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ON THE LAWS OF CERTAIN VARIETIES OF GROUPS

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Let m and n be coprime positive integers. The variety $\underline{\underline{A}}_m\underline{\underline{A}}_n$ (consisting of all groups G such that for some normal subgroup H of G, H is abelian of exponent dividing m and G/H is abelian of exponent dividing n) and the variety $\underline{\underline{A}}_n\underline{\underline{A}}_m$ both satisfy the following three laws:

all elements have order dividing mn;

the commutator of two mth powers has order dividing m;

the commutator of two nth powers has order dividing n.

It is proved that any law which holds in both these varieties (notably that commutators commute) is a consequence of the above three laws.

1. Preliminaries

We follow the notation of the book [2] of Neumann; in particular (if $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are varieties)

$\underline{\underline{XY}}$ consists of those G such that for some $H \triangleleft G$, $H \in \underline{\underline{X}}$ and $G/H \in \underline{\underline{Y}}$,

$\underline{\underline{X}} \vee \underline{\underline{Y}}$ is the variety defined by the laws which hold in both $\underline{\underline{X}}$ and $\underline{\underline{Y}}$,

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$\underline{\underline{A}}_n$ is the variety of abelian groups of exponent dividing n .

Throughout the paper m and n will be fixed coprime positive integers. It is easily proved (Proposition 2.2 below) that

$$\underline{\underline{A}}_m\underline{\underline{A}}_n = \operatorname{var}\{x^{mn} = 1,\ [x, y]^m = 1,\ [x^n, y^n] = 1\} ,$$

and symmetrically

$$\underline{\underline{A}}_n\underline{\underline{A}}_m = \operatorname{var}\{x^{mn} = 1,\ [x, y]^n = 1,\ [x^m, y^m] = 1\} .$$

Hence it seems natural to investigate whether the laws

(i) $x^{mn} = 1$,

(ii) $[x^m, y^m]^m = 1$,

(iii) $[x^n, y^n]^n = 1$,

define the variety $\underline{\underline{A}}_m\underline{\underline{A}}_n \vee \underline{\underline{A}}_n\underline{\underline{A}}_m$. This paper is devoted to proving that they do.

THEOREM 1. *Laws* (i), (ii) *and* (iii) *above form a basis for the laws of the variety* $\underline{\underline{A}}_m\underline{\underline{A}}_n \vee \underline{\underline{A}}_n\underline{\underline{A}}_m$.

This improves the theorem of [1] where an additional law was needed.

2. Proof of Theorem 1

Since m and n are coprime the following is trivial.

LEMMA 2.1. *Suppose that the group* G *satisfies law* (i), *and let* $g \in G$. *Then* $g^m = 1$ *if and only if* $g = h^n$ *for some* $h \in G$. □

Interchanging m and n we see that, under the hypotheses of Lemma 2.1, $g^n = 1$ if and only if $g = h^m$ for some $h \in G$. This symmetry between m and n persists throughout the paper, and in applications of the various lemmas we sometimes interchange m and n without explicitly mentioning it.

PROPOSITION 2.2. $\underline{\underline{A}}_m\underline{\underline{A}}_n = \operatorname{var}\{x^{mn} = 1,\ [x, y]^m = 1,\ [x^n, y^n] = 1\}$.

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Proof. It is trivial that the given laws hold in $\underline{\underline{A}}_m\underline{\underline{A}}_n$. Conversely, suppose that a group G satisfies these laws. By Lemma 2.1 commutators in G are nth powers and hence commute. So G is metabelian, and the rest is clear. □

LEMMA 2.3. *If* a *and* b *are elements of any group* G *then* $[b^{-1}, a, b, a]$ *is conjugate in* G *to* $[a, b^{-1}, a^{-1}, b]$.

Proof.

$$[a, b^{-1}, a^{-1}, b] = [[a, b^{-1}]^{-1}[a, b^{-1}]^{a^{-1}}, b]$$
$$= [b^{-1}, a]^{a^{-1}}[a, b^{-1}][b^{-1}, a]^{b}[a, b^{-1}]^{a^{-1}b} .$$

Thus

$$[a, b^{-1}, a^{-1}, b]^{a} = [b^{-1}, a][a, b^{-1}]^{a}[b^{-1}, a]^{ba}[a, b^{-1}]^{a^{-1}ba}$$
$$= [b^{-1}, a][a, b^{-1}]^{a}[b^{-1}, a]^{ba}[a, b^{-1}]^{b} ,$$

which is conjugate to

$$[a, b^{-1}]^{b}[b^{-1}, a][a, b^{-1}]^{a}[b^{-1}, a]^{ba}$$
$$= [[b^{-1}, a]^{-1}[b^{-1}, a]^{b}, a] = [b^{-1}, a, b, a] . \qquad \square$$

Let $\underline{\underline{V}}$ be the variety defined by the laws (i), (ii) and (iii). It is trivial that laws (i), (ii) and (iii) hold in $\underline{\underline{A}}_m\underline{\underline{A}}_n$ and in $\underline{\underline{A}}_n\underline{\underline{A}}_m$, and hence

LEMMA 2.4. $\underline{\underline{A}}_m\underline{\underline{A}}_n \vee \underline{\underline{A}}_n\underline{\underline{A}}_m \leq \underline{\underline{V}}$. □

Throughout the rest of this paper G will denote an arbitrary group in $\underline{\underline{V}}$.

LEMMA 2.5. *If* $a, b \in G$ *and* $a^m = b^n = 1$ *then*

(a) $[b^{-1}, a, b, a] = 1 = [a, b^{-1}, a^{-1}, b]$,

(b) $[[a, b], [a, b^{-1}]]^n = 1$.

Proof. *(a)* We have

(1) $$[b^{-1}, a, b]^m = [b(b^{-1})^a, b]^m = [(b^{-1})^a, b]^m = 1$$

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by law (ii) and Lemma 2.1 (since $b^n = 1$). Hence by law (iii) and Lemma 2.1, $[b^{-1}, a, b, a]^n = 1$. By Lemma 2.3 we deduce that $[a, b^{-1}, a^{-1}, b]^n = 1$. But

$$[a, b^{-1}, a^{-1}]^n = [a^{-1}a^{b^{-1}}, a^{-1}]^n = [a^{b^{-1}}, a^{-1}]^n = 1$$

by law (iii), and therefore (by law (ii)) $[a, b^{-1}, a^{-1}, b]^m = 1$. Since m and n are coprime, $[a, b^{-1}, a^{-1}, b] = 1$, and (by Lemma 2.3) $[b^{-1}, a, b, a] = 1$ also.

(b)

$$\begin{aligned}[[a, b], [a, b^{-1}]] &= [[b^{-1}, a]^b, [a, b^{-1}]] \\ &= [[a, b^{-1}][b^{-1}, a]^b, [a, b^{-1}]] \\ &= [[b^{-1}, a, b], [a, b^{-1}]] \\ &= [b^{-1}, a, b, a^{-1}a^{b^{-1}}] \\ &= [b^{-1}, a, b, a^{b^{-1}}]\end{aligned}$$

since $[b^{-1}, a, b]$ commutes with a by (a). But (1) above gives $[b^{-1}, a, b, a^{b^{-1}}]^n = 1$ by law (iii). Thus $[[a, b], [a, b^{-1}]]^n = 1$. □

LEMMA 2.6. *If* $a, b \in G$ *with* $a^n = b^n = 1$ *and* $[a, a^b] = 1$ *then* $[a, b] = 1$.

Proof. We have $[a, b]^m = 1$ (law (ii)). But since a commutes with a^b, $(a^{-1}a^b)^n = (a^{-1})^n(a^b)^n = 1$, and since m and n are coprime, $[a, b] = 1$. □

LEMMA 2.7. *Suppose that* $a, b, c \in G$ *with* $a^n = b^m = [a, b] = c^n = 1$. *Then* $[a^c, b] = 1$.

Proof. By law (ii) we have $[a, c]^m = 1$ and so law (iii) gives

(2) $$[a, c, b]^n = 1$$

and

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(3) $$[[a, c]^b, [a, c]]^n = 1 .$$

By (2) and law (iii),

(4) $$[a, c, b, a]^m = 1$$

and

(5) $$[a, c, b, a^c]^m = 1 .$$

By law (ii), $[a^c, a]^m = 1$, and so, by law (iii),

(6) $$[b^{-a^c}, [a^c, a]]^n = 1 .$$

But

$$\begin{aligned}[a, c, b, a] &= [a^{-1}a^c, b, a] \\ &= [a^c, b, a] \quad (\text{since } [a, b] = 1) \\ &= [(b^{-1})^{a^c}b, a] \\ &= [(b^{-1})^{a^c}, a]^b \quad (\text{since } [a, b] = 1) \\ &= [(b^{-1})^{a^c}, (a^{-1})^{a^c}a]^b \quad (\text{since } [b^{a^c}, a^{a^c}] = 1) \\ &= [(b^{-1})^{a^c}, [a^c, a]]^b ,\end{aligned}$$

and since m and n are coprime, (4) and (6) give

(7) $$[a, c, b, a] = 1 .$$

Therefore

$$\begin{aligned}[a, c, b, a^c] &= [a, c, b, a^{-1}a^c] \\ &= [a^{-1}a^c, b, a^{-1}a^c] \\ &= [(a^{-1}a^c)^b, a^{-1}a^c] .\end{aligned}$$

By (3) and (5) therefore $[(a^{-1}a^c)^b, a^{-1}a^c] = 1$. Now $(a^{-1}a^c)^m = b^m = 1$, and so Lemma 2.6 gives $[a^{-1}a^c, b] = 1$. Since a and b commute this gives $[a^c, b] = 1$.

COROLLARY 2.8. *If* $d, e, f \in G$ *with* $d^n = e^m = [d, e] = f^m = 1$

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then $[d^f, e] = 1$.

Proof. From Lemma 2.7, by interchanging m and n , we have $[d, e^{f^{-1}}] = 1$ and hence $[d^r, e] = 1$. □

LEMMA 2.9. *If* $a, b \in G$ *with* $a^n = b^m = [a, b] = 1$ *then all conjugates of* a *commute with* b .

Proof. Let $g \in G$ and let p, q be integers with $pm + qn = 1$. Then $g = cf$ where $c = g^{pm}$ and $f = g^{qn}$. By law (i), $c^n = 1$, and so Lemma 2.7 gives $[a^c, b] = 1$. But $f^m = 1$ (law (i)); so Corollary 2.8 with a^c in place of d and b in place of e gives $[a^{cf}, b] = 1$. □

LEMMA 2.10. *If* $a, b, c \in G$ *with* $a^n = 1$ *and* $[a, a^b] = [a, a^c] = 1$ *then* $[a^b, a^c] = 1$.

Proof. Since a commutes with a^b and a^c it commutes with $[a^b, a^c]$. But $[a^b, a^c]^m = 1$ (law (ii)); so by Lemma 2.9 all conjugates of a commute with $[a^b, a^c]$. Thus

$$1 = [[a^b, a^c], a^b] = [(a^b)^{a^c}, a^b]$$

and, by Lemma 2.6, $[a^b, a^c] = 1$. □

LEMMA 2.11. *If* $a, b, c \in G$ *with* $[a, a^b] = [a, a^c] = 1$ *then* $[a^b, a^c] = 1$. *The set* $T = \{t \in G \mid [a, a^t] = 1\}$ *is a subgroup of* G .

Proof. Let a_1, a_2 be powers of a with $a_1^n = a_2^m = 1$ and $a_1a_2 = a$. Since $[a_1, a_2] = 1$ it follows from Lemma 2.9 that $\left[a_1^b, a_2^c\right] = \left[a_1^c, a_2^b\right] = 1$. Since $[a, a^b] = 1$, $[a^r, (a^r)^b] = 1$ for all integers r . So $\left[a_1, a_1^b\right] = \left[a_2, a_2^b\right] = 1$, and similarly $\left[a_1, a_1^c\right] = \left[a_2, a_2^c\right] = 1$. By Lemma 2.10 we obtain $\left[a_1^b, a_1^c\right] = \left[a_2^b, a_2^c\right] = 1$. Hence $[a^b, a^c] = \left[a_1^ba_2^b, a_1^ca_2^c\right] = 1$.

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Since $[a^b, a^c] = 1$ yields $[a, a^{cb^{-1}}] = 1$ we see that $b, c \in T$ implies $cb^{-1} \in T$. Since $1 \in T$, T is a subgroup. □

LEMMA 2.12. *Suppose that* $a, b \in G$ *satisfy* $a^m = b^n = (ab)^m = 1$, *and let* H *be the subgroup of* G *generated by* $\{a, b\}$. *Then the elements* b^{a^i}, $0 \le i \le m-1$, *generate an abelian normal subgroup* B *of* H. *The set* $\{b^{-1}b^{a^i} \mid 1 \le i \le m-1\}$ *also generates* B.

Proof. $[a, b]^n = [a, ab]^n = 1$ by law (iii), since $a^m = (ab)^m = 1$. But $[a, b^{-1}] = b[a, b]^{-1}b^{-1}$; so $[a, b^{-1}]^n = 1$, and, by law (ii), $[[a, b], [a, b^{-1}]]^m = 1$. But, by Lemma 2.5 *(b)*, $[[a, b], [a, b^{-1}]]^n = 1$, and hence $[[a, b], [a, b^{-1}]] = 1$. So $[[a, b], [a, b]^{b^{-1}}] = 1$, and by Lemma 2.6 we deduce that $[a, b]$ commutes with b. Thus $a \in \{t \in G \mid [b, b^t] = 1\}$, and by Lemma 2.11 it follows that $[b, b^{a^i}] = 1$ for all i. So B is an abelian normal subgroup of H.

Since $(ab)^m = 1$ we obtain $b^{a^{m-1}} \ldots b^{a^2} b^a b = 1$, and hence $b^{-m} = (b^{-1}b^a)(b^{-1}b^{a^2}) \ldots (b^{-1}b^{a^{m-1}})$. Therefore the subgroup generated by $\{b^{-1}b^{a^i} \mid 1 \le i \le m-1\}$ contains b, hence contains b^{a^i} for all i, hence equals B.

LEMMA 2.13. *If* c, d, e G *with* $c^n = d^n = e^m = 1$ *then* $[c, d, e, c] = 1$.

Proof. Let $a = [c, d]$, $b = [a, e]$. Then $a^m = 1$ (by law (ii)), $b^n = 1$ (by law (iii)), and $(ab)^m = (a^e)^m = 1$. By Lemma 2.12 therefore there exists an abelian subgroup B such that

(8) $$\{b^{-1}b^{a^i} \mid 1 \le i \le m-1\} \text{ generates } B,$$

(9) $$b^{a^i} \in B \text{ for } i = 0, 1, \ldots, m-1.$$

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We have $a^m = 1$, $c^n = 1$ and $(ca)^n = \left(c^d\right)^n = 1$. Hence by Lemma 2.12 with m and n interchanged the elements a^{c^r} , $0 \le r \le n-1$, generate an abelian subgroup. So, for each $i \in \{1, 2, \ldots, m-1\}$,

$$\begin{aligned}\left(a^i c^{-1}\right)^n &= a^i \left(a^i\right)^c \ldots \left(a^i\right)^{c^{n-1}} \\ &= \left(a^{c^{n-1}} \ldots a^c a\right)^i \\ &= \left((ca)^n\right)^i \\ &= 1 ,\end{aligned}$$

and hence, by law (ii),

$$[b, c]^m = \left[b, a^i c^{-1}\right]^m = 1 . \tag{10}$$

Let $i \in \{1, 2, \ldots, m-1\}$ and let $g = \left[b^c, b^{-1} b^{a^i}\right]$. Since $b^n = 1$ and b commutes with b^{a^i} , law (ii) gives $g^m = 1$. But

$$\begin{aligned}g &= b^{-c} b \left(b^{-1}\right)^{a^i} b^c b^{-1} b^{a^i} \\ &= \left[b^{-1} b^c , b^{-c} b^{a^i}\right] \\ &= \left[[b, c], \left[b, a^i c^{-1}\right]^c\right] ,\end{aligned}$$

so that (10) gives $g^n = 1$. Hence $g = 1$; that is, b^c centralizes $b^{-1} b^{a^i}$. Since this holds for all i , (8) and (9) yield $\left[b^c, b\right] = 1$. But $b^n = c^n = 1$; so Lemma 2.6 gives $[b, c] = 1$, as required. □

LEMMA 2.14. *If* $a, b, f \in G$ *with* $a^n = b^n = f^m = 1$ $[a, b, f] = 1$.

Proof. By Lemma 2.13 with a, b, f in place of c, d, e we have

$$[a, b, f]^a = [a, b, f] . \tag{11}$$

But by Lemma 2.13 with $b, a^{-1}, f^{a^{-1}}$ in place of c, d, e we have that b centralizes $\left[b, a^{-1}, f^{a^{-1}}\right] = \left[[a, b]^{a^{-1}}, f^{a^{-1}}\right] = [a, b, f]$ (by (11)). Since a and b both centralize $[a, b, f]$, so does $[a, b]$. Hence

$$[[a, b]^f, [a, b]] = 1 ,$$

and since $f^m = [a, b]^m = 1$ (law (ii)), Lemma 2.6 gives $[a, b, f] = 1$. □

LEMMA 2.15. *If $H \in \underline{\underline{V}}$ satisfies $h^m = 1$ for all $h \in H$ then H is abelian.*

Proof. Let $a, b \in H$. By hypothesis $[a, b]^m = 1$. But $a^m = b^m = 1$; so by Lemma 2.1 and law (iii), $[a, b]^n = 1$. Hence $[a, b] = 1$. □

LEMMA 2.16. *Let H be the subgroup of G generated by $S = \{[a, b] \mid a, b \in G, a^n = b^n = 1\}$. Then H is abelian, $h^m = 1$ for all $h \in H$, $H \lhd G$ and $G/H \in \underline{\underline{A}}_n\underline{\underline{A}}_m$.*

Proof. Clearly $S^g = S$ for all $g \in G$, and hence $H \lhd G$. If $f \in S$ then $f^m = 1$ by law (ii). So, by Lemma 2.14, f commutes with $[a, b]$ whenever $a^n = b^n = 1$; that is, f commutes with all elements of S . Hence H is abelian.

Since H is abelian and its generating set S consists of elements of order dividing m it follows that $h^m = 1$ for all $h \in H$.

Let L be the subgroup of G generated by $T = \{a \in G \mid a^n = 1\}$. Then clearly $H \le L \lhd G$. Moreover if $a, b \in T$ then $[a, b] \in H$, and so L/H is abelian. But L/H is generated by elements of order dividing n ; so $L/H \in \underline{\underline{A}}_n$. Finally, if $g \in G$ then, by law (i), $g^m \in L$, whence $G/L \in \underline{\underline{A}}_m$ (by Lemma 2.15), whence $G/H \in \underline{\underline{A}}_n\underline{\underline{A}}_m$. □

LEMMA 2.17. *$G \in \underline{\underline{A}}_n\underline{\underline{A}}_m \vee \underline{\underline{A}}_m\underline{\underline{A}}_n$.*

Proof. Let H be as in Lemma 2.16, K the subgroup of G generated by $\{[a, b] \mid a, b \in G, a^m = b^m = 1\}$. By Lemma 2.16 if $g \in H$ then $g^m = 1$, and dually if $g \in K$ then $g^n = 1$. Hence $H \cap K = 1$. But, by Lemma 2.16, $G/H \in \underline{\underline{A}}_n\underline{\underline{A}}_m \le \underline{\underline{A}}_n\underline{\underline{A}}_m \vee \underline{\underline{A}}_m\underline{\underline{A}}_n$, and dually $G/K \in \underline{\underline{A}}_m\underline{\underline{A}}_n \le \underline{\underline{A}}_n\underline{\underline{A}}_m \vee \underline{\underline{A}}_m\underline{\underline{A}}_n$. Since $H \cap K = 1$ the homomorphism

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$x \mapsto (xH, xK)$ is an embedding of G in $G/H \times G/K \in \underline{\underline{A}}_n\underline{\underline{A}}_m \vee \underline{\underline{A}}_m\underline{\underline{A}}_n$, whence the result. □

Lemma 2.17 shows that $\underline{\underline{V}} \leq \underline{\underline{A}}_n\underline{\underline{A}}_m \vee \underline{\underline{A}}_m\underline{\underline{A}}_n$. The reverse inclusion is obvious since laws (i), (ii) and (iii) hold in both $\underline{\underline{A}}_n\underline{\underline{A}}_m$ and $\underline{\underline{A}}_m\underline{\underline{A}}_n$. Thus Theorem 1 is proved.

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