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FUNDAMENTAL, PICARD, AND CLASS GROUPS OF RINGS OF INVARIANTS

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Let G be an affine algebraic group over the algebraically closed field k, and let V be an affine, normal algebraic variety over k on which G acts. Suppose that the ring of invariants $k[V]^G$ is finitely generated over k, and let W be the affine variety with $k[W] = k[V]^G$. The purpose of this paper is to show that the induced homomorphism from the étale fundamental group of V to that of W is surjective, and to examine the consequences of this observation in terms of the relations between the Picard and divisor class groups of $k[V]$ and $k[W]$. We show that if n is an integer prime to the characteristic of k and if every unit in $k[V]$ is a unit of $k[W]$ times an n-th power of a unit of $k[V]$, then the inclusion $k[V] \subset k[W]$ induces a monomorphism on the n-torsion part of their Picard groups. If, in addition, the inverse image of the singular locus of W in V has no component of codimension one, then the inclusion also induces a monomorphism on the n-torsion part of the class groups of the rings. The condition on units will be automatically satisfied for linear actions, and in these cases we can conclude that there is no n-torsion in the Picard group, and if the hypothesis on the singular locus holds, no n-torsion in the class group of the ring of invariants. We give examples to show that it is still possible that the class group is non-trivial, and may even have torsion. These examples are given for $G = GL_1(k)$, but apply to any group having a non-trivial character. (It is well-known that the ring of invariants of a linear action of a group with no non-trivial characters is always factorial.)

We adopt the following conventions: k is fixed as above, "variety" entails "irreducible," and all our algebraic groups are linear. We identify varieties with their k-rational points.

PROPOSITION 1. *Let $p : X \to Y$ be a dominant morphism of normal k-varieties, and suppose $p^*k(Y)$ is algebraically closed in $k(X)$. Then p induces a surjective homomorphism $\Pi_1(X) \to \Pi_1(Y)$ of fundamental groups.*

Proof. The proof is a straightforward generalization of [**4**, Theorem 11, p. 43]. By [**5**, 5.2.1, p. 94], it must be shown that if Z is a connected étale covering space of Y, then Xx_YZ is connected. Thus let W be a connected component of Xx_YZ. We may assume Z is Galois over Y with group Γ, and hence that W is Galois over X with group Γ' contained in $1 \times \Gamma$. Since X and Y are normal, Z

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and W are irreducible, and we have the following diagram of function fields:

$$\begin{array}{ccc} & k(W) & \\ \diagup & & \diagdown \\ k(X) & & k(Z) \\ \diagdown & & \diagup \\ & k(Y) & \end{array}$$

It is clear that $k(W)$ is the composition of $k(Z)$ and $k(X)$. Now $k(X) \cap k(Z)$ is algebraic over $k(Y)$, being a subextension of $k(Z)$ over $k(Y)$, and $k(X) \cap k(Z)$ is contained in $k(X)$. Thus since $k(Y)$ is algebraically closed in $k(X)$, $k(X) \cap k(Z) = k(Y)$. Now by [**3**, Theorem 4, p. 196], it follows that the restriction of Γ' to $k(Z)$ induces an isomorphism of Γ' with Γ, so that $\Gamma' = 1 \times \Gamma$. But then $W = Xx_YZ$, and hence Xx_YZ is connected.

In order to apply the preceding result to rings of invariants, we need to verify the hypothesis of algebraic closure. The next proposition shows that this will happen for connected groups.

PROPOSITION 2. *Let G be a connected algebraic group over k, and let V be a normal affine k-variety on which G acts. Let K be the quotient field of $k[V]^G$. Then K is algebraically closed in $k(V)$.*

Proof. Let S denote $k[V]^G$, and let y in $k(V)$ be algebraic over K. Then there is an s in S such that $x = sy$ is integral over S, so, since V is normal, x is in $k[V]$. It follows that $H = \{g \in G | gx = x\}$ is a closed subgroup of G. Now choose $f \in S[t]$ monic such that $f(x) = 0$. Then since the coefficients of f are in S, for all g in G, $0 = gf(x) = f(gx)$. Since the G-translates of x all appear among the roots of f, there are only finitely many such translates, and their number is the index of H in G. But since G is connected, its only closed subgroup of finite index is itself, so $H = G$ and $x \in S$. Thus $y = x/s$ is in K.

For later use we combine Propositions 1 and 2 in the following corollary.

COROLLARY 3. *Let $p : X \to Y$ be a dominant morphism of affine k-varieties with X normal. Suppose the connected algebraic group G acts on X and p induces an isomorphism of $k[Y]$ with $k[K]^G$. Let U be an open subset of Y. Then the restriction of p induces a surjective homomorphism $\Pi_1(p^{-1}U) \to \Pi_1(U)$.*

Proof. By Proposition 2, $k(U) = k(Y)$ is algebraically closed in $k(p^{-1}U) = k(X)$. We can apply Proposition 1 since Y, and hence U, is normal since $k[X]^G$ is integrally closed.

In order to apply the above results on fundamental groups to the calculation of class groups, we recall some facts about the étale cohomology of the Kummer sequence (see [**1**, 1.1, p. 102]) : let p be the characteristic of k, let n be an integer prime to p, let V be a k-variety, and let G_m be the sheaf (in the étale topology on V) of non-vanishing functions. Then the n-th power map on G_m gives a short

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exact sequence of sheaves in the étale topology

$$1 \longrightarrow \mathbf{Z}/n\mathbf{Z} \xrightarrow{(\ )^n} G_m \longrightarrow G_m \longrightarrow 1$$

which induces the (étale) cohomology sequence

$$G_m(V) \to G_m(V) \to H^1(V, \mathbf{Z}/n\mathbf{Z}) \to H^1(V, G_m) \to H^1(V, G_m)$$

where the extreme maps are n-th power. Now $H^1(V, \mathbf{Z}/n\mathbf{Z})$ can be identified with $\mathrm{Hom}(\Pi_1(V), \mathbf{Z}/n\mathbf{Z})$ and $H^1(V, G_m)$ with $\mathrm{Pic}(V)$ (the Picard group of V). Thus we have a short exact sequence

$$(*) \quad 1 \to G_m(V)/[G_m(V)]^n \to \mathrm{Hom}(\Pi_1(V), \mathbf{Z}/n\mathbf{Z}) \to \mathrm{Pic}^{(n)}(V) \to 1$$

where $\mathrm{Pic}^{(n)}(V)$ means the n-torsion in $\mathrm{Pic}(V)$. The sequence (*) is clearly contra-functorial in V.

LEMMA 4. *Let $p : V \to W$ be a morphism of k-varieties and n an integer prime to the characteristic of k. Suppose:*

i) *every f in $G_m(V)$ is of the form $f = (p^*g)(h)^n$ for g in $G_m(W)$ and h in $G_m(V)$,*

ii) *p induces a surjection $\Pi_1(V) \to \Pi_1(W)$.*

Then p induces an injection $\mathrm{Pic}^{(n)}(W) \to \mathrm{Pic}^{(n)}(V)$.

Proof. From p and (*) we have the following commutative diagram.

$$\begin{array}{ccccccccc} 1 \to & G_m(W)/G_m(W)^n & \to & \mathrm{Hom}(\Pi_1(W), \mathbf{Z}/n\mathbf{Z}) & \to & \mathrm{Pic}^{(n)}(W) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & G_m(V)/G_m(V)^n & \to & \mathrm{Hom}(\Pi_1(V), \mathbf{Z}/n\mathbf{Z}) & \to & \mathrm{Pic}^{(n)}(V) & \to 1. \end{array}$$

Hypothesis i) implies the left map is onto and hypothesis ii) implies the middle map is one-one. The snake lemma then implies that the right map is injective.

We know by Corollary 3 that hypothesis ii) of Lemma 4 will hold for rings of invariants. Hypothesis i) will hold if, for instance, all the non-vanishing functions are constant, for instance, if the group is acting on a polynomial ring. This leads to the following:

COROLLARY 5. *Let G be a connected algebraic group over k, and suppose G acts linearly on the k-vector space V such that the ring of invariants $k[V]^G$ is finitely generated. Then* $\mathrm{Pic}(k[V]^G)$ *has no torsion prime to the characteristic of k.*

Proof. Let W be the affine k-variety such that $k[W] = k[V]^G$. Then $G_m(V) = k^*$ so hypothesis i) of Lemma 4 holds, and by Corollary 3 (with $U = W$), hypothesis ii) holds. Then for n prime to the characteristic of k, $\mathrm{Pic}^{(n)}(W) \to \mathrm{Pic}^{(n)}(V)$ is injective by Lemma 4, and since $\mathrm{Pic}^{(n)}(V)$ is zero, the result follows.

For regular rings, the Picard group and the divisor class group coincide, so factorality questions can be studied using the Picard group. In general, rings of invariants will not be regular. Their class groups, however, can sometimes be calculated from the Picard groups of their non-singular loci, as the following theorem shows.

THEOREM 6. *Let $p : X \to Y$ be a dominant morphism of affine k-varieties with X normal. Suppose the connected algebraic group G acts on X, that p induces an isomorphism of $k[Y]$ with $k[X]^G$, and that if F is the singular locus of Y, every irreducible component of $p^{-1}(F)$ has codimension at least two in X. Let n be an integer prime to the characteristic of k. If every f in $G_m(V)$ is of the form $f = (p^*g)(h)^n$ for g in $G_m(Y)$ and h in $G_m(V)$, then the n-torsion in the class group of $k[Y]$ injects into the n-torsion in the class group of $k[X]$.*

Proof. Let $U = Y - F$. Since Y is normal, every irreducible component of F has codimension at least two in Y, so $G_m(U) = G_m(Y)$ and $\mathrm{Pic}(U) = \mathrm{Cl}(k[Y])$ (the class group of Y). The assumption on $p^{-1}(F)$ guarantees that $G_m(p^{-1}U) = G_m(X)$ and $\mathrm{Pic}(p^{-1}U) = \mathrm{Cl}(k[X])$. By Corollary 3, $\Pi_1(p^{-1}U) \to \Pi_1(U)$ is surjective, and the above equalities of G_m's show that hypothesis i) of Lemma 4 holds, so that by Lemma 4, $\mathrm{Pic}^{(n)}(U) \to \mathrm{Pic}^{(n)}(p^{-1}U)$ is an injection. The theorem now follows from the identification of Picard and class groups already noted.

The assumption on non-vanishing functions in Theorem 6 will be automatically satisfied in the case of linear actions, as before. We state this as a corollary.

COROLLARY 7. *Let G be a connected algebraic group over k, and suppose G acts linearly on the k-vector space V such that the ring of invariants $R = k[V]^G$ is finitely generated. Let I be the intersection of all the maximal ideals M of R such that R_M is not regular, and suppose that no height one prime of $k[V]$ contains I. Then $\mathrm{Cl}(R)$ has no torsion prime to the characteristic of k.*

Proof. Let Y be the affine k-variety with $k[Y] = R$. The singular locus F of Y is the zeros of I, and the assumption on I guarantees that the inverse image of F has no component of codimension one. Since $G_m(V) = k^*$, the assumption on units of Theorem 6 is satisfied for $V \to Y$, and so Theorem 6 gives an injection on class groups. Since $k[V]$ is factorial, the result follows.

Corollary 7 gives a condition on linear actions of algebraic groups which implies that their rings of invariants have no torsion prime to the characteristic of the base field in their class groups. The following example shows that if the condition does not hold the class groups may, in fact, have such torsion.

Let $G = GL_1(k)$ act on $V = k^{(3)}$ by $t(x, y, z) = (tx, t^2y, t^{-4}z)$. Identify $k[V]$ with $k[x, y, z]$, so that G acts by: $x \to tx$, $y \to t^2y$, $z \to t^{-4}z$. The ring of invariants can be computed by the techniques of [**6**, Theorem 1, p. 218]: it is generated by the invariant monomials, and these in turn are generated by the three monomials x^4z, y^2z and x^2yz, which satisfy the identity $(x^2yz)^2 = (x^4z)(y^2z)$. We have a homomorphism $k[a, b, c]/(c^2 - ab) \to k[x, y, z]^G$ given by $a \to x^4z$, $b \to y^2z$, and

$c \to x^2yz$ which for dimension reasons must be an isomorphism. By [**3**, Proposition 11.4, p. 51], the class group of $k[a, b, c]/(c^2 - ab)$ is cyclic of order 2 and hence, if k has odd characteristic, $k[x, y, z]^G$ does have torsion prime to the characteristic of k. Regarding $k[a, b, c]/(c^2 - ab)$ as the coordinate ring of the surface $c^2 = ab$ in $k^{(3)}$, we can see that the surface is singular only at the origin. Thus the inverse image in V of the singular locus is the zeros of the ideal (x^2z, y^2z, x^2yz), which is the same as the zeros of the ideal (xz, yz, xyz), and this closed algebraic subset of V consists of the three lines $x = y = 0$, $x = z = 0$, $y = z = 0$, and the plane $z = 0$. This plane is then a component of the inverse image of the singular locus of codimension one, which explains why Corollary 7 does not apply to this example. If G is any algebraic group with a non-trivial character, then G can be made to act on V through this character and the action of $GL_1(k)$ just studied, so that the ring of invariants is the same as the above and hence has torsion in its class group.

The techniques of this paper only yield information about the torsion in the class group. The following example shows that even when Corollary 7 applies the class group may be non-trivial.

Let $G = GL_1(k)$ act on $V = k^{(4)}$ by $t(x, y, z, w) = (tx, t^2y, t^{-1}z, t^{-2}w)$. Identify $k[V]$ with $k[x, y, z, w]$ so that G acts by $x \to tx$, $y \to t^2y$, $z \to t^{-1}z$, $w \to t^{-2}w$. Then it turns out that $k[V]^G$ is generated by the monomials xz, yz, x^2w, yz^2 which satisfy the relation $(xz)^2(yw) = (x^2w)(yz^2)$. Thus there is an isomorphism of $k[a, b, c, d]/(a^2b - cd)$ with $k[V]^G$ given by $a \to xz$, $b \to yw$, $c \to x^2w$, and $d \to yz^2$. Regarding the first ring as the coordinate ring of the threefold $a^2b = cd$ in $k^{(4)}$, we see that the singular locus of the threefold is the intersection of the three hyperplanes $a = 0$, $c = 0$ and $d = 0$ with the threefold. It follows that the inverse image in V of the singular locus is the zeros of the ideal (xz, x^2w, yz^2) which is the same as the zeros of the ideal (xz, xw, yz), which is the union of the three planes $x = y = 0$, $x = z = 0$, and $z = w = 0$. Since these all have codimension two in V, by Corollary 7, $k[V]^G$ has no torsion prime to the characteristic of k in its class group. On the other hand, $k[a, b, c, d]/(a^2b - cd)$ is clearly not a unique factorization domain and hence its class group is non-trivial.

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