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NONLINEAR MULTIPOINT BOUNDARY VALUE PROBLEMS FOR WEAKLY COUPLED SYSTEMS

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We establish existence results concerning solutions to multipoint boundary value problems for weakly coupled systems of second order ordinary differential equations with fully nonlinear boundary conditions.

1. Introduction

In this work we investigate multipoint boundary value problems for a system of second order ordinary differential equations of the form

$$y'' = f(x, y, y'), \quad 0 \leqslant x \leqslant 1, \tag{1}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the system (1) is coupled in y, but not in y'. This means that we can express the i-th component of $f(x,y,y')$ as $f_i(x,y,y_i')$, where $y = (y_1, \dots, y_n)$ and

$$f(x,y,y') = \big(f_1(x,y,y_1'), \dots, f_n(x,y,y_n')\big).$$

The system (1) is said to be weakly coupled.

By a solution of (1) we mean a twice continuously differentiable function y satisfying (1) everywhere.

We consider boundary conditions of the form

$$G\big(y(0), y(1), y(c), y(d), y'(0), y'(1)\big) = 0, \quad 0 < c \leqslant d < 1 \tag{2}$$

where $G = (g_0, g_1)$, $g_i = (g_{i1}, \dots, g_{in})$ and the g_{ij} are continuous and nonlinear, for $i = 0, 1$, $j = 1, \dots, n$. We shall refer to conditions of the form (2) as fully nonlinear boundary conditions.

In Section Two we introduce the necessary definitions and preliminary results concerned with this work.

Thompson's notion of compatibility of fully nonlinear two point boundary conditions G with the lower and upper solutions [2] is extended to our boundary conditions in Section Three.

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Section Four contains our main existence result: If the boundary conditions G are compatible with α and β and f satisfies a Bernstein-Nagumo condition, then there exist solutions y of (1) and (2) satisfying $\alpha \leqslant y \leqslant \beta$ on [0,1]. Finally, we compare our results with the special cases investigated by Ehme and Henderson.

The contribution this work makes over previous knowledge is the extension of the results of Ehme and Henderson [**1**] concerning weakly coupled systems of second order ordinary differential equations for three and four point boundary value problems with linear boundary conditions to the nonlinear case.

2. Definitions and Preliminary Results

In order to state our results we need some notation.

We denote the boundary of a set A by ∂A and the closure of A by $\overline{A}$. We denote the space of continuous functions mapping from A to B by $C(A;B)$. If $B = \mathbb{R}$ then we omit the B.

Let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$. We say $y \leqslant z$ in the vector sense if $y_i \leqslant z_i$ for all $i = 1, \dots, n$. If $y \leqslant z$ we set $[y,z] = \{q \in \mathbb{R}^n : y \leqslant q \leqslant z\}$ and set $(y,z) = \{q \in \mathbb{R}^n : y_i < q_i < z_i, \quad 1 \leqslant i \leqslant n\}$.

If A is a bounded, open subset of $\mathbb{R}^n$, $p \in \mathbb{R}^n$, $f \in C\left(\overline{A};\mathbb{R}^n\right)$ and $p \notin f(\partial A)$ we denote the Brouwer degree of f on A at p by $d(f, A, p)$.

Modification of f is common practice for existence proofs of boundary value problems. We shall make the necessary modifications by using the following functions.

Definition 1: A function $\alpha \in C^2\left([0,1];\mathbb{R}^n\right)$ is a lower solution for (1), if for $1 \leqslant i \leqslant n$

$$\alpha_i''(x) \geqslant f_i(x, \sigma, \alpha_i')$$

for all $\alpha(x) \leqslant \sigma$ with $\sigma_i = \alpha_i(x)$, for $0 < x < 1$. Similarly, define a function $\beta \in C^2\left([0,1];\mathbb{R}^n\right)$ to be an upper solution for (1), if for $1 \leqslant i \leqslant n$

$$\beta_i''(x) \leqslant f_i(x, \sigma, \beta_i')$$

for all $\sigma \leqslant \beta(x)$ with $\sigma_i = \beta_i(x)$, for $0 < x < 1$.

We shall always assume there exist lower and upper solutions α, β respectively for (1) with $\alpha \leqslant \beta$ on $[0,1]$ and will set

$$\Delta_c = \left(\alpha(c), \beta(c)\right) \text{ for each } c \in (0,1),$$

$$\alpha_{im} = \min_{x\in[0,1]} \left\{\alpha_i(x)\right\} \text{ and } \beta_{iM} = \max_{x\in[0,1]}\left\{\beta_i(x)\right\}, \text{ for } 1 \leqslant i \leqslant n, \text{ and}$$

$$\Delta = \left(\alpha(0), \beta(0)\right) \times \left(\alpha(1), \beta(1)\right).$$

We call the pair α, β nondegenerate if $\Delta \neq \emptyset$,

If $r < s$ are given, let $\pi : \mathbb{R} \to [r,s]$ be (the retraction) given by

$$\pi(t, r, s) = \max\left\{\min\{s,t\}, r\right\}.$$

Let $p(x,y) = (p_1, \dots, p_n)$ be given by

$$p_i(x,y) = \pi\Big(y_i, \alpha_i(x), \beta_i(x)\Big), \quad 1 \leqslant i \leqslant n.$$

For each $\varepsilon > 0$, let $K \in C\Big(\mathbb{R} \times (0,\infty); [-1,1]\Big)$ satisfy

(i) $K(\cdot, \varepsilon)$ is odd,

(ii) $K(t,\varepsilon) = 0$ if and only if $t = 0$,

(iii) $K(t,\varepsilon) = 1$ for all $t \geqslant \varepsilon$.

Definition 2: If $r \leqslant s$ and $\varepsilon > 0$ are given, let $T \in C(\mathbb{R})$ be given by

$$T(t,r,s,\varepsilon) = K\Big(t - \pi(t,r,s), \varepsilon\Big).$$

Let

$$Q(x,t) = \begin{cases} (1-x)t & \text{for } 0 \leqslant t \leqslant x \leqslant 1 \\ (1-t)x & \text{for } 0 \leqslant x \leqslant t \leqslant 1 \end{cases}$$

and

$$w(y_0, y_1)(x) = y_0(1-x) + y_1 x, \ \text{for } 0 \leqslant x \leqslant 1$$

where $w = (w_1, \dots, w_n)$ and $y_j = (y_{j1}, \dots, y_{jn})$, for $j = 0, 1$.

Let $X = C^1\Big([0,1]; \mathbb{R}^n\Big) \times \mathbb{R}^n \times \mathbb{R}^n$ with the usual product norm.

Define $\mathcal{C} : C\Big([0,1]; \mathbb{R}^n\Big) \to C\Big([0,1]; \mathbb{R}^n\Big)$ by

$$\mathcal{C}(y)(x) = -\int_0^1 Q(x,t) y(t)\, dt$$

for all $y \in C\Big([0,1]; \mathbb{R}^n\Big)$ and $x \in [0,1]$.

Clearly $\mathcal{C}$ is completely continuous.

The following Lemma establishes a priori bounds on y' for solutions of (1) when f satisfies the Nagumo-Bernstein condition.

Lemma 3. *Let α, $\beta : [0,1] \to \mathbb{R}^n$ be continuous and $\alpha \leqslant \beta$ on $[0,1]$. Suppose $y \in C^2\Big([0,1]; \mathbb{R}^n\Big)$ satisfies*

$$\alpha \leqslant y \leqslant \beta \text{ and } |y_i''| \leqslant h_i\Big(|y_i'|\Big), \text{ on } [0,1], \text{ and } \quad \int_{d_i}^N \frac{s\, ds}{h_i(s)} > \beta_{iM} - \alpha_{im},$$

for some $N > 0$, where $h_i : C\Big([0,\infty); (0,\infty)\Big)$ and $d_i = \max\Big\{\Big|\beta_i(1) - \alpha_i(0)\Big|, \Big|\beta_i(0) - \alpha_i(1)\Big|\Big\}$, for $1 \leqslant i \leqslant n$. Then

$$|y_i'| \leqslant N \text{ on } [0,1],$$

for $1 \leqslant i \leqslant n$.

3. Compatibility and Nonlinear Boundary Conditions

Definition 4: We call the vector field $\Psi = (\psi_0, \psi_1) \in C\left(\overline{\Delta}; \mathbb{R}^{2n}\right)$ strongly inwardly pointing on Δ if for all $(C, D) \in \partial\Delta$

$$\begin{aligned} \psi_{0i}(C,D) &> \alpha_i'(0) \text{ if } C_i = \alpha_i(0) \\ \psi_{0i}(C,D) &< \beta_i'(0) \text{ if } C_i = \beta_i(0) \\ \psi_{1i}(C,D) &< \alpha_i'(1) \text{ if } D_i = \alpha_i(1) \\ \psi_{1i}(C,D) &> \beta_i'(1) \text{ if } D_i = \beta_i(1) \end{aligned}$$

for $1 \leqslant i \leqslant n$, where $\psi_0 = (\psi_{01}, \dots, \psi_{0n})$, and $\psi_1 = (\psi_{11}, \dots, \psi_{1n})$.

We call Ψ inwardly pointing if the strict inequalities are replaced by weak inequalities in the above definition.

Definition 5: Let $G \in C\left(\overline{\Delta} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}; \mathbb{R}^{2n}\right)$ and $0 < c \leqslant d < 1$. We say G is strongly compatible with α and β if for all strongly inwardly pointing vector fields Ψ on $\overline{\Delta}$, $\Phi_c \in C\left(\overline{\Delta}_0, \overline{\Delta}_c\right)$, and $\Phi_d \in C\left(\overline{\Delta}_1, \overline{\Delta}_d\right)$

$$\begin{aligned} \mathcal{G}(C,D) &\neq 0 \quad \text{for all } (C,D) \in \partial\Delta \\ d(\mathcal{G}, \Delta, 0) &\neq 0 \end{aligned}$$

where

$$\mathcal{G}(C,D) = G\Big(\big(C, D, \Phi_c(C), \Phi_d(D)\big); \Psi(C,D)\Big)$$

for all $(C, D) \in \partial\Delta$. We say G is compatible with α and β if there is a sequence $G_j \in C\left(\overline{\Delta} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}; \mathbb{R}^{2n}\right)$ strongly compatible with α and β and converging uniformly to G on a compact subsets of $\overline{\Delta} \times \mathbb{R}^n \times \mathbb{R}^n$.

4. Existence of Solutions

We now present our main existence theorem.

Theorem 1. *Assume that there exist nondegenerate lower and upper solutions $\alpha \leqslant \beta$ for (1), that f satisfies the Berstein-Nagumo condition and that $G \in C\left(\overline{\Delta} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}; \mathbb{R}^{2n}\right)$ is compatible with α and β. Then problem (1) and (2) has a solution y lying between α and β.*

Proof: Consider

(3) $$y'' = f(x, y, y'), \qquad 0 \leqslant x \leqslant 1.$$

Assume first that G is stongly compatible with the lower and upper solutions α and β, respectively. We modify the right hand side of the differential equation, f, for $y_i < \alpha_i$ and $y_i > \beta_i$ to acquire a second set of lower and upper solutions for (3), then we show that a solution of the modified problem lies in the region where f is unmodified and thus

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is the required solution. Choose $L > \max\left\{\left|\alpha_i'(x)\right|, \left|\beta_i'(x)\right| : 0 \leqslant x \leqslant 1,\ 1 \leqslant i \leqslant n\right\}$ and $\varepsilon > 0$ such that for $1 \leqslant i \leqslant n$

$$\int_{d_i}^{L} \frac{s\,ds}{h_i(s)+\varepsilon} > \beta_{iM} - \alpha_{im} + 2\varepsilon. \tag{4}$$

Let $J = (J_1, \dots, J_n)$ with

$$J_i(x, y, y_i') = f_i\Big(x, p(x, y), \pi(y_i', -L, L)\Big).$$

Let $k = (k_1, \dots, k_n)$ where

$$\begin{aligned} k_i(x, y, y_i') = \Big(1 - \Big|T\Big(y_i, \alpha_i(x), \beta_i(x), \varepsilon\Big)\Big|\Big) J_i(x, y, y_i') \\ + T\Big(y_i, \alpha_i(x), \beta_i(x), \varepsilon\Big)\Big(\Big|J_i(x, y, y_i')\Big| + \varepsilon\Big). \end{aligned}$$

Hence k_i is a bounded, continuous function on $[0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ and satifies

$$\Big|k_i(x, y, p)\Big| \leqslant h_i\Big(|p|\Big) + \varepsilon$$

for all $p \in \mathbb{R}$ with $|p| \leqslant L$.

Consider

$$y'' = k(x, y, y'), \qquad 0 \leqslant x \leqslant 1 \tag{5}$$

together with boundary conditions (2). We show Problem (5) and (2) has a solution y satisfying $\alpha \leqslant y \leqslant \beta$ and $|y_i'| \leqslant L$ on $[0,1]$, for $i = 1, \dots, n$. Since f and k agree in this region this is the required solution of Problem (1) and (2). Let

$$\begin{aligned} \alpha_{i\varepsilon} &= \alpha_{im} - \varepsilon, \quad \alpha_\varepsilon = (\alpha_{1\varepsilon}, \dots, \alpha_{n\varepsilon}) \\ \beta_{i\varepsilon} &= \beta_{iM} + \varepsilon \quad \beta_\varepsilon = (\beta_{1\varepsilon}, \dots, \beta_{n\varepsilon}). \end{aligned}$$

Now

$$\begin{aligned} T(\alpha_{i\varepsilon}, \alpha_i, \beta_i, \varepsilon) &= K\Big(\alpha_{i\varepsilon} - \pi\Big(\alpha_{i\varepsilon}, \alpha_i(x), \beta_i(x)\Big), \varepsilon\Big) \\ &= K\Big(\alpha_{i\varepsilon} - \alpha_i(x), \varepsilon\Big) \\ &= -1. \end{aligned}$$

Hence

$$\begin{aligned} \alpha_{i\varepsilon}'' = 0 &> -\Big(\Big|J_i(x, \sigma, \alpha_{i\varepsilon}')\Big| + \varepsilon\Big) \\ &= k_i(x, \sigma, \alpha_{i\varepsilon}'), \text{ for } \sigma \geqslant \alpha_\varepsilon, \text{ with } \sigma_i = \alpha_{i\varepsilon}. \end{aligned}$$

Hence α_ε is a lower solution for (5). Similarly, β_ε is an upper solution for (5). Also from (4)

$$\int_{d_i}^{L} \frac{s\,ds}{h_i(s)+\varepsilon} > \beta_{i\varepsilon} - \alpha_{i\varepsilon}.$$

Now suppose y is a solution of (5) and $\big(y(0), y(1)\big) \in \overline{\Delta}$. We show that y is also a solution of (3) and $\alpha \leqslant y \leqslant \beta$ on [0,1]. We argue by contradiction. Suppose $y_i(t) < \alpha_i(t)$ for some $t \in [0,1]$ and $i \in \{1, \dots, n\}$. We may assume from continuity and the boundary conditions that $\alpha_i - y_i$ attains its positive maximum at $t \in (0,1)$. Thus $\alpha_i'(t) = y_i'(t)$ so that $|y_i'(t)| < L$ and $\alpha_i''(t) \leqslant y_i''(t)$. Now $\alpha_i(t) - y_i(t) > 0$ and hence

$$T\Big(y_i(t), \alpha_i(t), \beta_i(t), \varepsilon\Big) = K\Big(y_i(t) - \alpha_i(t), \varepsilon\Big) < 0.$$

Therefore

$$\begin{aligned}
y_i''(t) &= k_i\Big(t, y(t), y_i'(t)\Big) \\
&= \Big(1 - \Big|T\Big(y_i(t), \alpha_i(t), \beta_i(t), \varepsilon\Big)\Big|\Big) J_i(t, y, y_i') \\
&\quad + T\Big(y_i(t), \alpha_i(t), \beta_i(t), \varepsilon\Big)\Big(\Big|J_i(t, y, y_i')\Big| + \varepsilon\Big) \\
&= \Big(1 - \Big|K\Big(y_i(t) - \alpha_i(t), \varepsilon\Big)\Big|\Big) f_i\Big(t, p(t, y), y_i'\Big) \\
&\quad + K\Big(y_i(t) - \alpha_i(t), \varepsilon\Big)\Big(\Big|f_i\Big(t, p(t, y), y_i'\Big)\Big| + \varepsilon\Big) \\
&< f_i\Big(t, p(t, y), y_i'\Big), \\
&\leqslant \alpha_i''(t), \text{ since } p_i(t, y) = \alpha_i(t),\ \alpha(t) \leqslant p\Big(t, y(t)\Big) \leqslant \beta(t), \text{ and } \alpha_i'(t) = y_i'(t);
\end{aligned}$$

a contradiction. Thus $y \geqslant \alpha$ and similarly $y \leqslant \beta$ on [0,1]. Now, $|y_i'| < L$ on $[0,1]$, for $1 \leqslant i \leqslant n$, by Lemma 3 so that y is the required solution.

Let

$$\Omega_\varepsilon = \Big\{y \in C^1\Big([0,1]; \mathbb{R}^n\Big) : \ y \in (\alpha_\varepsilon, \beta_\varepsilon),\ |y_i'| < L, \text{ on } [0,1]\Big\}$$

and

$$\Gamma_\varepsilon = \Omega_\varepsilon \times \Delta.$$

Define $\mathcal{K} : C^1\Big([0,1]; \mathbb{R}^n\Big) \to C\Big([0,1]; \mathbb{R}^n\Big)$ by

$$\mathcal{K}(y)(x) = k\Big(x, y(x), y'(x)\Big), \quad 0 \leqslant x \leqslant 1.$$

Let Ψ be a strongly inwardly pointing vector field on Δ, $\Phi_c \in C\Big(\overline{\Delta}_0, \overline{\Delta}_c\Big)$, and $\Phi_d \in C\Big(\overline{\Delta}_1, \overline{\Delta}_d\Big)$.

Define $\mathcal{H} : \overline{\Gamma}_\varepsilon \times [0,1] \to X$ by

$$\mathcal{H}(y, C, D, \lambda) = \left(y - 3\lambda w(C, D) - (1 - 3\lambda)\frac{(\alpha_\varepsilon + \beta_\varepsilon)}{2}, \mathcal{G}(C, D)\right)$$

for $0 \leqslant \lambda \leqslant 1/3$

$$\mathcal{H}(y, C, D, \lambda) = \left(y + \mathcal{C}3\Big(\lambda - \frac{1}{3}\Big)\mathcal{K}(y) - w(C, D), \mathcal{G}(C, D)\right)$$

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for $1/3 \leqslant \lambda \leqslant 2/3$, and

$$\mathcal{H}(y, C, D, \lambda) = \Big(y + \mathcal{C}\mathcal{K}(y) - w(C, D), \mathcal{S}(y, C, D, \lambda)\Big)$$

for $2/3 \leqslant \lambda \leqslant 1$ where $\mathcal{S}$ is defined by

$$\begin{aligned}\mathcal{S} &= \mathcal{S}(y, C, D, \lambda)\\ &= G\left(\left(C, D, 3\Big(\lambda - \frac{2}{3}\Big)\right)\Big(y(c), y(d)\Big) + 3(1-\lambda)\Big(\Phi_c(C), \Phi_d(D)\Big),\right.\\ &\qquad\qquad \left.\left(3\Big(\lambda - \frac{2}{3}\Big)\Big(y'(0), y'(1)\Big) + 3(1-\lambda)\Psi(C, D)\right)\right).\end{aligned}$$

Clearly $\mathcal{H}$ is completely continuous. It is easy to see that y is a solution of (5) and (2) with $\big(y, y(0), y(1)\big) \in \Gamma_\varepsilon$ if and only if

$$\mathcal{H}(y, y(0), y(1), 1) = 0.$$

Now if there is a solution with $\big(y, y(0), y(1)\big) \in \partial\Gamma_\varepsilon$ then there is nothing to prove so we assume there is no solution in $\partial\Gamma_\varepsilon$. We show $\mathcal{H}$ is a homotopy for the Schauder degree on Γ_ε at 0. We argue by contradiction and assume solutions exist to $\mathcal{H}(y, C, D, \lambda) = 0$ with $\lambda \in [0, 1]$ and $(y, C, D) \in \partial\Gamma_\varepsilon$. We investigate the cases $\lambda \in [2/3, 1]$ and $[1/3, 2/3)$; the case $\lambda \in [0, 1/3)$ is trivial.

Case (i) $\lambda \in [2/3, 1]$.

By assumption there is no solution with $\lambda = 1$, so we assume there is a solution (y, C, D) with $\lambda \in [2/3, 1)$ and $\alpha \leqslant y \leqslant \beta$ on $[0, 1]$. Thus $y(0) = C$ and $y(1) = D$. Assume that $(C, D) \in \partial\Delta$. If $C_i = y_i(0) = \alpha_i(0)$ for some $i = 1, \dots, n$, then $y_i'(0) \geqslant \alpha_i'(0)$. Thus

$$3\left(\lambda - \frac{2}{3}\right) y_i'(0) + 3(1-\lambda)\psi_{0i}\Big(y_i(0), y_i(1)\Big) > \alpha_i'(0)$$

since Ψ is strongly inwardly pointing. It follows from strong compatiblility that

$$\mathcal{S}(y, y(0), y(1), \lambda) \neq 0,$$

a contradiction.

Similarly, the other cases $(C, D) = \big(y(0), y(1)\big) \in \partial\Delta$ lead to a contradiction. Thus $(C, D) \notin \partial\Delta$. Assume that $y \in \partial\Omega_\varepsilon$. Again by Lemma 3, $|y_i'| < L$ on $[0, 1]$ for $1 \leqslant i \leqslant n$, so either $y_i(t) = \alpha_{i\varepsilon}(t)$ or $y_i(t) = \beta_{i\varepsilon}(t)$ for some $t \in [0, 1]$ and $i \in \{1, \dots, n\}$. Assume $y_i(t) = \alpha_{i\varepsilon}(t)$ for some $t \in [0, 1]$. From the boundary conditions we see that $t \in (0, 1)$ and thus $y_i'(t) = \alpha_{i\varepsilon}'(t) = 0$ while $y_i''(t) \geqslant \alpha_{i\varepsilon}''(t) = 0$. Now

$$\begin{aligned} y_i''(t) &= k_i\Big(t, y(t), y'(t)\Big)\\ &= \Big(1 - \Big|K\Big(y_i(t) - \alpha_i(t), \varepsilon\Big)\Big|\Big) f_i\Big(t, p(x, y), \alpha_i'(t)\Big)\\ &\qquad + K\Big(y_i(t) - \alpha_i(t), \varepsilon\Big)\Big(\Big|f_i\Big(t, p(x, y), y_i'(t)\Big)\Big| + \varepsilon\Big)\\ &= -\Big(\Big|f_i(t, p(x, y), 0)\Big| + \varepsilon\Big) < 0, \end{aligned}$$

a contradiction. Similarly the assumption $y_i(t) = \beta_{i\varepsilon}(t)$ for some $t \in [0,1]$ leads also to a contradiction so that $y \notin \partial\Omega_\varepsilon$. Thus there are no solutions of $\mathcal{H}(y, C, D, \lambda) = 0$ with $\lambda \in [2/3, 1]$ and $(y, C, D) \in \partial\Gamma_\varepsilon$.

Case (ii) $\lambda \in [1/3, 2/3]$.

Since Ψ is strongly inwardly pointing and G is strongly compatible, by the compatibility conditions there are no solutions (y, C, D) with $(C, D) \in \partial\Delta$. The proof of the case $y \in \partial\Omega_\varepsilon$ leads to a contradiction in a similar way as for $\lambda \in [2/3, 1)$. Thus $\mathcal{H}$ is a homotopy for the Schauder degree and since $\mathcal{H}(\cdot, 0) = (I - b, \mathcal{G})$ where I is the identity on $C^1\big([0,1]; \mathbb{R}^n\big)$ and $b \in \Omega_\varepsilon$ is a constant it follows that

$$\begin{aligned} d(\mathcal{H}(\cdot, 1), \Omega_\varepsilon, 0) &= d(\mathcal{H}, (\cdot, 0), \Omega_\varepsilon, 0) \\ &= d(\mathcal{G}, \Delta, 0) \neq 0. \end{aligned}$$

Thus there is a solution $(y, C, D) \in \Gamma_\varepsilon$ of $\mathcal{H}(y, C, D, 1) = 0$, and hence a solution $y \in C^2\big([0,1]; \mathbb{R}^n\big)$ of Problem (5) and (2).

Suppose now that G is compatible with α and β. Then there is a sequence $\{G_j\}$ strongly compatible with α and β which converges uniformly to G on compact subsets of $\mathbb{R}^{6n}$ to G. Let y_j be the corresponding solutions. By compactness there is a subsequence of y_j which converges in $C^2\big([0,1]; \mathbb{R}^n\big)$, to the desired solution. ∎

We now show that the boundary conditions (2.4) of Ehme and Henderson [**1**] are compatible if and only if

$$\begin{aligned} &\alpha(0) \leqslant \alpha(c), \qquad \alpha(1) \leqslant \alpha(d), \\ &\beta(0) \geqslant \beta(c), \qquad \beta(1) \geqslant \beta(d). \end{aligned} \tag{6}$$

Lemma 6. *Let $\alpha \leqslant \beta$ be non-degenerate lower and upper solutions, respectively, for (1) and let the boundary conditions be given by*

$$\begin{aligned} G_0\big(C, D, y(c), y(d), y'(0), y'(1)\big) &= C - y(c), \\ G_1\big(C, D, y(c), y(d), y'(0), y'(1)\big) &= D - y(d). \end{aligned} \tag{7}$$

Then G is compatible if and only if the inequalities (6) hold.

Proof: (i) Necessity:

Assume $\alpha(0) > \alpha(c)$. Let

$$\Phi_c(C) = \alpha(c), \;\; \Phi_d(D) = \alpha(d), \;\text{for all } (C, D) \in \overline{\Delta}$$

and let Ψ be strongly inwardly pointing on Δ. Then

$$\begin{aligned} \mathcal{G}(C, D) &= \big(C - \Phi_c(C), D - \Phi_d(D)\big) \\ &= \big(C - \alpha(c), D - \alpha(d)\big). \end{aligned}$$

This implies

$$d\big(\mathcal{G}, \Delta, (0,0)\big) = d\big(\mathcal{G}_0, \big(\alpha(0), \beta(0)\big), 0\big) d\big(\mathcal{G}_1, \big(\alpha(1), \beta(1)\big), 0\big) = 0$$

since $\mathcal{G}_0(C) > 0$ for all $C \in \big[\alpha(0), \beta(0)\big]$ and thus $\alpha(0) \leqslant \alpha(c)$.

Similarly, $\beta(0) \geqslant \beta(c)$, $\alpha(1) \leqslant \alpha(d)$ and $\beta(1) \geqslant \beta(d)$.

(ii) Sufficiency:

Suppose (6) holds. Let $G = (g_{0i}, g_{1i})$, where

$$g_{0i} = C - y(c) + \left(C - \frac{\alpha(0) + \beta(0)}{2}\right) \frac{1}{i}, \text{ and}$$
$$g_{1i} = D - y(d) + \left(D - \frac{\alpha(1) + \beta(1)}{2}\right) \frac{1}{i}.$$

Let Φ_c, Φ_d and Ψ be given as in Definition 5. Then

$$\mathcal{G}_{0i} = C - \Phi_c(C) + \left(C - \frac{\alpha(0) + \beta(0)}{2}\right) \frac{1}{i}$$
$$\mathcal{G}_{1i} = D - \Phi_d(D) + \left(D - \frac{\alpha(1) + \beta(1)}{2}\right) \frac{1}{i}.$$

This implies

$$\mathcal{G}_{0i}\big(\alpha(0), D\big) < 0, \quad \mathcal{G}_{1i}\big(C, \alpha(1)\big) < 0$$
$$\mathcal{G}_{0i}\big(\beta(0), D\big) > 0, \quad \mathcal{G}_{1i}\big(C, \beta(1)\big) > 0$$

and hence $d\big(\mathcal{G}_i, \Delta, (0,0)\big) = 1 \neq 0$. Thus G_i is strongly compatible. Now G_i converges uniformly to G on compact subsets of $\mathbb{R}^{6n}$ and thus G is compatible with α and β. $\square$

If G is given by (7), then it follows from our Theorem 1 that Problem (1) and (2) has a solution if the inequalities (6) hold. This represents an improvement over [**1**, Theorem 2.4] for their boundary conditions (2.4). Ehme and Henderson's boundary conditions (2.3) are given by

$$\begin{aligned} &G_0\big(C, D, y(c), y(d), y'(0), y'(1)\big) = y'(0), \text{ and} \\ (8) \qquad &G_1\big(C, D, y(c), y(d), y'(0), y'(1)\big) = D - y(d). \end{aligned}$$

By an argument similar to Lemma 6 one can show that these boundary conditions are compatible if and only if

$$\begin{aligned} &\alpha'(0) \geqslant 0, \qquad \alpha(1) \leqslant \alpha(d) \\ (9) \qquad &\beta'(0) \leqslant 0, \qquad \beta(1) \geqslant \beta(d). \end{aligned}$$

If G is given by (8), again it follows from our Theorem 1 that Problem (1) and (2) has a solution if the inequalities (9) hold. This represents an improvement over [**1**, Theorem

2.4] for their boundary conditions (2.3). Thus our result is an improvement over that of Ehme and Henderson, for their boundary conditions (2.3) and (2.4). Moreover it applies to a very broad range of boundary conditions including nonlinear boundary conditions and is easy to apply. As an illustration of this we give the following example.

REMARK 7. If

$$G_0\big(C, D, y(c), y(d), y'(0)y'(1)\big) = C - y(c) - 3c$$
$$G_1\big(C, D, y(c), y(d), y'(0)y'(1)\big) = D - y(d)$$

then G is compatible if and only if

$$\alpha(0) \leqslant \alpha(c) + 3c, \quad \alpha(d) \geqslant \alpha(1)$$

$$\beta(0) \geqslant \beta(c) + 3c, \quad \beta(d) \leqslant \beta(1).$$

This follows by a similar argument to that in Lemma 6. Moreover, if c and $1 - d$ are small the solutions approximate the solutions of Equation (1) with boundary conditons

$$y'(0) = 3, \quad y'(1) = 0.$$

As is to be expected from results involving degree theory solutions may exist even if the boundary conditions are not compatible with the lower and upper solutions and a Bernstein-Nagumo condition is satisfied. However it is not difficult to construct examples of Problem (1) and (2) where G is not compatible and for which the Problem has no solutions.

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