x - x^T) = 0$. Replacing s by $s(x - x^T)$ and using (13) and Lemma 3 implies $(x - x^T)\big\{[s(x - x^T), r][x, t] + \big[[x, s(x - x^T)], r\big]t\big\}(x - x^T) = (x - x^T)\{s[x - x^T, r][x, t] + [x, s][x - x^T, r]t\}(x - x^T) = 0$ or $(x - x^T)\{s(x - x^T)r[x, t] - [x, s]r(x - x^T)t\}(x - x^T) = 0$. But $(x - x^T)[x, sr](x - x^T) = 0$ implies that $(x - x^T)\{s(x - x^T)r[x, t] + s[x, r](x - x^T)t\}(x - x^T) = (x - x^T)s\{(x - x^T)r[x, t] + [x, r](x - x^T)t\}(x - x^T) = 0$. So if $x \neq x^T$, $(x - x^T)r[x, t](x - x^T) + [x, r](x - x^T)t(x - x^T) = 0$. Since $(x - x^T)[x, rt](x - x^T) = 0$, this becomes $-(x - x^T)[x, r]t(x - x^T) + [x, r](x - x^T)t(x - x^T) = \{-(x - x^T)[x, r] + [x, r](x - x^T)\}t(x - x^T) = 0$. So if $(x - x^T) \neq 0$,

(15) $$(x - x^T)[x, r] = [x, r](x - x^T) \text{ for all } r \text{ in } R.$$

Letting r be rs gives $(x - x^T)(r[x, s] + [x, r]s) = (r[x, s] + [x, r]s)(x - x^T)$ and then replacing r by $r(x - x^T)$ implies $(x - x^T)r(x - x^T)[x, s] = [x, r](x - x^T)s(x - x^T)$. But using (15), this implies $(x - x^T)\{r[x, s] - [x, r]s\}(x - x^T) = 2(x - x^T)r[x, s](x - x^T) = 0$. Hence x is in Z.

PROOF OF THE THEOREM. Since T is nontrivial on U, there must be an x in U such that $x \neq x^T$. By Lemma 5, x is in Z. Let y be in U and y not be in Z. Then by Lemma 5, $y = y^T$. But then $(x + y)^T = x^T + y^T = x^T + y \neq x + y$. Hence $x + y$ is in Z but this is impossible since y was assumed not to be in Z. Hence for all y in U, y must be in Z and so U is contained in Z.

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