Ergod. Th. & Dynam. Sys. (1982), **3**, 119–127
Printed in Great Britain

Characteristic exponents of dynamical systems in metric spaces

YURI KIFER
Institute of Mathematics, Hebrew University, Jerusalem, Israel

(*Received* 15 *July* 1982)

Abstract. We introduce for dynamical systems in metric spaces some numbers which in the case of smooth dynamical systems turn out to be the maximal and the minimal characteristic exponents. These numbers have some properties similar to the smooth case. Analogous quantities are defined also for invariant sets.

Section 1

Let S^t be a one-parameter continuous group of homeomorphisms of a metric separable space X with the distance $d(\cdot,\cdot)$, where t is a continuous parameter $-\infty<t<\infty$ or a discrete one $t=\ldots,-1,0,1,\ldots$.

Assume that X has no isolated points then all sets

$$B_x(\delta,T)=\{y\in X\backslash x: d(S^tx,S^ty)\le\delta \text{ for all } t\in[0,T]\}$$

are non-empty. Set

$$A_\delta(x,t)=\sup_{y\in B_x(\delta,t)} d(S^tx,S^ty)/d(x,y) \tag{1}$$

and

$$\alpha_\delta(x,t)=\inf_{y\in B_x(\delta,t)} d(S^tx,S^ty)/d(x,y). \tag{2}$$

Let μ be a Borel S^t-invariant probability measure such that

$$\int_X \sup_{-1\le u\le 1}|\ln A_\delta(x,u)|\mu(dx)+\sup_{-\infty<t<\infty}\frac{1}{|t|}\int_X|\ln A_\delta(x,t)|\mu(dx)<\infty. \tag{3}$$

This condition is satisfied, for instance, when there is $K\ge 1$ such that

$$\sup_{-1\le u\le 1} d(S^ux,S^uy)/d(x,y)\le K \tag{4}$$

for any $x,y\in X$, $x\ne y$ (some kind of Lipschitz condition on the transformations S^t). Indeed, from (4) it follows that

$$K^{-|t|-1}\le d(S^tx,S^ty)/d(x,y)\le K^{|t|+1}$$

for any $x,y\in X$, $x\ne y$ and all $t\in(-\infty,\infty)$. Hence, $|\ln A_\delta(x,t)|\le(|t|+1)\ln K$ that gives (3).

Define

$$\Lambda_\delta^\pm(x)=\limsup_{t\to\pm\infty}\frac{1}{|t|}\ln A_\delta(x,t) \tag{5}$$

and

$$\lambda_\delta^{\pm}(x)=\limsup_{t\to\pm\infty}\frac{1}{|t|}\ln\alpha_\delta(x,t). \tag{6}$$

THEOREM 1. *Let μ be a Borel S^t-invariant probability measure satisfying the condition* (3). *Then*

$$\Lambda_\delta^{+}(x)=\lim_{t\to\infty}\frac{1}{t}\ln A_\delta(x,t)=-\lim_{t\to\infty}\frac{1}{|t|}\ln\alpha_\delta(x,t)=-\lambda_\delta^{-}(x) \tag{7}$$

and

$$\lambda_\delta^{+}(x)=\lim_{t\to\infty}\frac{1}{t}\ln\alpha_\delta(x,t)=-\lim_{t\to-\infty}\frac{1}{|t|}\ln A_\delta(x,t)=-\Lambda_\delta^{-}(x) \tag{8}$$

μ-almost everywhere (μ-a.e.). Moreover $\Lambda_\delta^{\pm}(x)$ and $\lambda_\delta^{\pm}(x)$ are μ-a.e. invariant under S^t and so if μ is ergodic then $\Lambda_\delta^{\pm}(x)$ and $\lambda_\delta^{\pm}(x)$ are constants μ-a.e.

If (3) *is satisfied for all $\delta\in(0,\delta_0)$ with some $\delta_0>0$ then there are*

$$\Lambda^{\pm}(x)=\lim_{\delta\to0}\Lambda_\delta^{\pm}(x)\quad\text{and}\quad\lambda^{\pm}(x)=\lim_{\delta\to0}\lambda_\delta^{\pm}(x). \tag{9}$$

Remark 1. The assumption that $\Lambda^+(x)>0$ and $\lambda^+(x)<0$ for any x (or for x from a set of positive measure) can be interpreted as a very weak kind of hyperbolicity condition. In the smooth case $\Lambda^+(x)$ and $\lambda^+(x)$ turn out to be the maximal and the minimal characteristic exponent, respectively (see § 3).

Remark 2. The part of theorem 1, concerning positive t (i.e. $t\to\infty$) can be proved for the case when S^t is just a semigroup.

Proof. Since

$$B_x(\delta,t+u)\subset B_x(\delta,t)\quad\text{and}\quad B_x(\delta,t+u)\subset B_{S^tx}(\delta,u)$$

then

$$\begin{aligned}A_\delta(x,t+u)&=\sup_{y\in B_x(\delta,t+u)}d(S^{t+u}x,S^{t+u}y)/d(x,y)\\&\le\sup_{y\in B_x(\delta,t)}d(S^tx,S^ty)/d(x,y)\sup_{z\in B_{S^tx}(\delta,u)}d(S^{t+u}x,S^uz)/d(S^tx,z)\\&\le A_\delta(x,t)A_\delta(S^tx,u).\end{aligned} \tag{10}$$

Set

$$Z_{ut}^{\delta}(x)=\ln A_\delta(S^ux,t-u)\quad\text{for } t>u>0\quad\text{or}\quad t<u<0.$$

By (10) it follows that

$$Z_{ut}^{\delta}(x)\le Z_{uv}^{\delta}(x)+Z_{vt}^{\delta}(x)$$

for any $t>v>u>0$ or $t<v<u<0$. Hence Z_{ut}^{δ} is a subadditive process in the sense of Kingman [**2**].

Next, for any numbers $C_1,\dots,C_n$ and $u_1,t_1;u_2,t_2;\dots;u_n,t_n$ one has

$$\begin{aligned}\mu\{x:Z^{\delta}_{u_1+\vartheta t_1+\vartheta}(x)\le C_1,\dots,Z^{\delta}_{u_n+\vartheta t_n+\vartheta}(x)\le C_n\}\\=\mu(x:A_\delta(S^{u_1+\vartheta}x,t_1-u_1)\le e^{C_1},\dots,A_\delta(S^{u_n+\vartheta}x,t_n-u_n)\le e^{C_n}\}\\=\mu\{x:A_\delta(S^{u_1}x,t_1-u_1)\le e^{C_1},\dots,A_\delta(S^{u_n}x,t_n-u_n)\le e^{C_n}\}\\=\mu\{x:Z^{\delta}_{u_1t_1}(x)\le C_1,\dots,Z^{\delta}_{u_nt_n}(x)\le C_n\}\end{aligned}$$

for each ϑ, since μ is S^t-invariant. Therefore $Z_{ut}^{\delta}(x)$ is a stationary process in the sense of [**2**].

https://doi.org/10.1017/S0143385700001838 Published online by Cambridge University Press

Since μ is S^t-invariant then by (3),

$$\int_X \sup_{0\le u<f=v\le 1} |Z^\delta_{uv}(x)|\mu(dx) = \int_X \sup_{0\le u\le v\le 1} |\ln A_\delta(S^u x, v-u)|\mu(dx)$$
$$= \int_X \sup_{0\le u\le v\le 1} |\ln A_\delta(x, v-u)|\mu(dx)$$
$$= \int_X \sup_{0\le s\le 1} |\ln A_\delta(x, s)|\mu(dx) < \infty. \tag{11}$$

The inequalities (3) and (11) guarantee all conditions of theorem 5 from [**2**] and theorem 4 from [**3**]. Thus the limits $\lim_{t\to\pm\infty} (1/|t|)Z^\delta_{0t}(x)$ exist μ-a.e. This proves the left hand side of (7) and the right hand side of (8).

As in (10) one can see that

$$\alpha_\delta(x, t+u) \ge \alpha_\delta(x, t)\cdot \alpha_\delta(S^t x, y) \tag{12}$$

and so $z^\delta_{ut}(x) = -\ln \alpha_\delta(S^u x, t-u)$ is a subadditive and, as above, stationary process in the sense of [**2**]. Since

$$\alpha_\delta(x, t) = \inf_{y\in B_x(\delta,t)} d(S^t x, S^t y)/d(x, y)$$
$$= \Big(\sup_{y\in B_x(\delta,t)} d(x, y)/d(S^t x, S^t y)\Big)^{-1}$$
$$= A_\delta^{-1}(S^t x, -t) \tag{13}$$

and μ is S^t-invariant then by (3),

$$\int_X \sup_{-1\le u\le 1} |\ln \alpha_\delta(x, u)|\mu(dx) = \sup_{-\infty<t<\infty} \frac{1}{|t|}\int_X |\ln \alpha_\delta(x, t)|\mu(dx) < \infty. \tag{14}$$

Therefore, as above, we can apply theorem 5 of [**2**] and theorem 4 of [**3**] to obtain that μ-a.e. the limits $\lim_{t\to\pm\infty} (1/|t|)z^\delta_{0t}(x)$ exist. This proves the right hand side of (7) and the left hand side of (8).

By theorem 3 of [**2**] (see also [**1**]) the functions $\Lambda^\pm_\delta(x)$ and $\lambda^\pm_\delta(x)$ are invariant with respect to the dynamical system S^t and for any S^t-invariant Borel set Q

$$\int_Q \Lambda^\pm_\delta(x)\mu(dx) = \lim_{t\to\pm\infty} \frac{1}{|t|}\int_Q \ln A_\delta(x, t)\mu(dx) \tag{15}$$

and

$$\int_Q \lambda^\pm_\delta(x)\mu(dx) = \lim_{t\to\pm\infty} \frac{1}{|t|}\int_Q \ln \alpha_\delta(x, t)\mu(dx). \tag{16}$$

But μ is S^t-invariant and so by (13),

$$\int_Q \ln \alpha_\delta(x, t)\mu(dx) = -\int_Q \ln A_\delta(S^t x, -t)\mu(dx) = -\int_Q \ln A_\delta(x, -t)\mu(dx). \tag{17}$$

By (15)–(17) we obtain that for any S^t-invariant set Q,

$$\int_Q \Lambda^\pm_\delta(x)\mu(dx) = -\int_Q \lambda^\mp_\delta(x)\mu(dx)$$

and since $\Lambda_\delta^\pm(x)$ and $\lambda_\delta^\pm(x)$ are invariant with respect to S^t then

$$\Lambda_\delta^\pm(x) = -\lambda_\delta^\mp(x) \quad (\mu\text{-a.e.}) \quad ;$$

that completes the proof of (7) and (8). If $\delta_1 < \delta_2$ then, clearly,

$$B_x(\delta_1, t) \subset B_x(\delta_2, t)$$

and so

$$A_{\delta_1}(x, t) \le A_{\delta_2}(x, t) \quad \text{and} \quad \alpha_{\delta_1}(x, t) \ge \alpha_{\delta_2}(x, t)$$

for any $x \in X$ and $t \in (-\infty, \infty)$. It follows from here that

$$\Lambda_{\delta_1}^\pm(x) \le \Lambda_{\delta_2}^\pm(x) \quad \text{and} \quad \lambda_{\delta_1}^\pm(x) \ge \lambda_{\delta_2}^\pm(x)$$

i.e. $\Lambda_\delta^\pm(x)$ increases in δ and $\lambda_\delta^\pm(x)$ decreases in δ. Therefore the limits in (9) exist, completing the proof of theorem 1. □

Section 2

Let Q be an S^t-invariant set and define

$$B_Q(\delta, T) = \{y \in X \backslash Q : d(S^t y, Q) \le \delta \text{ for all } t \in [0, T]\},$$

where $d(x, Q) = \inf_{y \in Q} d(x, y)$. Assume that Q is not an isolated set then $B_Q(\delta, T)$ is not empty for all $\delta > 0$ and $T \in (-\infty, \infty)$.

Set

$$A_\delta(Q, t) = \sup_{y \in B_Q(\delta, t)} \frac{d(S^t y, Q)}{d(y, Q)} \tag{18}$$

and

$$\alpha_\delta(Q, t) = \inf_{y \in B_Q(\delta, t)} \frac{d(S^t y, Q)}{d(y, Q)}. \tag{19}$$

Assume that

$$R \equiv \limsup_{u \to 0} |\ln A_\delta(Q, u)| < \infty. \tag{20}$$

This condition is satisfied if, for instance, (4) is true. Indeed, if (4) holds then

$$K^{-2} \le \frac{d(S^u x, S^u y)}{d(x, y)} \le K^2$$

for any $x, y \in X$ and $u \in [-1, 1]$. But then for any $z \in Q$

$$\frac{d(S^u x, Q)}{d(x, z)} \le K^2 \quad \text{and} \quad \frac{d(S^u x, S^u z)}{d(x, Q)} \ge K^{-2}$$

and so

$$K^{-2} \le \frac{d(S^u x, Q)}{d(x, Q)} \le K^2$$

for any $x \in X$ and $u \in [-1, 1]$ that gives (20).

THEOREM 2. *If* (20) *is true then there exist*

$$\Lambda_\delta^+(Q) = \lim_{t \to \infty} \frac{1}{t} \ln A_\delta(Q, t) = -\lim_{t \to -\infty} \frac{1}{|t|} \ln \alpha_\delta(Q, t) = -\lambda_\delta^-(Q) \tag{21}$$

https://doi.org/10.1017/S0143385700001838 Published online by Cambridge University Press

and

$$\lambda_\delta^+(Q)=\lim_{t\to\infty}\frac{1}{t}\ln\alpha_\delta(Q,t)=-\lim_{t\to-\infty}\frac{1}{|t|}\ln A_\delta(Q,t)=-\Lambda_\delta^-(Q). \tag{22}$$

If (20) *is satisfied for all δ small enough then there are*

$$\Lambda^\pm(Q)=\lim_{\delta\to0}\Lambda_\delta^\pm(Q)\text{ and }\lambda^\pm(Q)=\lim_{\delta\to0}\lambda^\pm(Q). \tag{23}$$

Remark 3. If t is a discrete parameter then (20) is an extra condition.

Proof. If t and u are both positive or both negative then

$$\begin{aligned}A_\delta(Q,t+u)&=\sup_{y\in B_Q(\delta,t+u)}\frac{d(S^{t+u}y,Q)}{d(S^ty,Q)}\frac{d(S^ty,Q)}{(d(y,Q)}\\&\le\sup_{y\in B_Q(\delta,t)}\frac{d(S^ty,Q)}{d(y,Q)}\sup_{z\in B_Q(\delta,u)}\frac{d(S^uz,Q)}{d(z,Q)}\\&=A_\delta(Q,t)\cdot A_\delta(Q,u)\end{aligned} \tag{24}$$

since

$$B_Q(\delta,t+u)\subset B_Q(\delta,t)\quad\text{and}\quad B_Q(\delta,t+u)\subset B_Q(\delta,u). \tag{25}$$

Therefore by the standard result on subadditive functions there are some limits

$$\Lambda_\delta^\pm(Q)\equiv\lim_{n\to\pm\infty}\frac{1}{n}\ln A_\delta(Q,n) \tag{26}$$

for n running over the integers.

By (20) there is $q>0$ such that

$$\sup_{-q\le u\le q}|\ln A_\delta(Q,u)|\le 2R. \tag{27}$$

From (24) and (27) one gets

$$\sup_{-1\le u\le1}|\ln A_\delta(Q,u)|\le\left(\frac{1}{q}+1\right)\sup_{-q\le u\le q}|\ln A_\delta(Q,u)|\le2\left(\frac{1}{q}+1\right)R. \tag{28}$$

Hence by (24) and (28),

$$\ln A_\delta(Q,[t]+t)-2R\left(\frac{1}{q}+1\right)\le\ln A_\delta(Q,t)\le\ln A_\delta(Q,[t])+2R\left(\frac{1}{q}+1\right) \tag{29}$$

if $t>0$, and

$$\ln A_\delta(Q,[t])-2R\left(\frac{1}{q}+1\right)\le\ln A_\delta(Q,t)\le\ln A_\delta(Q,[t]+1)+2R\left(\frac{1}{q}+1\right) \tag{30}$$

if $t<0$, where $[t]$ is the integral part of t.

Now from (26), (29) and (30) it follows that the limits in the left hand side of (21) and in the right hand side of (22) exist.

Next,

$$\begin{aligned}\alpha_\delta(Q,t)&=\inf_{y\in B_Q(\delta,t)}\frac{d(S^ty,Q)}{d(y,Q)}\\&=\left(\sup_{y\in B_Q(\delta,t)}\frac{d(y,Q)}{d(S^ty,Q)}\right)^{-1}=A_\delta^{-1}(Q,-t)\end{aligned}$$

and so

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln \alpha_\delta(Q, t) = -\lim_{t\to\pm\infty} \frac{1}{|t|} \ln A_\delta(Q, -t) \quad ;$$

that completes the proof of (21) and (22).

Clearly, $\Lambda_\delta^\pm(Q)$ and $\lambda_\delta^\pm(Q)$ are monotonic in δ, which gives (23). □

We say that the trajectory of the point x is forward stable (in the sense of Lyapunov) if for all $\varepsilon > 0$ there is $\delta > 0$ such that $d(S^t x, S^t y) \le \varepsilon$ for all $t \ge 0$ provided $d(x, y) \le \delta$. If, in addition, $d(S^t x, S^t y) \to 0$ as $t \to \infty$ then we say this trajectory is asymptotically forward stable. Replacing x in these definitions by an S^t-invariant set Q we shall get the definitions of the forward stability and the asymptotic forward stability of the invariant set Q.

The next result follows easily from the definitions.

THEOREM 3

(i) *If* $\Lambda^+(x) < 0$ *and the trajectory of* x *is forward stable then it is asymptotically forward stable*;

(ii) *If* $\lambda^+(x) > 0$ *then the trajectory of* x *is not forward stable*;

(iii) *The same is true when* x *is replaced by an invariant set* Q.

Remark 3. In all definitions one can replace t by $-t$ to get similar results for negative time.

Proof. If $\Lambda^+(x) < 0$ then by (9) there is $\varepsilon > 0$ such that

$$\Lambda_\varepsilon^+(x) \le \tfrac{1}{2}\Lambda^+(x) < 0. \tag{31}$$

Then by (7) one can find $t_0 > 0$ so that

$$\frac{1}{t} \ln A_\varepsilon(x, t) \le \tfrac{1}{4}\Lambda^+(x) < 0 \quad \text{for all } t \ge t_0. \tag{32}$$

On the other hand, by the definition of forward stability, there is $\delta > 0$ such that if $d(y, x) \le \delta$ then $d(S^t y, S^t x) \le \varepsilon$. Without loss of generality we can assume that $\delta \le \varepsilon$. Then for any $t \ge 0$,

$$B_x(\varepsilon, t) \supset \{y : d(y, x) \le \delta\} \equiv U_x(\delta). \tag{33}$$

Therefore by (1) and (32) it follows for $t \ge t_0$ that

$$\begin{aligned} e^{\frac{1}{4}\Lambda^+(x)t} \ge A_\varepsilon(x, t) &= \sup_{y \in B_x(\varepsilon, t)} d(S^t x, S^t y)/d(x, y) \\ &\ge \sup_{y \in U_x(\delta)} d(S^t x, S^t y)/d(x, y) \end{aligned} \tag{34}$$

i.e. $d(S^t x, S^t y) \to 0$ as $t \to \infty$ for any $y \in U_x(\delta)$, which proves (i).

To prove (ii) assume that the trajectory of x is forward stable. If $\lambda^+(x) > 0$ then by (8) and (9), there is $\varepsilon > 0$ and $t_0 > 0$ such that

$$\frac{1}{t} \ln \alpha_\varepsilon(x, t) \ge \lambda^+(x) > 0 \quad \text{for all } t \ge t_0. \tag{35}$$

By the forward stability of x one can choose δ so that (33) holds for any $t \geq 0$. Then by (33) and (35)

$$e^{\frac{1}{4}\lambda^+(x)t} \leq \alpha_\varepsilon(x, t) = \inf_{y \in B_x(\varepsilon, t)} d(S^t x, S^t y)/d(x, t)$$
$$\leq \inf_{y \in U_x(\delta)} d(S^t x, S^t y)/d(x, y) \tag{36}$$

for $t \geq t_0$. Thus $d(S^t x, S^t y) \to \infty$ as $t \to \infty$ which contradicts the assumption that the trajectory of x is stable ahead and so it is not true, proving (ii).

The proof of (iii) is done in the same way. □

Section 3

Now let X be a compact Riemannian manifold and S^t be a group of diffeomorphisms on it. Denote by DS^t_x the differential of S^t at x and by T_x the tangent space at x.

THEOREM 4. *Let μ be a Borel S^t-invariant probability measure on X. Then μ-almost everywhere*

$$\Lambda^+(x) = \lim_{t \to \infty} \frac{1}{t} \ln \|DS^t_x\| \tag{37}$$

and

$$\lambda^+(x) = \lim_{t \to \infty} \frac{1}{t} \ln \inf_{\xi \in T_x, \|\xi\| = 1} \|DS^t_x \xi\| = -\lim_{t \to \infty} \frac{1}{t} \ln \|DS^{-t}_{S^t x}\|, \tag{38}$$

where $\Lambda^+(x)$ and $\lambda^+(x)$ are the same as in theorem 1 *and $\|\cdot\|$ denotes the norms of operators and vectors in the tangent bundle.*

Remark 4. The same is true for negative t.

Proof. It is easy to see that

$$A_\delta(x, t) \geq \|DS^t_x\|. \tag{39}$$

Indeed, let $\xi \in T_x$, $\|\xi\| = 1$ and $\|DS^t_x \xi\| = \|DS^t_x\|$. If $\operatorname{Exp}_x : T_x \to X$ is the exponential map then, clearly,

$$\lim_{\varepsilon \to 0} \frac{d(S^t x, S^t \operatorname{Exp}_x(\varepsilon\xi))}{d(x, \operatorname{Exp}_x(\varepsilon\xi))} = \|DS^t_x \xi\| = \|DS^t_x\|$$

that gives (39).

Next, it is easy to see that there is a function $\alpha(\delta) > 0$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$ and for any $x \in X$, $t > 0$ and $y \in B_x(\delta, t)$,

$$\|DS^t_y\| \leq (1 + \alpha(\delta))^t \|DS^t_x\|. \tag{40}$$

Besides, for any $\delta > 0$ one can find $\varepsilon > 0$ such that if $y \in B_x(\varepsilon, t)$ then $y = \operatorname{Exp}_x(\rho\zeta)$, for some $\zeta \in T_x$, $0 < \rho \leq \varepsilon$ and $\operatorname{Exp}_x(u\zeta) \in B_x(\delta, t)$ for all $u \in [0, \rho]$. Then by (40)

$$d(S^t x, S^t y) \leq \int_0^\rho \|DS^t_{\operatorname{Exp}_x(u\zeta)}\| \, du$$
$$\leq (1 + \alpha(\delta))^t \|DS^t_x\| \, d(x, y).$$

Thus $A_\varepsilon(x, t) \leq (1 + \alpha(\delta))^t \|DS^t_x\|$. That together with (39), (7) and (9) yields (37).

Next it is easy to show that

$$\alpha_\delta(x,t) \le \inf_{\xi\in T_x, \|\xi\|=1} \|DS_x^t\xi\| \equiv \Theta_x(t). \tag{41}$$

Indeed, let $\eta \in T_x$, $\|\eta\|=1$ and

$$\|DS_x^t\eta\| = \Theta_x(t), \tag{42}$$

then for ε small enough

$$\mathrm{Exp}_x\,(\varepsilon\eta) \in B_x(\delta, t)$$

and so

$$d(S^tx, S^t\,\mathrm{Exp}_x\,(\varepsilon\eta))/d(x, \mathrm{Exp}_x\,(\varepsilon\eta)) \ge \alpha_\delta(x,t).$$

On the other hand

$$\lim_{\varepsilon\to 0} d(S^tx, S^t\,\mathrm{Exp}_x\,(\varepsilon\eta))/d(x, \mathrm{Exp}_x\,(\varepsilon\eta)) = \|DS_x^t\eta\|$$

that together with (42) gives (41).

As above one can find a positive function $\beta(\delta) < 1$ such that $\beta(\delta)\to 0$ as $\delta\to 0$ and for any $x\in X$, $t>0$ and $y\in B_x(\delta,t)$,

$$\Theta_y(t) \ge (1-\beta(\delta))^t\Theta_x(t), \tag{43}$$

where $\Theta_y(t)$ is defined in (41).

Besides, for any $\delta>0$ one can find $\varepsilon>0$ such that if $y\in B_x(\varepsilon,t)$ then there is a geodesic line $\gamma(u)$ such that:

$$\gamma(0) = S^tx,\ \gamma(d(S^tx, S^ty)) = S^ty,$$

$$S^{-t}\gamma(u) \in B_x(\delta,t) \quad \text{for all } u\in[0, d(S^tx, S^ty)].$$

Then by (43),

$$\begin{aligned} d(x,y) &\le \int_0^{d(S^tx,S^ty)} \|DS_{\gamma(u)}^{-t}\|du \\ &= \int_0^{d(S^tx,S^ty)} (\Theta_{S^{-t}\gamma(u)}(t))^{-1}\,du \\ &\le (1-\beta(\delta))^{-t}(\Theta_x(t))^{-1}\,d(S^tx, S^ty), \end{aligned} \tag{44}$$

since, clearly,

$$\|DS_{S^tx}^{-t}\| = (\Theta_x(t))^{-1}. \tag{45}$$

By (44) one obtains

$$\alpha_\varepsilon(x,t) \ge (1-\beta(\delta))\Theta_x(t) \tag{46}$$

that together with (41), (8) and (9) give the first equality in (38). The second one follows from (45).

From [**4**], [**5**] and theorem 4 one easily gets

COROLLARY. *For almost all points x, with respect to any S^t-invariant measure, $\Lambda^+(x)$ and $\lambda^+(x)$ are the maximal and the minimal characteristic exponents at x, respectively.*

This work was supported by a grant from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

REFERENCES

[1] Y. Derriennic. Sur le theoreme ergodique sous-additif. *C.R. Acad. Sc. Paris* **281 A** (1975), 985–988.

[2] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc.* **B30** (1968), 499–510.

[3] J. F. C. Kingman. Subbadditive ergodic theory. *Ann. Prob.* **1** (1973), 883–909.

[4] V. I. Oseldec. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19** (1968), 197–221.

[5] Ya. B. Pesin. Lyapunov characteristic exponents and smooth ergodic theory. *Russian Math. Surveys* **32** (1977), 55–114.