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A Remark on Certain Integral Operators of Fractional Type

Pablo Alejandro Rocha

Abstract. For $m, n \in \mathbb{N}$, $1 < m \le n$, we write $n = n_1 + \cdots + n_m$ where $\{n_1, \dots, n_m\} \subset \mathbb{N}$. Let $A_1, \dots, A_m$ be $n \times n$ singular real matrices such that

$$\bigoplus_{i=1}^{m} \bigcap_{1 \le j \ne i \le m} \mathcal{N}_j = \mathbb{R}^n,$$

where $\mathcal{N}_j = \{x : A_j x = 0\}$, $\dim(\mathcal{N}_j) = n - n_j$, and $A_1 + \cdots + A_m$ is invertible. In this paper we study integral operators of the form

$$T_r f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-n_1 + \alpha_1} \cdots |x - A_m y|^{-n_m + \alpha_m} f(y)\, dy,$$

$n_1 + \cdots + n_m = n$, $\frac{\alpha_1}{n_1} = \cdots = \frac{\alpha_m}{n_m} = r$, $0 < r < 1$, and the matrices A_i's are as above. We obtain the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of T_r for $0 < p < \frac{1}{r}$ and $\frac{1}{q} = \frac{1}{p} - r$.

1 Introduction

For $0 \le \alpha < n$ and $m > 1$, $(m \in \mathbb{N})$, let $T_{\alpha,m}$ be the integral operator defined by

$$T_{\alpha,m} f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \cdots |x - A_m y|^{-\alpha_m} f(y)\, dy, \tag{1.1}$$

where $\alpha_1, \dots, \alpha_m$ are positive constants such that $\alpha_1 + \cdots + \alpha_m = \alpha - n$, and $A_1, \dots, A_m$ are $n \times n$ invertible matrices such that $A_i \ne A_j$ if $i \ne j$. We observe that for the case $\alpha > 0$, $m = 1$, and $A_1 = I$, $T_{\alpha,1}$ is the Riesz potential I_α. Thus for $0 < \alpha < n$, the operator $T_{\alpha,m}$ is a kind of generalization of the Riesz potential. The case $\alpha = 0$ and $m > 1$ was studied under the additional assumption that $A_i - A_j$ are invertible if $i \ne j$. The behavior of this class of operators and their generalizations on the spaces of functions $L^p(\mathbb{R}^n)$, $L^p(w)$, $H^p(\mathbb{R}^n)$, and $H^p_{<\infty}(w^p)$ was studied in [1, 2, 4, 5, 7, 8].

If $0 < \alpha < n$ and $m > 1$, then the operator $T_{\alpha,m}$ has the same behavior as the Riesz potential on $L^p(\mathbb{R}^n)$. Indeed

$$|T_{\alpha,m} f(x)| \le C \sum_{j=1}^{m} \int_{\mathbb{R}^n} |A_j^{-1} x - y|^{\alpha - n} |f(y)|\, dy = C \sum_{j=1}^{m} I_\alpha(|f|)(A_j^{-1} x),$$

for all $x \in \mathbb{R}^n$. This pointwise inequality implies that $T_{\alpha,m}$ is a bounded operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and it is of type weak $(1, n/n - \alpha)$.

It is well known that the Riesz potential I_α is bounded from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$ for $0 < p \le 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (see [3, 11]). In [8], the author jointly with M. Urciuolo

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proved the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of the operator $T_{\alpha,m}$ and we also showed that the $H^p(\mathbb{R}) - H^q(\mathbb{R})$ boundedness does not hold for $0 < p \le \frac{1}{1+\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and $T_{\alpha,m}$ with $0 \le \alpha < 1$, $m = 2$, $A_1 = 1$, and $A_2 = -1$. This is a significant difference with respect to the case $0 < \alpha < 1$, $n = m = 1$, and $A_1 = 1$.

In this note we will prove that if we consider certain singular matrices in (1.1), then such an operator is still bounded from H^p into L^q. More precisely, for $m, n \in \mathbb{N}$, $1 < m \le n$, we write $n = n_1 + \cdots + n_m$, where $\{n_1, \dots, n_m\} \subset \mathbb{N}$. We also consider $n \times n$ singular real matrices $A_1, \dots, A_m$ such that $\bigoplus_{i=1}^m \bigcap_{1 \le j \ne i \le m} \mathcal{N}_j = \mathbb{R}^n$, where $\mathcal{N}_j = \{x : A_j x = 0\}$, $\dim(\mathcal{N}_j) = n - n_j$, $A_1 + \cdots + A_m$ is invertible. Given $0 < r < 1$ and $n_1, \dots, n_m$ such that $n_1 + \cdots + n_m = n$, let $\alpha_1, \dots, \alpha_m$ be positive constants such that $\frac{\alpha_1}{n_1} = \cdots = \frac{\alpha_m}{n_m} = r$. For such parameters we define the integral operator T_r by

$$T_r f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-n_1+\alpha_1} \cdots |x - A_m y|^{-n_m+\alpha_m} f(y)\, dy, \tag{1.2}$$

where the matrices A_i are as above.

We observe that the operator defined in (1.2) can be written as in (1.1), taking the matrices A_i there to be singular. In fact, $T_r = T_{\beta,m}$ with $\beta_i = n_i - \alpha_i$ for each $i = 1, 2, \dots, m$ and $\beta = nr$.

Our main result is the following theorem.

Theorem 1.1 *Let T_r be the integral operator defined in (1.2). If $0 < r < 1$, $0 < p < \frac{1}{r}$, and $\frac{1}{q} = \frac{1}{p} - r$, then T_r can be extended to an $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ bounded operator.*

In Section 2 we state two auxiliary lemmas to get the main result in Section 3. We conclude this note with an example in Section 4.

Throughout this paper, c will denote a positive constant, not necessarily the same at each occurrence. The symbol $A \lesssim B$ stands for the inequality $A \le cB$ for some constant c.

2 Preliminary Results

Let K be a kernel in $\mathbb{R}^n \times \mathbb{R}^n$. We formally define the integral operator T_K by $T_K f(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$.

We start with the following lemma.

Lemma 2.1 *Let $n, m \in \mathbb{N}$, with $1 < m \le n$, and let $n_1, \dots, n_m$ be natural numbers such that $n_1 + \cdots + n_m = n$. For each $i = 1, \dots, m$ let K_i be non-negative kernels in $\mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$ such that the operator T_{K_i} is bounded from $L^p(\mathbb{R}^{n_i})$ into $L^q(\mathbb{R}^{n_i})$ with $1 < p \le q < \infty$. Then the operator $T_{K_1 \otimes \cdots \otimes K_m}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.*

Proof Since $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, let $x = (x^1, \dots, x^m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. Now the operator $T_{K_1 \otimes \cdots \otimes K_m}$ is given by

$$T_{K_1 \otimes \cdots \otimes K_m} f(x) = \int_{\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}} K_1(x^1, y^1) \cdots K_m(x^m, y^m) f(y^1, \dots, y^m)\, dy^1 \cdots dy^m.$$

Using that the kernels K_i define bounded operators for $1 \le i \le m$, the lemma follows from an iterative argument and Minkowski's inequality for integrals. ■

Lemma 2.2 *Let $m, n \in \mathbb{N}$, with $1 < m \le n$, and let $n_1, \dots, n_m$ be natural numbers such that $n_1 + \cdots + n_m = n$. If $A_1, \dots, A_m$ are $n \times n$ singular real matrices such that $\bigoplus_{i=1}^{m} \bigcap_{1 \le j \ne i \le m} \mathcal{N}_j = \mathbb{R}^n$, where $\mathcal{N}_j = \{x : A_j x = 0\}$, $\dim(\mathcal{N}_j) = n - n_j$, and $A_1 + \cdots + A_m$ is invertible, then there exist two $n \times n$ invertible matrices B and C such that $B^{-1} A_j C$ is the canonical projection from $\mathbb{R}^n$ on $\{0\} \times \cdots \times \mathbb{R}^{n_j} \times \cdots \times \{0\}$ for each $j = 1, \dots, m$.*

Proof It is easy to check that

$$\bigoplus_{i=1}^{m} \bigcap_{1 \le j \ne i \le m} \mathcal{N}_j = \mathbb{R}^n \Longrightarrow \bigoplus_{1 \le i \ne k \le m} \bigcap_{1 \le j \ne i \le m} \mathcal{N}_j = \mathcal{N}_k.$$

So

$$A_k \Big(\bigcap_{1 \le j \ne k \le m} \mathcal{N}_j \Big) = \mathcal{R}(A_k), \quad k = 1, \dots, m. \tag{2.1}$$

Since $\dim(\mathcal{N}_k) = n - n_k$, $\dim(\bigcap_{1 \le j \ne k \le m} \mathcal{N}_j) = \dim(\mathcal{R}(A_k)) = n_k$. Let $\{y_1^k, \dots, y_{n_k}^k\}$ be a basis of $\bigcap_{1 \le j \ne k \le m} \mathcal{N}_j$. Thus $\{y_1^1, \dots, y_{n_1}^1, \dots, y_1^m, \dots, y_{n_m}^m\}$ is a basis for $\mathbb{R}^n$. Let C be the $n \times n$ matrix whose columns are the elements of the above basis. Since $A_1 + \cdots + A_m$ is invertible, we have that $B = (A_1 + \cdots + A_m) C$ is invertible. So (2.1) gives that $B^{-1} A_j C$ is the canonical projection from $\mathbb{R}^n$ on $\{0\} \times \cdots \times \mathbb{R}^{n_j} \times \cdots \times \{0\}$ for each $j = 1, \dots, m$. ∎

3 The Main Result

Proof of Theorem 1.1 We begin by obtaining the $L^p - L^q$ boundedness of the operator T_r for $1 < p < \frac{1}{r}$ and $\frac{1}{q} = \frac{1}{p} - r$, and then with this result we will prove the $H^p - L^q$ boundedness of T_r for $0 < p \le 1$ and $\frac{1}{q} = \frac{1}{p} - r$.

$L^p - L^q$ boundedness. If A is an $n \times n$ invertible matrix, we put $f_A(x) = f(A^{-1}x)$. Let B and C be the matrices give by Lemma 2.2. Then

$$\begin{aligned}
&[T_r(f_C)]_{B^{-1}}(x) \\
&\quad = \int_{\mathbb{R}^n} |Bx - A_1 y|^{-n_1 + \alpha_1} \cdots |Bx - A_m y|^{-n_m + \alpha_m} f(C^{-1} y)\, dy \\
&\quad = |\det(C)| \int_{\mathbb{R}^n} |B(x - B^{-1} A_1 C y)|^{-n_1 + \alpha_1} \cdots |B(x - B^{-1} A_m C y)|^{-n_m + \alpha_m} f(y)\, dy.
\end{aligned}$$

Since B is invertible, there exists a positive constant c such that $c|x| \le |Bx|$ for all $x \in \mathbb{R}^n$. Thus

$$\begin{aligned}
&\big| [T_r(f_C)]_{B^{-1}}(x) \big| \\
&\quad \lesssim \int_{\mathbb{R}^n} |x - B^{-1} A_1 C y|^{-n_1 + \alpha_1} \cdots |x - B^{-1} A_m C y|^{-n_m + \alpha_m} |f(y)|\, dy \\
&\quad \lesssim \int_{\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}} |x^1 - y^1|^{-n_1 + \alpha_1} \cdots |x^m - y^m|^{-n_m + \alpha_m} |f(y^1, \dots, y^m)|\, dy^1 \dots dy^m.
\end{aligned}$$

The second inequality follows from Lemma 2.2 and from that $|x^j - y^j| \le |x - P_j y|$, where $P_j = B^{-1} A_j C$ is the canonical projection from $\mathbb{R}^n$ on $\{0\} \times \cdots \times \mathbb{R}^{n_j} \times \cdots \times \{0\}$. Since $\gamma(\alpha_j)^{-1} |x^j - y^j|^{-n_j + \alpha_j}$, for an appropriate constant $\gamma(\alpha_j)$ (see [9, p. 117]), is the kernel of the Riesz potential on $\mathbb{R}^{n_j}$, then [9, Theorem 1] and Lemma 2.1 give the $L^p - L^q$ boundedness of the operator T_r for $1 < p < \frac{1}{r}$ and $\frac{1}{q} = \frac{1}{p} - r$.

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$H^p - L^q$ boundedness. Let $0 < p \leq 1$. We recall that a p-atom is a measurable function a supported on a ball B of $\mathbb{R}^n$ satisfying $\|a\|_\infty \leq |B|^{-1/p}$ and $\int y^\beta a(y)\,dy = 0$ for every multiindex β with $|\beta| \leq \lfloor n(p^{-1}-1)\rfloor$, ($\lfloor\cdot\rfloor$ denotes the integer part).

Let $0 < r < 1$, $0 < p \leq 1 < p_0 < \frac{1}{r}$, and $\frac{1}{q} = \frac{1}{p} - r$. Given $f \in H^p(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$, from [10, Theorem 2, p.107], we have that there exists a sequence of real numbers $\{\lambda_j\}_{j=1}^\infty$, a sequence of balls $B_j = B(z_j, \delta_j)$ centered at z_j with radius δ_j and p-atoms a_j supported on B_j satisfying

$$\sum_{j=1}^{\infty} |\lambda_j|^p \lesssim \|f\|_{H^p}^p, \tag{3.1}$$

such that f can be decomposed as $f = \sum_{j=1}^\infty \lambda_j a_j$, where the convergence is in H^p and L^{p_0} (for the convergence in L^{p_0}, see [6, Theorem 5]). So the $H^p - L^q$ boundedness of T_r will be proved if we show that there exists $c > 0$ such that

$$\|T_r a_j\|_{L^q} \leq c, \tag{3.2}$$

with c independent of the p-atom a_j. Indeed, since $f = \sum_{j=1}^\infty \lambda_j a_j$ in L^{p_0} and T_r is an $L^{p_0} - L^{\frac{p_0}{1-rp_0}}$ bounded operator, we have that $|T_r f(x)| \leq \sum_{j=1}^\infty |\lambda_j|\,|T_r a_j(x)|$ for almost all x; this pointwise inequality, the inequality in (3.2), together with the inequality

$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^{\min\{1,q\}}\Big)^{\frac{1}{\min\{1,q\}}} \leq \Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}}$$

and (3.1) allow us to conclude that $\|T_r f\|_q \leq c\|f\|_{H^p}$, for all $f \in H^p(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$. So the theorem follows from the density of $H^p(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$ in $H^p(\mathbb{R}^n)$.

We will prove the estimate in (3.2). We define $D = \max_{1\leq i\leq m} \max_{|y|=1} |A_i(y)|$. Let a_j be a p-atom supported on a ball $B_j = B(z_j, \delta_j)$, and for each $1 \leq i \leq m$ let $B_{ji}^* = B(A_i z_j, 4D\delta_j)$. Since T_r is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$ for $1 < p_0 < \frac{1}{r}$ and $\frac{1}{q_0} = \frac{1}{p_0} - r$, the Hölder inequality gives

$$\begin{aligned}\int_{\bigcup_{1\leq i\leq m} B_{ji}^*} |T_r a_j(x)|^q\,dx &\leq \sum_{1\leq i\leq m} \int_{B_{ji}^*} |T_r a_j(x)|^q\,dx \\ &\leq c \sum_{1\leq i\leq m} |B_{ji}^*|^{1-\frac{q}{q_0}} \|T_r a_j\|_{q_0}^q \leq c\delta_j^{n-\frac{nq}{q_0}} \|a_j\|_{p_0}^q \\ &\leq c\delta_j^{n-\frac{nq}{q_0}} \Big(\int_{B_j} |a_j|^{p_0}\Big)^{\frac{q}{p_0}} \leq c\delta_j^{n-\frac{nq}{q_0}} \delta_j^{-\frac{nq}{p}} \delta_j^{\frac{nq}{p_0}} = c.\end{aligned} \tag{3.3}$$

We denote $k(x,y) = |x - A_1 y|^{-n_1+\alpha_1} \cdots |x - A_m y|^{-n_m+\alpha_m}$, and we put $N - 1 = \lfloor n(p^{-1}-1)\rfloor$. In view of the moment condition of a_j we have, for $x \in \mathbb{R}^n \smallsetminus (\bigcup_{i=1}^m B_{ji}^*)$, that

$$T_r a_j(x) = \int_{B_j} k(x,y) a_j(y)\,dy = \int_{B_j} (k(x,y) - q_{N,j}(x,y)) a_j(y)\,dy,$$

where $q_{N,j}$ is the degree $N-1$ Taylor polynomial of the function $y \to k(x,y)$ expanded around z_j. By the standard estimate of the remainder term in the Taylor expansion,

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there exists ξ between y and z_j such that

$$|k(x,y) - q_{N,j}(x,y)| \lesssim |y - z_j|^N \sum_{k_1+\cdots+k_n=N} \Big| \frac{\partial^N}{\partial y_1^{k_1}\cdots\partial y_n^{k_n}} k(x,\xi) \Big|$$
$$\lesssim |y - z_j|^N \Big(\prod_{i=1}^{m} |x - A_i\xi|^{-n_i+\alpha_i} \Big) \Big(\sum_{l=1}^{m} |x - A_l\xi|^{-1} \Big)^N.$$

Now we decompose the set $R_j := \mathbb{R}^n \setminus (\bigcup_{i=1}^m B_{ji}^*)$ by $R_j = \bigcup_{k=1}^m R_{jk}$ where

$$R_{jk} = \{x \in R_j : |x - A_k z_j| \le |x - A_i z_j| \text{ for all } i \ne k\}.$$

If $x \in R_j$, then $|x - A_i z_j| \ge 4D\delta_j$, for all $i = 1, 2, \dots, m$. Since $\xi \in B_j$, it follows that $|A_i z_j - A_i\xi| \le D\delta_j \le \frac{1}{4}|x - A_i z_j|$, so

$$|x - A_i\xi| = |x - A_i z_j + A_i z_j - A_i\xi| \ge |x - A_i z_j| - |A_i z_j - A_i\xi| \ge \frac{3}{4}|x - A_i z_j|.$$

If $x \in R_j$, then $x \in R_{jk}$ for some k. Since $\sum_{i=1}^m (-n_i + \alpha_i) = -n(1-r)$, we obtain

$$|k(x,y) - q_{N,j}(x,y)| \lesssim |y - z_j|^N \Big(\prod_{i=1}^{m} |x - A_i z_j|^{-n_i+\alpha_i} \Big) \Big(\sum_{l=1}^{m} |x - A_l z_j|^{-1} \Big)^N$$
$$\lesssim |y - z_j|^N |x - A_k z_j|^{-n(1-r)-N},$$

if $x \in R_{jk}$ and $y \in B_j$. This inequality gives

$$\begin{aligned}
(3.4)\qquad & \int_{R_j} \Big| \int_{B_j} k(x,y) a_j(y)\, dy \Big|^q dx \\
&= \int_{R_j} \Big| \int_{B_j} [k(x,y) - q_{N,j}(x,y)] a_j(y)\, dy \Big|^q dx \\
&\lesssim \sum_{k=1}^{m} \int_{R_{jk}} \Big(\int_{B_j} |y - z_j|^N |x - A_k z_j|^{-n(1-r)-N} |a_j(y)|\, dy \Big)^q dx \\
&\lesssim \Big(\int_{B_j} |y - z_j|^N |a_j(y)|\, dy \Big)^q \sum_{k=1}^{m} \int_{(B_{jk}^*)^c} |x - A_k z_j|^{-n(1-r)q-Nq}\, dx \\
&\lesssim \delta_j^{qN - n\frac{q}{p} + nq} \int_{4D\delta_j}^{\infty} t^{-q(n(1-r)+N)+n-1}\, dt \le c
\end{aligned}$$

with c independent of the p-atom a_j, since $-q(n(1-r)+N)+n < 0$. Finally $\mathbb{R}^n = \bigcup_{i=1}^m B_{ji}^* \cup R_j$, so the inequality in (3.2) follows from (3.3) and (3.4). ∎

4 An Example

For $n = m = 3$, $n_1 = n_2 = n_3 = 1$, we consider the following 3×3 singular matrices

$$A_1 = \begin{pmatrix} 4 & 4 & -1 \\ 0 & 0 & 0 \\ -4 & -4 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & -1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

It is clear that

$$A_1 + A_2 + A_3 = \begin{pmatrix} 6 & 3 & -2 \\ -5 & 2 & 3 \\ -5 & -4 & 2 \end{pmatrix}$$

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is invertible. For each $j = 1, 2, 3$, let $\mathcal{N}_j = \{x \in \mathbb{R}^3 : A_j x = 0\}$. A computation gives $\mathcal{N}_1 = \langle (1,0,4), (0,1,4) \rangle$, $\mathcal{N}_2 = \langle (1,1,0), (0,0,1) \rangle$, and $\mathcal{N}_3 = \langle (1,0,1), (0,1,0) \rangle$. One can check that $\mathcal{N}_1 \cap \mathcal{N}_2 = \langle (1,1,8) \rangle$, $\mathcal{N}_1 \cap \mathcal{N}_3 = \langle (4,-3,4) \rangle$, and $\mathcal{N}_2 \cap \mathcal{N}_3 = \langle (1,1,1) \rangle$.

We observe that $\mathcal{N}_1 \cap \mathcal{N}_2 \oplus \mathcal{N}_1 \cap \mathcal{N}_3 \oplus \mathcal{N}_2 \cap \mathcal{N}_3 = \mathbb{R}^3$. As in the proof of Lemma 2.2, we define the matrices C and B by

$$C = \begin{pmatrix} 1 & 4 & 1 \\ 1 & -3 & 1 \\ 1 & 4 & 8 \end{pmatrix}, \quad B = (A_1 + A_2 + A_3)\, C = \begin{pmatrix} 7 & 7 & -7 \\ 0 & -14 & 21 \\ -7 & 0 & 7 \end{pmatrix}.$$

Both matrices are invertible with

$$B^{-1} = \begin{pmatrix} \frac{2}{21} & \frac{1}{21} & -\frac{1}{21} \\ \frac{1}{7} & 0 & \frac{1}{7} \\ \frac{2}{21} & \frac{1}{21} & \frac{2}{21} \end{pmatrix}.$$

Now it is easy to check that

$$B^{-1} A_1 C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^{-1} A_2 C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^{-1} A_3 C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, from Theorem 1.1, it follows that the operator T_r defined by

$$T_r f(x) = \int_{\mathbb{R}^3} |x - A_1 y|^{-1+r} |x - A_2 y|^{-1+r} |x - A_3 y|^{-1+r} f(y)\, dy,$$

with $0 < r < 1$, is a bounded operator from $H^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for $0 < p < 1/r$ and $\frac{1}{q} = \frac{1}{p} - r$.

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Universidad Nacional del Sur, Departamento de Matemática, INMABB (Conicet), (8000) Bahía Blanca, Buenos Aires, Argentina
e-mail: alinorocha@gmail.com