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0-DISTRIBUTIVE AND P-UNIFORM SEMILATTICES

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ABSTRACT. A counter-example is provided to the conjecture of Y. S. Pawar and N. K. Thakare that a semilattice S with 0 is 0-distributive if and only if for each filter F and each ideal I such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint from I. This shows that 0-distributivity is not equivalent to weak distributivity. A characterization is also given of finite P-uniform semilattices in terms of 0-distributivity.

§1. **Introduction.** Let S be a meet semilattice with zero. A non-empty subset F of S is a filter provided that (i) if $x \in F$ and $x \leq y$ then $y \in F$, and (ii) if $x, y \in F$, then $x \wedge y \in F$. A filter F is a prime filter if whenever $x \vee y \in F$ then either $x \in F$ or $y \in F$. An ideal of S is a non-empty subset I of S such that $b \in I$ and $a \leq b$ imply that $a \in I$, and whenever $a \vee b$ exists for $a, b \in I$, then $a \vee b \in I$. An ideal I is called prime if whenever $a \wedge b \in I$, then either $a \in I$ or $b \in I$. A semilattice S is called 0-distributive if whenever a, b, c are in S, $a \wedge b = 0$, $a \wedge c = 0$ and $b \vee c$ exists, then $a \wedge (b \vee c) = 0$. In [5], Y. S. Pawar and N. K. Thakare conjectured the following: "A semilattice S with 0 is 0-distributive if and only if for any filter F and any ideal I such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint from I". We shall show that this conjecture is false in general, and obtain some results on when it is true. In fact, this conjecture seems to be based on assuming the existence of a zero and weakening a distributive law in a theorem which appeared in an earlier paper (Balbes [1]). Finally, we shall relate 0-distributive and P-uniform finite semilattices. (For the definition of P-uniform, see below or [6]).

§2. **Counterexample.** The fact that the above conjecture is false follows from the following example.

EXAMPLE 2.1. Let $S = \{0, a, b, c, d, e, f, g, h, i\}$ with Hasse diagram shown below

Then $I = \{0, a, b, c\}$ is an ideal, and $F = \{h\}$ is a filter. Clearly $F \cap I = \emptyset$. It can

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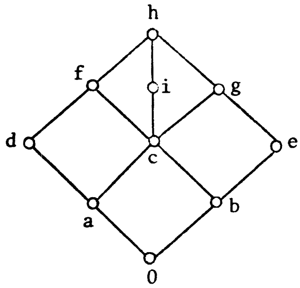

be checked that there is no prime filter which contains F and is disjoint from I. On the other hand, it is easy to check that the semilattice S is 0-distributive.

As we indicated, the conjecture probably arose out of a modification of a result in [1]. That result was stated incorrectly in [1]. The corrected version of that result (Theorem 2.2 of [1]) is as follows.

THEOREM 2.2. (Balbes). *In a semilattice L, the following are equivalent*

(i) *If* $x_1 \vee x_2 \vee \cdots \vee x_n$ *exists in L, then for each* $x \in L$, $(x \wedge x_1) \vee (x \wedge x_2) \vee \cdots \vee (x \wedge x_n)$ *exists and equals* $x \wedge (x_1 \vee \cdots \vee x_n)$.

(ii) *If F is a filter in L and I is a non-empty subset of L, disjoint from F and such that* $x_1 \vee \cdots \vee x_n$ *exists in I whenever* $x_1, x_2, \ldots, x_n \in I$, *then there exists a prime filter F′ such that* $F \subset F'$ *and* $F' \cap I = \emptyset$.

(iii) *If F is a filter in L and I is an ideal of L disjoint from F, then there exists a prime filter F′ such that* $F \subset F'$ *and* $F' \cap I = \emptyset$.

(iv) *If* $x \not\leq y$, *then there exists a prime filter F′ such that* $x \in F'$ *and* $y \notin F'$.

REMARKS (i) In [1], the statement (ii) above allowed I to be a set with a property which is weaker than that stated here. However, if one goes through the proof of the theorem there, one finds that I is required to have the property as stated in (ii) above. In fact, the condition on I is that of being an ideal. Consequently, we have added the equivalent condition (iii) above. It is easily checked that the correct statement of Balbes′ theorem is as we have formulated it here.

(ii) In [1], the semilattice is not required to have a zero.

(iii) Any semilattice satisfying the conditions of Balbes' theorem is called a prime semilattice.

DEFINITION 2.3. Let us call any semilattice satisfying condition (i) above weakly distributive, and if it satisfies condition (iii), let us call it prime separable.

REMARKS. (i) Balbes' theorem may be expressed as "A semilattice is weakly distributive if and only if it is prime separable".

(ii) A semilattice is distributive if $t \geq a \wedge b$ implies that $t = a' \wedge b'$ for some

$a' \geq a$, and $b' \geq b$ (Problem 99, page 329 of [2]). We shall show in §5 that a distributive semilattice is weakly distributive. Also a weakly distributive semilattice with 0 is 0-distributive. Example 2.1 shows that the converse is not true.

§3. Some characterizations. We shall consider Pawar and Thakare's conjecture under some further conditions. We shall also obtain some results when a semilattice is 0-distributive.

DEFINITION 3.1. Let a be an element of a semilattice S with 0. Then an element a^* of S is called a pseudocomplement of a if (i) $a \wedge a^* = 0$ (ii) $a \wedge b = 0$ implies that $b \leq a^*$. A semilattice in which every element has a pseudocomplement is said to be pseudocomplemented.

NOTATION. Given an element a of a semilattice S with 0, let $(a] = \{x \in S \mid x \leq a\}$ and let $\{a\}^\perp = \{x \in S \mid a \wedge x = 0\}$. The set $(a]$ is the principal ideal of S generated by a. We shall frequently write $a^\perp$ for $\{a\}^\perp$. If I is an ideal, we let $I^* = \{x \in S \mid x \wedge a = 0$ for all $a \in I\}$. We call I non-dense if $I^* \neq \{0\}$. In case S is pseudocomplemented, this means that for each $a \in I$, $(a^*] \neq \{0\}$.

It can be verified that in a 0-distributive semilattice, $\{a\}^\perp$ is an ideal. In fact, a semilattice S with 0 is 0-distributive if and only if $\{a\}^\perp$ is an ideal for all $a \in S$ (Theorem 5, [5]). Further, a semilattice S with 0 is pseudocomplemented if and only if $\{a\}^\perp$ is a principal ideal for each $a \in S$. Thus a pseudocomplemented semilattice is 0-distributive.

We shall need the following result.

LEMMA 3.2 (Theorem 7, [5]). *Let S be a semilattice with* 0. *Then S is* 0-*distributive if and only if for any filter F disjoint from $\{x\}^\perp$ (x in S), there exists a prime filter containing F and disjoint from $\{x\}^\perp$.*

NOTE. It is easily checked that in a pseudocomplemented semilattice, we have $(a^*] = (a]^*$ for all a. It is also true that $\{a\}^\perp = (a^*]$ for all a.

LEMMA 3.3. *Let S be a finite pseudocomplemented semilattice and let I be an ideal in S. Let $a = (\bigwedge_{x \in I} x^*)^*$. Then $a^* = \bigwedge_{x \in I} x^*$, and $a^{**} = a$.*

Proof. Use Theorem 1 of [8]. We have

$$a^* = \left(\bigwedge_{x \in I} x^*\right)^{**} = \bigwedge_{x \in I} x^{***} = \bigwedge_{x \in I} x^*$$

and hence $a^{**} = (\bigwedge_{x \in I} x^*)^* = a$.

LEMMA 3.4. *In a finite pseudocomplemented semilattice S, every minimal prime ideal I is principal, that is, I is of the form $I = (a]$ for some $a \in S$. In fact, we can choose $a = (\bigwedge_{x \in I} x^*)^*$, and hence $I = (a] = (a^{**}] = \{a^*\}^\perp$.*

Proof. If we choose $a = (\bigwedge_{x \in I} x^*)^*$, then $I \subset (a]$.

In fact, let $x \in I$. Then $\bigwedge_{x\in I} x^* \leq x^*$ and hence $x^{**} \leq (\bigwedge_{x\in I} x^*)^* = a$. Since $x \leq x^{**}$, we have $x \leq a$. On the other hand, for each $x \in I$, $x^* \notin I$ since I is minimal prime. (Lemma VI, [8]), and hence $\bigwedge_{x\in I} x^* \notin I$ because I is prime. Thus $a \in I$ because I is minimal prime.

Definition 3.5. An ideal I in a pseudocomplemented semilattice is called simple if for each $x \in I$, $x \vee x^*$ exists and $x \vee x^* = 1$.

Lemma 3.6. *In a finite pseudocomplemented semilattice, every non-dense simple prime ideal I is principal. In fact, if* $a = (\bigwedge_{x\in I} x^*)^*$, *then* $I = (a] = \{a^*\}^\perp$.

Proof. As in the proof of Lemma 3.4 we see that we need only the fact that I is an ideal to show that $I \subset (a] = \{a^*\}_\perp$. We may assume that I is proper. We claim that $I = (a] = \{a^*\}^\perp$. For if not, then there exists $y \in \{a^*\}^\perp$ such that $y \notin I$. Thus $a^* \wedge y = 0 \in I$. Since I is prime, and $y \notin I$, we have that $a^* \in I \subset (a]$. Thus $a^* \leq a$ and hence $a^* = a^* \wedge a = 0$. This means that $\bigwedge_{x\in I} x^* = 0 \in I$. Since I is prime, this means that there exists an element $x \in I$ such that $x^* \in I$ also. Since I is simple, it follows that $x \vee x^*$ exists and $x \vee x^* = 1$. But because I is an ideal and $x, x^* \in I$ and $x \vee x^*$ exists, we have that $1 = x \vee x^* \in I$. This contradicts the fact that I is proper, and completes the proof.

Theorem 3.7. *Let S be a finite pseudocomplemented semilattice. Then for any filter F and any non-dense simple prime ideal I such that* $F \cap I = \varnothing$, *there exists a prime filter containing F and disjoint from I.*

Proof. By Lemma 3.6 we have that $I = \{a^*\}^\perp$ for some $a \in S$. Since S is 0-distributive, by Lemma 3.2, we can find a prime filter containing F and disjoint from I.

Since in a pseudocomplemented semilattice, $\{x\}^\perp = (x^*]$ for all x, Lemma 3.2 may be reformulated as follows.

Lemma 3.8. *Let S be a pseudocomplemented semilattice. Then S is 0-distributive if and only if for all filters F and all principal ideals* $[x^*]$ *such that* $F \cap (x^*] = \varnothing$, *there exists a prime filter containing F and disjoint from* $(x^*]$.

Corollary 3.9. *Let S be a finite pseudocomplemented semilattice. Suppose that every principal ideal is prime, and that every prime ideal is non-dense and simple. Then S is* 0*-distributive if and only if for all filters F and all principal ideals I such that* $F \cap I = \varnothing$, *there exists a prime filter containing F and disjoint from I.*

Proof. Use Theorem 3.7 and Lemma 3.8.

§4. ***P*-uniform semilattices.** We give a characterization of finite P-uniform semilattices in terms of 0-distributivity.

Definition 4.1. ([6], page 17). A semilattice S is said to be primary uniform (P-uniform) if it satisfies the following condition: For any proper prime ideal F

of S, there exist an element e of F and an element f of $S-F$ such that $e\wedge S\cong f\wedge S$.

DEFINITION 4.2. If A is a set in a semilattice, we put $J_0(A)=\varnothing$ if A contains no ideal. Otherwise, $J_0(A)$ is the union of all ideals of S contained in A, and hence is the largest ideal contained in A.

We shall need one result from the theory of topological semigroups which we quote here.

THEOREM 4.3. (Numakura [3], or Theorem 1.5.4 of [4]). *Let S be a compact topological semigroup. Then each open prime ideal $P\neq S$ has the form $J_0(S-e)$ where e is a non-minimal idempotent. Conversely, if e is a non-minimal idempotent, then $J_0(S-e)$ is an open prime ideal.*

We shall be applying this theorem to finite meet semilattices. Hence compactness is satisfied, and every element is an idempotent.

We shall need a result on finite P-uniform semilattices. This is Theorem 8 of [6], page 23, and also the discussion on page 22 of [6] preceding the statement of that theorem.

THEOREM 4.4. (Shoji and Yamada [6]). *If a finite semilattice S is P-uniform, then $S=\{0\}$, or every element of S is a zero divisor. In fact, in the latter case, for each element x of S, there is an atom a such that $x\wedge a=0$.*

We make the observation that every finite semilattice automatically has a 0.

LEMMA 4.5. *Let $S\neq\{0\}$ be a semilattice with 0. Suppose that $a\in S$ is an atom such that $a^\perp$ is an ideal. Then $a^\perp=J_0(S-a)$.*

Proof. Since $a^\perp$ is an ideal, we have $a^\perp\subset J_0(S-a)$. On the other hand, if $x\in J_0(S-a)$ we claim that $x\in a^\perp$ also. For if not, then $x\notin a^\perp$ and hence $x\wedge a\neq 0$. But then $0<x\wedge a\leq a$, and since a is an atom, we must have that $x\wedge a=a$. Thus $a\leq x\in J_0(S-a)$. This implies that $a\in J_0(S-a)$ which is impossible.

LEMMA 4.6. *Let $S\neq\{0\}$ be a semilattice with 0, and let $a\in S$ be an atom such that $a^\perp$ is a prime ideal. Then $a^\perp=J_0(S-a)$ is a minimal prime ideal.*

Proof. We have seen by Lemma 4.5 that $a^\perp=J_0(S-a)$. Suppose that I is a prime ideal such that $I\subset a^\perp$. If $I\neq a^\perp$, then there exists an $x\in a^\perp$ such that $x\notin I$. We have $a\wedge x=0\in I$ and since $x\notin I$ and I is prime, we must have $a\in I\subset a^\perp$, which is impossible since a is an atom.

LEMMA 4.7. *Let $S\neq\{0\}$ be a finite semilattice. Suppose that S is P-uniform. Then every proper prime ideal of S is of the form $(a]=a_1^\perp=J_0(S-a_1)$ for some $a\in S$ and some atom a_1 such that $a\wedge a_1=0$. Thus every proper prime ideal is a minimal prime ideal.*

Proof. Let I be a proper prime ideal. Then $I=(a]$ for some a since S is finite. Since S is P-uniform, by Theorem 4.4, there is an atom a_1 such that $a \wedge a_1 = 0$. Hence $(a] \subset a_1^\perp$. If $(a] \neq a_1^\perp$, then there exists an element $x \in a_1^\perp$ such that $x \notin (a]$. Then $a_1 \wedge x = 0 \in (a]$, and since $(a]$ is prime and $x \notin (a]$, we must have $a_1 \in (a] \subset a_1^\perp$. This is impossible since a_1 is an atom.

We now prove our theorem relating P-uniformity and 0-distributivity.

THEOREM 4.8. *Let S be a finite semilattice without* 1. *Suppose that S has at least two atoms a_1, a_2 such that $a_1^\perp$ and $a_2^\perp$ are distinct ideals. Then S is P-uniform if and only if it is* 0-*distributive and every proper prime ideal of S is a minimal prime ideal.*

Proof. Suppose that S is P-uniform, and let I be a proper prime ideal of S. Then by Lemma 4.7, $I=(a]=a_i^\perp=J_0(S-a_i)$ for some $a \in S$ and some atom a_i such that $a \wedge a_i = 0$. Thus I is a minimal prime ideal. We now want to consider all the proper prime ideals of S. By Theorem 4.3, they are precisely those $J_0(S-a)$, for non-minimal elements a. Consider our two atoms a_1, a_2 such that $a_1^\perp$ and $a_2^\perp$ are distinct ideals. Thus we have $a_1^\perp = J_0(S-a_1)$ and $a_2^\perp = J_0(S-a_2)$ by Lemma 4.5. Then by Theorem 4.3, these are prime ideals; in fact, they are minimal prime ideals by Lemma 4.6. The intersection of all prime ideals is contained in $a_1^\perp \cap a_2^\perp$. But $a_1^\perp \cap a_2^\perp$ is also a prime ideal and is contained in the distinct minimal prime ideals $a_1^\perp$, $a_2^\perp$. It must follow then that $a_1^\perp \cap a_2^\perp = \{0\}$, and hence the intersection of all prime ideals is $\{0\}$. Hence by Theorem 2 of [5], it follows that S is 0-distributive. For the converse, suppose that S is 0-distributive and that every proper prime ideal of S is a minimal prime ideal. Let I be a proper prime ideal. Then by Theorem 4.3, $I=J_0(S-a)$ for some non-minimal element a. If a is an atom, then $I=J_0(S-a)=a^\perp$. If a is not an atom, then there is some atom b such that $a>b$. Since S is 0-distributive, by Theorem 5 of [5], we have that $b^\perp$ is an ideal, and hence since b is an atom, $b^\perp$ is a prime ideal, in fact $b^\perp = J_0(S-b)$. This follows by Lemma 4.5 and Theorem 4.3. Since $a>b$ we have $a^\perp \subset b^\perp$, that is, $J_0(S-a) \subset J_0(S-b)$. Since these are both prime ideals and hence are minimal prime ideals by hypothesis, we must have that $I=a^\perp=J_0(S-a)=J_0(S-b)=b^\perp$. Thus we can always assume that our original proper prime ideal I is of the form $J=J_0(S-a)=a^\perp$ for some atom a. Now let $Q=S-a^\perp$. It is easily checked that Q is a filter, in fact, a prime filter. Take an atom t of S lying in $a^\perp$. Then $t \wedge S = t \wedge (Q \cup a^\perp) = t \wedge Q \cup t \wedge a^\perp$. If $z \in Q$, then $t \wedge z = 0$ or t. In fact, if $z \in Q$ then $z \notin a^\perp$ and hence $z \wedge a \neq 0$. If $t \wedge z \neq 0$, then $0 < t \wedge z \leq t$. Since t is an atom, we must have $t \wedge z = t$. Similarly, if $z \in a^\perp$, then $t \wedge z = 0$ or t. Since $t \in a^\perp$, we thus have $t \wedge S = \{0, t\}$. On the other hand, consider $a \wedge S$. We have $a \wedge S = a \wedge (Q \cup a^\perp) = a \wedge Q \cup a \wedge a^\perp$. Clearly $a \wedge a^\perp = \{0\}$. Also, since a is an atom and $a \in Q$, we have that $a \wedge Q = \{a\}$. Thus $a \wedge S = \{0, a\}$. Thus we have found elements $t \in I$, $a \in S-I$ such that $t \wedge S \cong a \wedge S$. Thus S is P-uniform.

§5. **Distributive implies weakly distributive.** We prove here our earlier assertion that a distributive semilattice is weakly distributive.

DEFINITION 5.1. If x, a are elements of a semilattice S, then $\langle x, a\rangle = \{y \in S \mid x \wedge y \leq a\}$.

DEFINITION 5.2. A closed ideal of a semilattice S is a non-empty subset I of S such that

(i) $y \leq x$ and $x \in I$ imply that $y \in I$

(ii) for any x, y in I, there exists z in I such that $z \geq x$ and $z \geq y$.

Thus, a closed ideal is also an ideal in our sense.

DEFINITION 5.3. Let a be an element of a semilattice S. Then S is called a-distributive if $\langle x, a\rangle$ is a closed ideal for all $x \in S$.

In [7], J. C. Varlet proved the following result.

THEOREM 5.4. (Theorem 3.3 of [7]). *A semilattice S is distributive if and only if it is a-distributive for all $a \in S$.*

NOTE. What we call a closed ideal is what Varlet [7] called an ideal.

DEFINITION 5.5. If F_1, F_2 are filters of a semilattice S, then $F_1 + F_2 = \{z \in S \mid z \geq f_1 \wedge f_2$ for $f_1 \in F_1,\ f_2 \in F_2\}$. Then $F_1 + F_2$ is also a filter of S.

THEOREM 5.6. *A distributive semilattice S is weakly distributive.*

Proof. According to Balbes' Theorem, we need only verify condition (iv) of Theorem 2.2 above. Suppose that $a \not\leq b$. We can always find a filter which contains a but not b, for example, $[a)$. By Zorn's Lemma, there exists a maximal filter F containing a but not b. We claim that F is prime, thus proving our result. For, if F is not prime, then we can find elements x, y of S such that $x \notin F$, $y \notin F$ but $x \vee y \in F$. We observe that $\langle x, b\rangle \cap F \neq \varnothing$. For, since $x \notin F$, we have that the filter $F + [x)$ properly contains F. Since $a \in F + [x)$, and since F is maximal with respect to the property of containing a but not b, it follows that $b \in F + [x)$. Thus, we can find an element $z \in F$, and an element $z_1 \geq x$ such that $b \geq z \wedge z_1$. This means that $z \wedge x \leq z \wedge z_1 \leq b$. Hence $z \in \langle x, b\rangle \cap F$. Similarly, we can show that $\langle y, b\rangle \cap F \neq \varnothing$. Thus we can find an element $t \in \langle y, b\rangle \cap F$, that is, $t \in F$ and $y \wedge t \leq b$. Thus, $z \wedge x \wedge t \leq b \wedge t \leq b$, and $z \wedge y \wedge t \leq z \wedge b \leq b$. This means that $x, y \in \langle z \wedge t, b\rangle$. Since S is distributive and hence is b-distributive by Theorem 5.4, it follows that $\langle z \wedge t, b\rangle$ is a closed ideal of S. Since $x \vee y$ exists, it follows that $x \vee y \in \langle z \wedge t, b\rangle$. Thus $z \wedge t \wedge (x \vee y) \leq b$. But z, t and $x \vee y$ are elements of a filter F and hence $z \wedge t \wedge (x \vee y) \in F$. Hence $b \in F$. This is a contradiction, and proves our theorem.

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