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ERGODIC EXTENSIONS OF ENDOMORPHISMS

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Abstract

We examine a class of ergodic transformations on a probability measure space (X, μ) and show that they extend to representations of $\mathcal{B}(L^2(X, \mu))$ that are both implemented by a Cuntz family and ergodic. This class contains several known examples, which are unified in our work. During the analysis of the existence and uniqueness of this Cuntz family, we find several results of independent interest. Most notably, we prove a decomposition of X for N-to-one local homeomorphisms that is connected to the orthonormal bases of certain Hilbert modules.

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1. Introduction

There is considerable interest in endomorphisms of nonselfadjoint operator algebras. As in the theory of automorphisms of C*-algebras, where one asks if a representation of the algebra admits a unitary operator which implements the automorphism, in the case of endomorphisms there are various ways in which an endomorphism might be implemented. In this paper, we examine the case when an endomorphism of $C(X)$, arising from a local homeomorphism, admits a representation in which the endomorphism is realised by a family of Cuntz isometries. In other words, if α is the endomorphism of $C(X)$ associated with a representation π, when does there exist a family $S_1, \ldots, S_N$ of Cuntz isometries such that

$$\pi(\alpha(f)) = \sum_{j=1}^{N} S_j \pi(f) S_j^* \quad \text{for all } f \in C(X)?$$

Such a quantisation of the dynamical system gives rise to a number of questions relating the intrinsic properties of α to operator theoretic properties of the algebras related to $\pi(C(X))$ and $S_1, \ldots, S_N$. In this paper, we address the problem of ergodicity.

We are motivated by and extend the recent work of Courtney *et al.* [1], in association with earlier work of Laca [7] on ergodic transformations of $\mathcal{B}(H)$. In contrast to our

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previous work [6], which focuses on the abstract operator algebraic point of view, here we analyse a class of particular transformations $\varphi\colon X \to X$ of a probability measure space (X, μ). This leads to an analysis of a class of transformations of (X, μ) and connects to the existence of a basis for Hilbert modules.

There are several well known examples (including the backward shift on infinite words on N symbols, where N is finite, and finite Blaschke products with N factors) where such transformations yield an endomorphism $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ that is implemented by a Cuntz family. Our goal here is two-fold. Firstly, we give conditions under which a transformation $\varphi\colon X \to X$ defines such an endomorphism $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ so that $\alpha(f) = f \circ \varphi$ (Proposition 2.2). Secondly, we show that ergodicity of $\varphi\colon X \to X$ (as a transformation of a probability measure space) implies ergodicity of the induced $\alpha_S : \mathcal{B}(L^2(X, \mu)) \to \mathcal{B}(L^2(X, \mu))$ as a representation of a von Neumann algebra (Theorem 3.2).

The existence of a Cuntz family implementing $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ is connected to a decomposition of the space X based on a maximal family of sets (Lemma 3.1). One may ask whether different decompositions yield the same extension. We show that the answer to this question is connected to the existence of an orthonormal basis of a suitable W*-module (Proposition 4.5). As a consequence, we obtain a complete invariant on multiplicity n crossed products after Stacey [12] (Corollary 4.6).

A useful tool for the study of the endomorphism $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ is the *intertwining Hilbert module* $\mathcal{E}(X, \mu)$ used in Section 4. This construction appears in the work of Muhly and Solel [10]; however, we will not require the technology of the induced representations. Under certain conditions on the transformation $\varphi\colon X \to X$, there is a transfer operator, and our setting encompasses several cases, including that described in [1, Theorem 5.2]. We conclude by showing that the existence of a basis for the Hilbert module $\mathcal{E}(X, \mu)$ is equivalent to the existence of a Cuntz family implementing $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ and, in turn, is equivalent to the existence of a basis for $L^\infty(X, \mu)$ viewed as a Hilbert module, where the inner product is defined by the transfer operator (Theorem 5.2).

Hilbert modules may not have a well-defined (up to unitary equivalence) basis, in contrast to Hilbert spaces. Therefore, it is central to our analysis to achieve a well-defined basis. For example, $\mathcal{O}_2$ is unitarily equivalent to $\sum_{k=1}^n \mathcal{O}_2$ for all $n \in \mathbb{N}$, as Hilbert modules over $\mathcal{O}_2$ (Remark 4.2). This phenomenon is also connected to the multiplicity of multivariable C*-dynamics [5] and produces an obstacle for the classification of these objects. To tackle this problem, Gipson [4] develops the notion of the invariant basis number for C*-algebras, along with an in-depth analysis of C*-algebras that do (or do not) attain such a number.

2. Preliminaries

Let us begin with a general comment on $*$-endomorphisms α_S of $\mathcal{B}(H)$ that are implemented by a Cuntz family $\{S_1, \ldots, S_N\}$: that is,

$$\alpha_S(T) = \sum_{i=1}^{N} S_i T S_i^* \quad \text{for all } T \in \mathcal{B}(H).$$

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We write $\mathcal{O}_N = \mathrm{C}^*(S_1, \ldots, S_N)$ for the Cuntz algebra [2] inside $\mathcal{B}(H)$. Both α_S and the restriction $\alpha_S|_{\mathcal{O}_N}$ of α_S to $\mathcal{O}_N$ are injective, but they are not onto for $N > 1$. Indeed, if there is a $T \in \mathcal{B}(H)$ such that $\alpha_S(T) = 0$, then

$$T = S_1^* S_1 T S_1^* S_1 = S_1^* \alpha_S(T) S_1 = 0.$$

Furthermore, if there is a $T \in \mathcal{O}_N$ such that $\alpha_S(T) = S_1$, then

$$I = S_1^* S_1 = S_1^* \alpha_S(T) = T S_1^*,$$

and hence S_1 is a unitary, which holds if and only if $N = 1$.

Let (X, μ) and (Y, ν) be compact Hausdorff spaces, endowed with their Borel structure and measures μ and ν. Then a continuous map $\varphi\colon (X, \mu) \to (Y, \nu)$ is a Borel homomorphism. However, the mapping

$$\alpha\colon (L^\infty(Y, \nu), \|\cdot\|_\infty) \to (L^\infty(X, \mu), \|\cdot\|_\infty) : f \mapsto f \circ \varphi,$$

where $\|\cdot\|_\infty$ is the essential sup-norm, may not even be well defined. In particular, one can show that α is well defined if and only if $\mu \circ \varphi^{-1} \ll \nu$ (that is, the set map φ^{-1} *preserves the ν-null sets*). When $\varphi(Y)$ is, in addition, a Borel set, α is well defined and injective if and only if $\nu(\varphi(Y)^c) = 0$ (that is, φ is *almost onto* X) and $\mu \circ \varphi^{-1} \sim \nu$.

In general, a Borel map $\varphi\colon X \to Y$ is said to *preserve the ν-null sets* if $\nu \circ \varphi \ll \mu$. In this case, $\nu \ll \mu \circ \varphi^{-1}$. Note here that, if $\nu \circ \varphi \ll \mu$, then $\varphi(E)$ is Borel for every Borel subset E of X. Indeed, a Borel subset E of X is the union of an F_σ set A and a μ-null set N. Then $\varphi(N)$ is a ν-null set, and compactness of X implies that A is σ-compact. Hence $\varphi(A)$ is Borel; thus $\varphi(E)$ is measurable.

Recall that, if $\varphi\colon X \to Y$ is a Borel map, then a mapping $\psi\colon \varphi(X) \to X$ is called *a Borel (cross) section of* φ, where ψ is a Borel map and $\varphi \circ \psi = \mathrm{id}_{\varphi(X)}$.

Proposition 2.1. *Let $\varphi\colon X \to Y$ be an onto map, such that φ and φ^{-1} preserve the null sets, and let $\psi\colon Y \to X$ be a Borel section of φ. Then $X_0 := \psi(Y)$ is Borel and there is an isometry $S : L^2(Y, \nu) \to L^2(X, \mu)$ such that*

$$M_{f\circ\varphi}|_{L^2(X_0, \nu|_{X_0})} = S M_f S^* \quad \textit{for all } f \in L^\infty(Y, \nu).$$

Proof. Observe that ψ preserves the null sets (which implies that X_0 is Borel). Since $\mu \circ \varphi^{-1} \ll \nu$, then $\mu \ll \nu \circ \varphi$. For a null set $E \subseteq Y$, we have $\nu \circ \varphi(\psi(E)) = \nu(E) = 0$, and thus $\mu \circ \psi(E) = 0$. Note that, since X_0 is Borel, then $\varphi|_{X_0}$ is a Borel isomorphism with $\psi\colon Y \to X_0$ as an inverse.

On the other hand, if $\mu \circ \psi(E) = 0$, then $\nu(E) = \nu \circ \varphi(\psi(E)) = 0$, since φ preserves the null sets. Therefore, ν is equivalent to $\mu \circ \psi = \mu|_{X_0} \circ \psi$ and the Radon–Nikodym derivative $u = d(\mu|_{X_0} \circ \psi)/d\nu$ is defined. It is a standard fact that the operator $S_0^*\colon L^2(X_0, \mu|_{X_0}) \to L^2(Y, \nu)$, defined by

$$S_0^*(g) = g \circ \psi \cdot u^{1/2} \quad \text{for all } g \in L^2(X_0, \mu|_{X_0}),$$

is a unitary such that

$$M_{f\circ\varphi}|_{L^2(X_0, \nu|_{X_0})} = S_0 M_f S_0^* \quad \text{for all } f \in L^\infty(Y, \nu).$$

Extend S_0^* trivially to S^* on $L^2(X, \mu) = L^2(X_0, \mu|_{X_0}) \oplus L^2(X_0^c, \mu|_{X_0^c})$. Then the adjoint S of S^* is an isometry and gives the required equation. ◻

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Proposition 2.2. *Let $\varphi\colon X \to Y$ be an onto map, such that φ and φ^{-1} preserve the null sets. Suppose that there is a family $\{\psi_1, \ldots, \psi_N\}$ of N Borel sections of φ such that $\psi_i(Y) \cap \psi_j(Y) = \emptyset$ for $i \neq j$, and $\bigcup_i \psi_i(Y)$ is almost equal to X. Then there is a Cuntz family that implements α.*

Proof. For every $i = 1, \ldots, N$, let $X_i = \psi_i(Y)$ and let S_i be constructed as above on $\mathcal{X}_i = L^2(X_i, \mu|_{X_i})$. Note that $\mathcal{X}_i \perp \mathcal{X}_j$ for $i \neq j$. Thus, if $X_0 = \bigcup_i X_i$ and $\mathcal{X}_0 = \bigoplus_i \mathcal{X}_i$, we get $M_{\mathbf{1}|_{X_i}} = M_{\alpha(\mathbf{1})}|_{\mathcal{X}_i} = S_i M_{\mathbf{1}} S_i^* = S_i S_i^*$, and therefore

$$M_{\alpha(f)}|_{\mathcal{X}_0} = \sum_{i=1}^{N} M_{\alpha(f)}|_{\mathcal{X}_i} = \sum_{i=1}^{N} S_i M_f S_i^*.$$

Since $X = \bigcup_i X_i$ a.e., we obtain $I_{\mathcal{X}_0} = M_{\mathbf{1}}|_{\mathcal{X}_0} = \sum_{i=1}^N M_{\mathbf{1}}|_{\mathcal{X}_i} = \sum_{i=1}^N S_i S_i^*$. Finally, X_0 is almost equal to X: hence $L^2(X, \nu) = \mathcal{X}_0$ and the proof is complete. □

3. Ergodic extensions

Let (X, μ) be a probability measure space such that X is a compact Hausdorff space and μ is a regular Borel measure on X. A measure preserving map $\varphi\colon X \to X$ induces an injective $*$-homomorphism $\alpha\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$. We are interested in the case where α is implemented by a Cuntz family $\{S_i\}_{i=1}^N$ in $\mathcal{B}(L^2(X, \mu))$. In this case, α extends to an injective $*$-endomorphism α_S of $\mathcal{B}(L^2(X, \mu))$. A natural question is whether ergodicity of the mapping φ implies ergodicity of the $*$-endomorphism α_S. Recall that α_S is *ergodic* if the von Neumann algebra $\mathcal{N}_{\alpha_S} := \{T \in \mathcal{B}(\mathcal{H}) \mid \alpha_S(T) = T\}$ is trivial. We aim to give a positive answer for a class of ergodic mappings that includes central examples.

Recall that a map $\varphi\colon X \to X$ is called a *local homeomorphism* if, for every point $x \in X$, there is a neighbourhood U such that $\varphi|_U$ is a homeomorphism onto its image. Clearly, local homeomorphisms are continuous and open. We begin with a decomposition lemma tailored to our study.

Lemma 3.1. *Let φ be a local homeomorphism of a compact Hausdorff space X such that $|\varphi^{-1}(x)| = N > 1$ for all $x \in X$. Then there exist pairwise disjoint open subsets $U_1, \ldots, U_N$ such that:*

(1) *$\varphi|_{U_i}$ is one-to-one for all $i = 1, \ldots, n$;*
(2) *$\varphi(U_i) = \varphi(U_j)$ for all $i, j = 1, \ldots, N$;*
(3) *$X = \bigcup_{i=1}^N (U_i \cup \partial U_i)$; and*
(4) *$X = \varphi(U_i) \cup \partial\varphi(U_i)$ for all $i = 1, \ldots, N$.*

Moreover, $\varphi(\partial U_i) \subseteq \partial\varphi(U_i)$ and $\varphi^{-1}(\partial\varphi(U_i)) = \bigcup_{j=1}^N \partial U_j$, for all $i = 1, \ldots, N$.

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PROOF. Firstly, let us construct a family that satisfies (1) and (2). Let $\mathcal{F}$ be the collection that consists of $\{U_1, \ldots, U_N\}$ such that U_i are open and disjoint, $\varphi|_{U_i}$ is one-to-one for all $i = 1, \ldots, N$ and $\varphi(U_i) = \varphi(U_j)$.

Claim. The collection $\mathcal{F}$ is nonempty.

Proof of Claim. Choose $y \in X$ and suppose that $x_1, \ldots, x_N$ are the N pre-images of y. Let V_i be a neighbourhood of x_i such that $\varphi|_{V_i}$ is one-to-one. Since X is a Hausdorff space, we can choose V_i to be disjoint. Moreover, $\varphi(U_i)$ are open sets, since φ is an open map. Let $V = \bigcap_{i=1}^N \varphi(V_i)$, which is open, and let $U_i = \varphi^{-1}(V) \cap V_i$. Then the U_i are disjoint and $\varphi|_{U_i}$ is one-to-one, since the U_i are subsets of the V_i. In addition

$$\varphi(U_i) = \varphi \circ \varphi^{-1}(V) \cap \varphi(V_i) = V \cap \varphi(V_i) = V,$$

and the proof of the claim is complete.

The collection $\mathcal{F}$ is endowed with the partial order '$\leq$' such that

$$\{U_1, \ldots, U_N\} \leq \{V_1, \ldots, V_N\} \quad \text{if } U_i \subseteq V_i, \quad \text{for all } i = 1, \ldots, N.$$

Let $\mathfrak{C} = \{\{U_1^k, \ldots, U_N^k\} \mid k \in I\}$ be a chain in $\mathcal{F}$, with the understanding that, when $\{U_1^k, \ldots, U_N^k\} \leq \{U_1^l, \ldots, U_N^l\}$, $U_i^k \subseteq U_i^l$ for all $i = 1, \ldots, N$. Then the element $\{\bigcup_k U_1^k, \ldots, \bigcup_k U_N^k\}$ is an upper bound for $\mathfrak{C}$ inside $\mathcal{F}$. Indeed, it suffices to prove that the $\bigcup_k U_i^k$ are disjoint (with respect to the indices i). If there were an x in two such unions, then there would be some $k, l \in I$ such that $x \in U_i^k \cap U_j^l$. Without loss of generality, assume $\{U_1^k, \ldots, U_N^k\} \leq \{U_1^l, \ldots, U_N^l\}$, so that $x \in U_i^k \cap U_j^l \subseteq U_i^l \cap U_j^l = \emptyset$, which is absurd. Thus the collection $\mathcal{F}$ has a maximal element by Zorn's Lemma. From now on, fix this maximal element to be $\{U_1, \ldots, U_N\}$. By definition, the sets $U_1, \ldots, U_N$ satisfy the properties (1) and (2) of the statement.

Secondly, we prove that $X = \varphi(U_i) \cup \partial\varphi(U_i)$, for $i = 1, \ldots, N$, where $\{U_1, \ldots, U_N\}$ is the maximal family constructed above. Since $X \setminus \varphi(U_i)$ is closed, it suffices to show that it has empty interior. To this end, let V be an open neighbourhood of some $y \in \operatorname{int}(X \setminus \varphi(U_i))$ with N pre-images $x_1, \ldots, x_N$. Then $\varphi^{-1}(V)$ is open, contains the x_i and $\varphi^{-1}(V) \cap (\bigcup_{i=1}^N U_i) = \emptyset$. Indeed, if $z \in \varphi^{-1}(V) \cap (\bigcup_{i=1}^N U_i)$, then $\varphi(z) = V \cap \varphi(U_i) = \emptyset$, which is absurd. As in the proof of the claim above, we can find neighbourhoods V_i of x_i inside $\varphi^{-1}(V)$ such that V_i are disjoint, $\varphi|_{V_i}$ is one-to-one and $\varphi(V_i) = V$, perhaps by passing to a sub-neighbourhood of y. Therefore, the family $\{U_1 \cup V_1, \ldots, U_N \cup V_N\}$ is in $\mathcal{F}$, which contradicts the maximality of $\{U_1, \ldots, U_N\}$.

Thirdly, we show that $X = \bigcup_{i=1}^N (U_i \cup \partial U_i)$. It suffices to show that the closed set $X \setminus (\bigcup_{i=1}^N U_i)$ has empty interior. Indeed, in this case, it will coincide with its boundary, and hence with $\partial(\bigcup_{i=1}^N U_i)$. Since the U_i are open and disjoint, this boundary will be $\bigcup_{i=1}^N \partial(U_i)$. To this end, let U be an open neighbourhood of an element x in the interior of $X \setminus (\bigcup_{i=1}^N (U_i \cup \partial U_i))$. If there were an $x' \in U$ such that $\varphi(x') \in \varphi(U_i)$, then $\varphi(x')$ would have $N + 1$ pre-images, which is a contradiction. Indeed, recall that $\varphi(U_i) = \varphi(U_j)$ and $U_i \cap U_j = \emptyset$. Therefore, $\varphi(U)$ is contained in the interior of $X \setminus \varphi(U_i)$. But $X \setminus \varphi(U_i)$ has empty interior, which gives the contradiction.

Finally, let $x \in \partial U_i$. If $\varphi(x) \in \varphi(U_i)$, then the element $\varphi(x)$ would have $N+1$ pre-images, which is a contradiction. Therefore, $\varphi(\partial U_i) \subseteq X \setminus \varphi(U_i) = \partial\varphi(U_i)$. Note, also, that by this construction we obtain

$$\varphi^{-1}(\partial\varphi(U_i)) = \varphi^{-1}(X \setminus \varphi(U_i)) = X \Big\backslash \bigcup_{j=1}^{N} U_j = \bigcup_{j=1}^{N} U_j,$$

for all $i = 1, \ldots, N$, and the proof of the lemma is complete. □

Let φ be as in Lemma 3.1, such that φ and φ^{-1} preserve the null sets. If $\{U_i\}_{i=1}^N$ is the family satisfying the properties of Lemma 3.1 and

the ∂U_i (equivalently the $\varphi(\partial U_i)$) are null sets,

then the $*$-endomorphism $\alpha\colon L^\infty(X,\mu) \to L^\infty(X,\mu) : f \to f \circ \varphi$ is implemented by a Cuntz family. Indeed, let $X_0 = \bigcup_{i=1}^N U_i$ and $Y_0 = \varphi(U_i)$. Then $\varphi_0 := \varphi|_{X_0}$ has N Borel sections ψ_i, for $i = 1, \ldots, N$, with

$$\psi_i = [\varphi|_{U_i}]^{-1}\colon Y_0 \to X_0.$$

Moreover, φ_0 and φ_0^{-1} preserve the null sets. By Proposition 2.2 there is a Cuntz family $\{S_i\}_{i=1}^N$ with

$$S_i\colon L^2(Y_0, \mu|_{Y_0}) \to L^2(X_0, \mu|_{X_0})$$

that implements the representation

$$L^\infty(X_0, \mu|_{X_0}) \ni f \mapsto f \circ \varphi_0 \in L^\infty(Y_0, \mu|_{Y_0}).$$

Since X_0 and Y_0 are almost equal to X, the family $\{S_i\}_{i=1}^N$ implements α.

Given a decomposition of X as above and a finite word $\boldsymbol{i} = i_1 \ldots i_k$ in $\{1, \ldots, N\}$ we can define the Borel sets

$$U_{i_1 i_2 \cdots i_k} = \{x \in X \mid x \in U_{i_1}, \ldots, \varphi^{k-1}(x) \in U_{i_k}\}.$$

This definition is extended to infinite words $\boldsymbol{i} = i_1 i_2 \ldots i_k \ldots$ with the understanding that $U_{\boldsymbol{i}} = \bigcap_k U_{i_1 \ldots i_k}$.

Theorem 3.2. *Let (X, μ, φ) be a dynamical system such that:*

(1) *φ is a local homeomorphism of X such that each point of X has $N > 1$ pre-images;*
(2) *$\{U_i\}_{i=1}^N$ is a decomposition of X as in Lemma 3.1 such that the ∂U_i are null sets;*
(3) *φ is ergodic and preserves the null sets; and*
(4) *the sets $U_{\boldsymbol{i}}$, for $\boldsymbol{i} \in \mathbb{F}_N^+$ generate the σ-algebra up to sets of measure zero.*

Then $\alpha\colon M_f \mapsto M_{f\circ\varphi}$ admits an extension α_S to $\mathcal{B}(L^2(X, m))$ which is ergodic. Furthermore, α_S defines (by restriction) an irreducible representation of $\mathcal{O}_N$.

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Proof. Under the assumptions, there is a Cuntz family $\{S_i\}_{i=1}^N$ that implements α and $\alpha_S(T) = \sum_{i=1}^N S_i T S_i^*$, the extension of α to $\mathcal{B}(L^2(X,\mu))$, is a weak*-continuous endomorphism of $\mathcal{B}(L^2(X,\mu))$. Consequently, $\mathcal{N}_{\alpha_S} = \{T \in \mathcal{B}(\mathcal{H}) \mid \alpha_S(T) = T\}$ is a von Neumann algebra. Fix a projection $P \in \mathcal{N}_{\alpha_S}$. Then $\alpha_S(P) = P$ implies that $S_i P = P S_i$ and $S_i^* P = P S_i^*$, for all $i = 1, \ldots, N$. In particular, P commutes with the range projections of the S_i and the products of the S_i. But these projections are the characteristic functions of the sets $U_{\boldsymbol{i}}$, for the words $\boldsymbol{i}$ on the symbols $\{1, \ldots, N\}$. Since the sets $U_{\boldsymbol{i}}$ generate the σ-algebra up to null sets, the linear span of these projections is weak*-dense in $L^\infty(X,\mu)$. It follows that P is in the MASA (maximal abelian subalgebra) $L^\infty(X,\mu)$, and hence $P = \chi_E$ for a measurable set E. However,

$$M_{\chi_E} = P = \alpha_S(P) = \alpha(P) = M_{\chi_E \circ \varphi}$$

and ergodicity of φ implies that E is either X or $\emptyset$. Thus we obtain $\mathcal{N}_{\alpha_S} = \mathbb{C}I$. The second part of the theorem follows by the comments after [7, Definition 3.2]. □

We give examples of dynamical systems that satisfy the conditions of Theorem 3.2.

Examples 3.3. The first example is the canonical Cuntz–Krieger example of a dynamical system associated with Cuntz isometries. Let $N \in \mathbb{N}$ and

$$X = \prod_{k=1}^{\infty} \{1, \ldots, N\}_k \text{ with measure } \mu = \prod_{k=1}^{\infty} \mu_k,$$

where each $\mu_k = \mu_j$ for all j, k and such that $\mu_k(A) = |A|/N$ for all $A \subset \{1, \ldots, N\}$. If we consider X as a compact abelian group, with 'odometer' addition, then μ is the Haar measure on X. Let φ denote the shift map $\varphi(i_1, i_2, \ldots) = (i_2, i_3, \ldots)$, which is an N-to-one local homeomorphism. Then φ is ergodic and the conditions of the theorem are satisfied for the cylinder sets $U_i := \{(i_1, i_2, \ldots) \mid i_1 = i\}$ (which are clopen so that $\partial U_i = \emptyset$).

A second example arises when X is the circle $\mathbb{T}$, μ is Lebesgue measure and φ is a finite Blaschke product with $N > 1$ factors and zero Denjoy–Wolf fixed point (that is, at least one of the Blaschke factors is z). Then φ is ergodic and the sets U_i are arcs on the circle, so the condition $\mu(\partial U_i) = 0$ is satisfied. This example is considered in [1].

In view of Theorem 3.2, one can ask whether the σ-algebra generated by the sets $U_{\boldsymbol{i}}$ with $\boldsymbol{i} \in \mathbb{F}_N^+$ always generates the full σ-algebra of measurable sets, up to measure zero. This is not true, as the following example shows.

Example 3.4. Let (X, μ, φ) be the canonical Cuntz–Krieger example as above. Also let τ be an irrational rotation on the circle $\mathbb{T}$ with Lebesgue measure. Set $Y = X \times \mathbb{T}$, $\sigma(x, z) = (\varphi(x), \tau(z))$ and $\nu = \mu \times \lambda$. Then (Y, ν, σ) is ergodic as the product of the mixing shift map with the ergodic irrational rotation. Let $U_i = \{(i_1, i_2, \ldots) \in X \mid i_1 = i\}$ and $V_i = U_i \times [0, 1]$. Then the V_i are as in Lemma 3.1, but the $V_{\boldsymbol{i}}$ with $\boldsymbol{i} \in \mathbb{F}_N^+$ do not suffice to generate the σ-algebra of measurable sets up to measure zero.

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4. Uniqueness of the extension

The reader is referred to the work of Paschke [11] for an introduction to W*-modules and to [8, 9] for the general theory of C*-modules.

Definition 4.1. Let $\mathcal{M}$ be a Hilbert module over a unital C*-algebra A. A subset $\{\xi_1, \ldots, \xi_N\}$ of $\mathcal{M}$ is said to be an *orthonormal basis for* $\mathcal{M}$ if $\xi_i \in \mathcal{M}$, $\langle \xi_i, \xi_j \rangle = \delta_{ij} 1_A$ and

$$\xi = \sum_{i=1}^{N} \xi_i \cdot \langle \xi_i, \xi \rangle \quad \text{for all } \xi \in \mathcal{M}.$$

In the case where $N = \infty$, the sum is understood as norm-convergent.

It follows that $\sum_{i=1}^{N} \theta_{\xi_i, \xi_i} = \mathrm{id}_{\mathcal{M}}$, with the understanding that the sum is convergent in the strong topology when $N = \infty$. When A is nonunital, we define the basis of $\mathcal{M}$ by using the unitisation $A^1 = A + \mathbb{C}$. Indeed, we can extend the right action to A^1 by

$$\xi \cdot (a + \lambda) = \xi \cdot a + \lambda \xi,$$

for all $a \in A$ and $\lambda \in \mathbb{C}$. Then the basis of $\mathcal{M}$ over A is defined as the basis of $\mathcal{M}$ over A^1. This is just to ensure that the formula $\langle \xi_i, \xi_j \rangle = \delta_{ij} 1_{A^1}$ makes sense.

Remark 4.2. In general, a Hilbert module may not have an orthonormal basis. However, W*-modules have a basis $\{\xi_i\}$ such that $\langle \xi_i, \xi_i \rangle$ is a projection [11, Theorem 3.12]. Moreover, the size of an orthonormal basis is not well defined, meaning that there may be bases $\{s_i\}_{i \in I}$ and $\{t_j\}_{j \in J}$ with $|I| \neq |J|$. The reason is that the uniqueness of the linear combinations is not guaranteed. For a counterexample, let $\mathcal{M} = \mathcal{O}_2$ be the trivial Hilbert module over itself, where $\mathcal{O}_2$ is the Cuntz algebra on two generators, say s_1 and s_2. Then the sets $\{1_{\mathcal{O}_2}\}$ and $\{s_1, s_2\}$ are both bases for the Hilbert module. Indeed, for $\xi \in \mathcal{O}_2$, we trivially have that $\xi = 1_{\mathcal{O}_2} \cdot \langle 1_{\mathcal{O}_2}, \xi \rangle$ and

$$\xi = (s_1 s_1^* + s_2 s_2^*)\xi = s_1 \cdot \langle s_1, \xi \rangle + s_2 \cdot \langle s_2, \xi \rangle,$$

since $s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{O}_2}$.

Therefore, the trivial Hilbert module $\mathcal{M} = \mathcal{O}_2$ over $\mathcal{O}_2$ is unitarily equivalent to the (interior) direct sum $\mathcal{M} + \mathcal{M}$ over $\mathcal{O}_2$ by the unitary $U = [s_1 \;\; s_2]$. Inductively, we get that $\mathcal{M}$ is unitarily equivalent to $\sum_{i=1}^{n} \mathcal{M}$ for all $n \in \mathbb{N}$.

Similarly, $\mathcal{O}_N$ is unitarily equivalent to $\sum_{i=1}^{n} \mathcal{O}_N$ for an infinite number of $n \in \mathbb{N}$. This fact was later considered and examined in much more generality by Gipson [4].

Remark 4.3. Nevertheless, when the Hilbert module is over a stably finite C*-algebra A, the size is unique. Indeed, let $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$ be two orthonormal bases of such a Hilbert module $\mathcal{M}$ and form the rectangular matrix $U = [\langle \xi_i, \eta_j \rangle]$. Then the (i, j)-entry of the $|I| \times |J|$ matrix UU^* is

$$\sum_{k=1}^{|J|} \langle \xi_i, \eta_k \rangle \langle \eta_k, \xi_j \rangle = \sum_k \langle \xi_i, \eta_k \langle \eta_k, \xi_j \rangle \rangle = \langle \xi_i, \xi_j \rangle = \delta_{ij} 1_A.$$

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Analogous computations for U^*U show that U is a unitary in $M_{|I|,|J|}(A)$. Since A is stably finite, we get that $|I| = |J|$. In fact,

$$[\eta_1, \ldots, \eta_N] = [\xi_1, \ldots, \xi_N][U_{ij}],$$

and the unitary U is in $M_N(A)$. In contrast to [7] the unitary U may not be in $M_N(\mathbb{C})$.

Let $\alpha\colon L^\infty(X,\mu) \to L^\infty(X,\mu)$ be a *-homomorphism and define the linear space

$$\mathcal{E}(X,\mu) = \{T \in \mathcal{B}(L^2(X,\mu)) \mid Ta = \alpha(a)T \text{ for all } a \in L^\infty(X,\mu)\}.$$

Then $\mathcal{E}(X,\mu)$ becomes a Hilbert module over $L^\infty(X,\mu)$ by defining

$$S \cdot a := Sa \quad \text{and} \quad \langle S, T\rangle := S^*T$$

for all $a \in L^\infty(X,\mu)$ and $S, T \in \mathcal{E}(X,\mu)$. Indeed, for $b \in L^\infty(X,\mu)$, we obtain

$$(Sa)b = Sab = Sba = (Sb)a = (\alpha(b)S)a = \alpha(b)(Sa),$$

and thus $Sa \in \mathcal{E}(X,\mu)$. Also, we have that

$$\langle S, T\rangle \cdot b = (S^*T)b = S^*Tb = S^*\alpha(b)T = bS^*T = b \cdot \langle S, T\rangle$$

for all $b \in L^\infty(X,\mu)$, which implies that $\langle S, T\rangle \in L^\infty(X,\mu)' = L^\infty(X,\mu)$. Thus, the inner product and the right action are well defined, and routine calculations show that $\mathcal{E}(X,\mu)$ is a Hilbert module over $L^\infty(X,\mu)$. In particular, the Hilbert module $\mathcal{E}(X,\mu)$ becomes a W*-correspondence over $L^\infty(X,\mu)$ by defining

$$a \cdot S = \alpha(a)S \quad \text{for all } a \in L^\infty(X,\mu) \text{ and } S \in \mathcal{E}(X,\mu).$$

Indeed, for $b \in L^\infty(X,\mu)$,

$$(a \cdot S)b = \alpha(a)Sb = \alpha(a)\alpha(b)S = \alpha(b)\alpha(a)S = \alpha(b)(a \cdot S),$$

and hence $a \cdot S \in \mathcal{E}(X,\mu)$.

It is evident that $\mathcal{E}(X,\mu)$ is a weak*-closed subspace of $\mathcal{B}(L^2(X,\mu))$. Hence, as a self-dual W*-correspondence, it receives a basis $\{S_i\}_{i\in\mathcal{I}}$ such that $\langle S_i, S_j\rangle = S_i^*S_j = 0$ when $i \neq j$, $\langle S_i, S_i\rangle = S_i^*S_i$ is a projection in $L^\infty(X,\mu)$, and $T = \sum_i S_iS_i^*T$ for all $T \in \mathcal{E}(X,\mu)$ [11, Theorem 3.12].

LEMMA 4.4. *Let $\{S_i\}_{i=1}^N$ be a basis for $\mathcal{E}(X,\mu)$ with $N < \infty$. Then the following are equivalent:*

(1) *$\{S_i\}_{i=1}^N$ is an orthonormal basis for $\mathcal{E}(X,\mu)$; and*
(2) *$\{S_i\}_{i=1}^N$ is a Cuntz family that implements α on $L^\infty(X,\mu)$.*

PROOF. For convenience, we write $I \in \mathcal{B}(L^2(X,\mu))$ also for the unit of $L^\infty(X,\mu)$. Since $\{S_i\}_{i=1}^N$ is a basis, we obtain $I = \sum_{i=1}^N \theta_{S_i,S_i} = S_iS_i^*$. Moreover, $S_i \in \mathcal{E}(X,\mu)$, and thus $S_ia = \alpha(a)S_i$ for all $a \in L^\infty(X,\mu)$. Hence,

$$\sum_{i=1}^N S_iaS_i^* = \alpha(a)\sum_{i=1}^N S_iS_i^* = \alpha(a).$$

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Conversely, if $\{S_i\}_{i=1}^N$ is a Cuntz family, then $\langle S_i, S_j\rangle = \delta_{ij} I$ and $\sum_{i=1}^N S_i S_i^* = I$, since $\alpha(I) = I$ for the unit $I \in L^\infty(X,\mu)$. If $\alpha(a) = \sum_{i=1}^N S_i a S_i^*$, then $S_i a = \alpha(a) S_i$ for all $a \in L^\infty(X,\mu)$, where $i = 1,\dots,N$. Thus we obtain $S_i \in \mathcal{E}(X,\mu)$. For $T \in \mathcal{E}(X,\mu)$, set $a_i = \langle S_i, T\rangle = S_i^* T$. Then

$$\sum_{i=1}^N S_i a_i = \sum_{i=1}^N S_i S_i^* T = T,$$

and the proof is complete. □

Let $\{S_1,\dots,S_N\}$ be an orthonormal basis of $\mathcal{E}(X,\mu)$, and let the extension α_S of α be given by

$$\alpha_S : \mathcal{B}(L^2(X,\mu)) \to \mathcal{B}(L^2(X,\mu)) : R \mapsto \sum_{i=1}^N S_i R S_i^*.$$

We can then define the linear space

$$\mathcal{H}_S := \{T \in \mathcal{B}(L^2(X,\mu)) \mid TR = \alpha_S(R) T \text{ for all } R \in \mathcal{B}(L^2(X,\mu))\}.$$

It becomes a Hilbert space endowed with the inner product

$$\langle T_1, T_2\rangle = T_1^* T_2 \quad \text{for all } T_1, T_2 \in \mathcal{H}_S.$$

Indeed, it is easy to check that $\langle T_1, T_2\rangle \in \mathcal{B}(L^2(X,\mu))' = \mathbb{C}$. Moreover, it has dimension N and the Cuntz family $\{S_i\}_{i=1}^N$ is in $\mathcal{H}_S$. The proof is the same as in Remark 4.2 taking into account that $\alpha_S(R) S_j = S_j R$, for all $R \in \mathcal{B}(L^2(X,\mu))$. These results were established by Laca [7].

Proposition 4.5. *Let $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ be two orthonormal bases for $\mathcal{E}(X,\mu)$. Then the following are equivalent:*

(1) *the unitary U that induces a pairing of the bases is in $M_N(\mathbb{C})$; and*
(2) *the extensions α_S and α_Q in $\mathcal{B}(L^2(X,\mu))$ coincide.*

Proof. For convenience, we write $I \in \mathcal{B}(L^2(X,\mu))$ also for the unit of $L^\infty(X,\mu)$.
(1) ⇒ (2): we compute

$$\begin{aligned}
\alpha_Q(R) = \sum_{i=1}^N Q_i R Q_i^* &= \sum_{i,j,k=1}^N S_j \langle S_j, Q_i\rangle R \langle Q_i, S_k\rangle S_k^* \\
&= \sum_{k,j=1}^N S_j R \sum_{i=1}^N \langle S_j, Q_i\rangle\langle Q_i, S_k\rangle S_k^* \\
&= \sum_{k,j=1}^N S_j R \delta_{j,k} S_k^* = \sum_{k=1}^N S_k R S_k^* = \alpha_S(R),
\end{aligned}$$

since $\sum_{i=1}^N \langle S_j, Q_i\rangle\langle Q_i, S_k\rangle$ is the (j,k)-entry of $UU^* = I$, and the entry $\langle S_j, Q_i\rangle$ of U is in $\mathbb{C}$.

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(2) ⇒ (1): if $\alpha_Q = \alpha_S$, then by definition $\mathcal{H}_S = \mathcal{H}_Q$. Thus,

$$S_i^* Q_k R = S_i^* \alpha_Q(R) Q_k = S_i^* \alpha_S(R) Q_k = R S_i^* Q_k$$

for all $R \in \mathcal{B}(L^2(X,\mu))$, and hence $S_i^* Q_k \in \mathcal{B}(L^2(X,\mu))' = \mathbb{C}$. □

As a consequence, we have a complete invariant for the multiplicity n crossed products on $L^\infty(X,\mu)$ [12]. Recall that, given a $*$-endomorphism $\alpha\colon A \to A$ of a C*-algebra A, *the multiplicity n crossed product* $A \times_\alpha^n \mathbb{N}$ is the enveloping C*-algebra generated by $\pi(A)$ and a Toeplitz–Cuntz family $\{Q_i\}_{i=1}^n$, such that π is a nondegenerate representation of A and $\pi(\alpha(a)) = \sum_{i=1}^n Q_i \pi(a) Q_i^*$, for all $a \in A$. When α is unital, nondegeneracy of π is redundant and $\{Q_i\}_{i=1}^n$ can be considered to be a Cuntz family [6, Section 3 and Proposition 3.1]. In [6, Subsection 3.3] we introduced the semicrossed product $A \times_\alpha \mathcal{T}_n^+$ as the noninvolutive subalgebra of $A \times_\alpha^n \mathbb{N}$ generated by $\pi(A)$ and $\{Q_i\}_{i=1}^n$.

Corollary 4.6. *Let α be a unital weak*-continuous isometric endomorphism of $L^\infty(X,\mu)$ and suppose that there is a representation $(\mathrm{id}, \{S_i\}_{i=1}^n)$ of Stacey's crossed product $L^\infty(X,\mu) \times_\alpha^n \mathbb{N}$ on $L^2(X,\mu)$. Then the following are equivalent:*

(1) *$L^\infty(X,\mu) \times_\alpha^n \mathbb{N} \simeq L^\infty(X,\mu) \times_\alpha^m \mathbb{N}$ via a $*$-isomorphism that fixes $L^\infty(X,\mu)$ elementwise;*
(2) *there is a representation $(\mathrm{id}, \{Q_i\}_{i=1}^m)$ of $L^\infty(X,\mu) \times_\alpha^m \mathbb{N}$ acting on $L^2(X,\mu)$;*
(3) *$n = m$; and*
(4) *$L^\infty(X,\mu) \times_\alpha \mathcal{T}_n^+ \simeq L^\infty(X,\mu) \times_\alpha \mathcal{T}_m^+$ via a completely isometric isomorphism that fixes $L^\infty(X,\mu)$ elementwise.*

Proof. The fact that α is an isometric endomorphism of a C*-algebra implies that it is a $*$-homomorphism of the C*-algebra $L^\infty(X,\mu)$ and the multiplicity n crossed products are well defined. The implication (3) ⇒ (4) is immediate.
(4) ⇒ (1): by [6, Theorem 3.13] the C*-algebra $L^\infty(X,\mu) \times_\alpha^n \mathbb{N}$ is the C*-envelope of $L^\infty(X,\mu) \times_\alpha \mathcal{T}_n^+$, and thus the completely isometric isomorphism of (4) extends to a $*$-isomorphism of the corresponding C*-algebras.
(1) ⇒ (2): if Φ is the $*$-isomorphism, let $Q_i := \Phi(S_i)$.
(2) ⇒ (3): let $(\mathrm{id}, \{S_i\}_{i=1}^n)$ and $(\mathrm{id}, \{Q_i\}_{i=1}^m)$ be two such representations. Then α is implemented by $\{S_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^m$, and thus they define a basis for $\mathcal{E}(X,\mu)$. Therefore $n = m$, by Remark 4.3. □

5. Existence of a transfer operator

In general, the mapping $C(X) \ni f \stackrel{C_\varphi}{\longmapsto} f \circ \varphi \in L^2(X,\mu)$ may not extend to an operator on the Hilbert space $L^2(X,\mu)$. However, if

$$c_0 \|\xi\|_2 \le \|C_\varphi \xi\|_2 \le c_1 \|\xi\|_2 \quad \text{for all } \xi \in L^2(X,\mu),$$

then C_φ is an injective operator in $\mathcal{B}(L^2(X,\mu))$, and φ^{-1} preserves the null sets. The map μ is called *φ-bounded* if there is a constant $K > 0$ such that $\mu(\varphi(E)) \le K\mu(E)$, for all measurable sets $E \subset X$. In this case, φ also preserves the μ-null sets.

Under these assumptions let $C_\varphi = S_\varphi a_\varphi$ be the polar decomposition of C_φ. Then S_φ is an isometry and a_φ is invertible. We can check that, by definition, $C_\varphi a = \alpha(a)C_\varphi$ for all $a \in L^\infty(X,\mu)$, and hence $C_\varphi \in \mathcal{E}(X,\mu)$. Therefore, $a_\varphi^2 = C_\varphi^* C_\varphi \in L^\infty(X,\mu)'$, so $a_\varphi \in L^\infty(X,\mu)$. Consequently, the isometry $S_\varphi = C_\varphi a_\varphi^{-1}$ is also in $\mathcal{E}(X,\mu)$ and the mapping

$$\mathcal{L}\colon L^\infty(X,\mu) \to L^\infty(X,\mu) : a \mapsto S_\varphi^* a S_\varphi$$

defines a *transfer operator of* α: that is, $\mathcal{L}$ is positive and $a\mathcal{L}(b) = \mathcal{L}(\alpha(a)b)$, for all $a, b \in L^\infty(X,\mu)$. Following Exel [3], let the semi-inner-product on the $L^\infty(X,\mu)$-module $L^\infty(X,\mu)_{\mathcal{L}}$ be given by

$$\langle \eta, \xi \rangle_{\mathcal{L}} = \mathcal{L}(\eta^* \xi), \quad \text{and} \quad \xi \cdot a = \xi \alpha(a),$$

for all $\eta, \xi, a \in L^\infty(X,\mu)$.

Proposition 5.1. *Assume that μ is φ-bounded and that C_φ is a bounded below operator of $\mathcal{B}(L^2(X,\mu))$. Then $L^\infty(X,\mu)_{\mathcal{L}}$ is a Hilbert module over $L^\infty(X,\mu)$ and, as a vector space, it coincides with $L^\infty(X,\mu)$.*

Proof. It suffices to show that the norm $\|\cdot\|_{\mathcal{L}}$ on the module $L^\infty(X,\mu)_{\mathcal{L}}$ is equivalent to the norm $\|\cdot\|$ of $L^\infty(X,\mu)$.

First, we show that there is a constant M such that $\|a\| \le M\|aS_\varphi\|$ for every $a \in L^\infty(X,\mu)$. Since $\|aS_\varphi\|^2 = \|\,|a|S_\varphi\|^2$ and $\|a\| = \|\,|a|\,\|$, it is enough to show that the relation $\|a\| \le M\|aS_\varphi\|$ holds for all positive a in the norm-dense subspace of simple functions. To this end, let $a = \sum_{i=1}^n d_i \chi_{E_i}$, where the sets E_i are disjoint and of positive measure, and $d_1 > d_2 > \cdots > d_n > 0$. Hence $\|a\| = d_1$. To compute the norm $\|aS_\varphi\|$ we let a act on unit vectors in the range of C_φ; equivalently with unit vectors in the range of S_φ. Let $E = E_1$ and $\xi = \mu(\varphi^{-1}(\varphi(E)))^{-1/2}\chi_{\varphi^{-1}(\varphi(E))}$. Then ξ is a unit vector in the range of S_φ. Also, the assumptions on S_φ and μ imply that $\mu(\varphi^{-1}(\varphi(E))) \le c_1^2\mu(\varphi(E)) \le c_1^2 K\mu(E)$. Therefore,

$$\begin{aligned}\|aS_\varphi\|^2 \ge \|a\xi\|_2^2 &= \int_X a^2|\xi|^2\,d\mu \ge \int_X d_1^2\chi_E|\xi|^2\,d\mu \\ &= \frac{1}{\mu(\varphi^{-1}(\varphi(E)))}\int_X d_1^2\chi_E\,d\mu = \frac{\mu(E)}{m(\varphi^{-1}(\varphi(E)))}d_1^2 \ge \frac{1}{c_1^2K}d_1^2.\end{aligned}$$

Since $\|a\| = d_1$, we have that $\|aS_\varphi\| \ge (1/c_1\sqrt{K})\|a\|$ on a norm-dense subspace.

The above inequality gives the equivalence of the norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|$. Indeed,

$$\begin{aligned}\frac{1}{M^2}\|a\|^2 &\le \|\,|a|S_\varphi\|^2 = \|S_\varphi^*|a|^2S_\varphi\| = \|\mathcal{L}(a^*a)\| \\ &= \|a\|_{\mathcal{L}}^2 = \|S_\varphi^*a^*aS_\varphi\| \le \|a^*a\| = \|a\|^2,\end{aligned}$$

where we have used the fact that S_φ is an isometry. The proof is now complete. □

The following theorem is the analogue of [1, Theorem 5.2].

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THEOREM 5.2. *Assume that μ is φ-bounded and that C_φ is a bounded below operator of $\mathcal{B}(L^2(X,\mu))$. Then the following are equivalent:*

(1) $\{\xi_i\}_{i=1}^N$ *is an orthonormal basis for the Hilbert module $L^\infty(X,\mu)_{\mathcal{L}}$;*
(2) $\{\xi_i S_\varphi\}_{i=1}^N$ *is an orthonormal basis for the Hilbert module $\mathcal{E}(X,\mu)$; and*
(3) $\{\xi_i S_\varphi\}_{i=1}^N$ *is a Cuntz family that implements α.*

PROOF. It will be convenient to denote $S_i := \xi_i S_\varphi$. Note that, by definition, $S_i \in \mathcal{E}(X,\mu)$ and recall that the equivalence (2) $\Leftrightarrow$ (3) is Lemma 4.4. We write $I \in \mathcal{B}(L^2(X,\mu))$ also for the unit of $L^\infty(X,\mu)$. The constant function of $L^2(X,\mu)$ will be denoted by $\mathbf{1}$.
(1) $\Rightarrow$ (3): First we have that the S_i have orthogonal ranges, since

$$S_i^* S_j = S_\varphi^* \xi_i^* \xi_j S_\varphi = \mathcal{L}(\xi_i^* \xi_j) = \langle \xi_i, \xi_j \rangle_{\mathcal{L}} = \delta_{ij} I.$$

Recall that the constant function $\mathbf{1}\colon X \to \mathbb{C}$ is a separating vector and $C_\varphi(\mathbf{1}) = \mathbf{1} \circ \varphi = \mathbf{1}$. Therefore,

$$\begin{aligned} S_i a S_i^*(\mathbf{1}) &= S_i a S_i^* C_\varphi(\mathbf{1}) = \xi_i S_\varphi a S_\varphi^* \xi_i^* S_\varphi a_\varphi(\mathbf{1}) \\ &= \alpha(a)\xi_i S_\varphi \mathcal{L}(\xi_i^* \alpha(a_\varphi))(\mathbf{1}) = \alpha(a)\xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a_\varphi)))S_\varphi(\mathbf{1}) \end{aligned}$$

for all $i = 1, \ldots, N$. Since $\{\xi_i\}_{i=1}^N$ defines a basis of $L^\infty(X,\mu)_{\mathcal{L}}$, it then follows that $a = \sum_{i=1}^N \xi_i \cdot \langle \xi_i, a \rangle_{\mathcal{L}} = \sum_{i=1}^N \xi_i \alpha(\mathcal{L}(\xi_i^* a))$ for all $a \in L^\infty(X,\mu)$. Thus, we compute

$$\begin{aligned} \sum_{i=1}^N S_i a S_i^*(\mathbf{1}) &= \sum_{i=1}^N \alpha(a)\xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a_\varphi)))S_\varphi(\mathbf{1}) \\ &= \alpha(a) \sum_{i=1}^N \xi_i \alpha(\mathcal{L}(\xi_i^* \alpha(a_\varphi)))S_\varphi(\mathbf{1}) \\ &= \alpha(a)\alpha(a_\varphi)S_\varphi(\mathbf{1}) = \alpha(a)S_\varphi a_\varphi(\mathbf{1}) = \alpha(a)C_\varphi(\mathbf{1}) = \alpha(a)(\mathbf{1}). \end{aligned}$$

Since $\mathbf{1}$ is a separating vector, we see that $\{S_i\}_{i=1}^N$ implements α.
(3) $\Rightarrow$ (1): Note that the functions ξ_i are orthonormal, since

$$\langle \xi_i, \xi_j \rangle_{\mathcal{L}} = \mathcal{L}(\xi_i^* \xi_j) = C_\varphi^* \xi_i^* \xi_j C_\varphi = S_i^* S_j = \delta_{ij} I.$$

To see that the $\{\xi_i\}_{i=1}^N$ span $L^\infty(X,\mu)_{\mathcal{L}}$, choose an element $a \in L^\infty(X,\mu)$ with $\langle \xi_i, a \rangle_{\mathcal{L}} = 0$ for all i. Then we obtain

$$\begin{aligned} (aS_\varphi)^* &= S_\varphi^* a^* \cdot \sum_{i=1}^N S_i S_i^* = S_\varphi^* a^* \cdot \sum_{i=1}^N \xi_i S_\varphi S_i^* \\ &= \sum_{i=1}^N (S_\varphi^* a^* \xi_i S_\varphi) S_i^* = \sum_{i=1}^N \langle a, \xi_i \rangle_{\mathcal{L}} S_i^* = 0, \end{aligned}$$

so that $aS_\varphi = 0$. Hence $aC_\varphi = 0$, and thus $a(\mathbf{1}) = aC_\varphi(\mathbf{1}) = 0$. Since $\mathbf{1}$ is a separating vector, we obtain $a = 0$. $\square$

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REMARK 5.3. Assume that $\varphi\colon X \to X$ has N Borel sections as in Proposition 2.2. Then the N isometries S_i of Proposition 2.2 can be written as

$$S_i = M_{\chi_{Y_i}} M_{u_i} C_\varphi a_\varphi = M_{\chi_{Y_i}} M_{u_i} \alpha(a_\varphi) C_\varphi = M_{\chi_{Y_i}} M_{u_i} M_{h\circ\varphi} C_\varphi,$$

where u_i are as in Proposition 2.1 for $\psi = \psi_i$ and $a_\varphi = M_h \in L^\infty(X,\mu)$. Therefore, the elements $\xi_i = M_{\chi_{Y_i}} M_{u_i} M_{h\circ\varphi} \in L^\infty(X,\mu)$ define a basis for $L^\infty(X,\mu)_{\mathcal{L}}$.

There is a converse of this scheme that works at the level of $*$-homomorphisms. We would like to thank Philip Gipson for bringing this to our attention. If there is a Cuntz family $\{S_i\}_{i=1}^N$ in $\mathcal{B}(L^2(X,\mu))$ that implements α, then $S_i^* a S_i \in L^\infty(X,\mu)$ for all $i = 1, \ldots, N$. This follows because $L^\infty(X,\mu)$ is a MASA, $S_i b = \alpha(b) S_i$ and

$$S_i^* a S_i \cdot b = S_i^* \alpha(b) a S_i = b \cdot S_i^* a S_i \quad \text{for all } b \in L^\infty(X,\mu).$$

Furthermore, $S_i S_i^*$ commutes with every $a \in L^\infty(X,\mu)$, so the $*$-homomorphisms $\beta_i\colon L^\infty(X,\mu) \to L^\infty(X,\mu)$, given by $\beta_i(a) = S_i^* a S_i$, are N left inverses for α.

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