Ergod. Th. & Dynam. Sys., (2023), **43**, 1760–1784

doi:10.1017/etds.2022.2

Fractal projections with an application in number theory

HAN YU

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, CB3 0WB, UK
(e-mail: hy351@maths.cam.ac.uk)

(*Received* 7 *September* 2020 *and accepted in revised form* 5 *January* 2022)

Abstract. In this paper, we discuss a connection between geometric measure theory and number theory. This method brings a new point of view for some number-theoretic problems concerning digit expansions. Among other results, we show that for each integer k, there is a number $M > 0$ such that if $b_1, \ldots, b_k$ are multiplicatively independent integers greater than M, there are infinitely many integers whose base $b_1, b_2, \ldots, b_k$ expansions all do not have any zero digits.

Key words: prime factors of binomial coefficients, Graham's question, fractal projection, Cantor set, Schanuel's conjecture
2020 Mathematics Subject Classification: 11A63, 11Z05, 11K55, 28A80 (Primary)

1. *Prime factors of binomial coefficients: Graham's question*

In the 1970s, Erdős, Graham, Ruzsa, and Straus proved that there are infinitely many integers n such that $\binom{2n}{n}$ is coprime with $3 \times 5 = 15$, see [6]. Later on, Graham asked the following question.

Question 1.1. (Graham's binomial coefficients problem) Are there infinitely many integers $n \geq 1$ such that the binomial coefficient $\binom{2n}{n}$ is coprime with $105 = 3 \times 5 \times 7$?

Remark 1.2. According to [16], Graham is offering \$1000 to the first person with a solution.

This problem turns out to be related to digit expansions of numbers in different bases. To be precise, let $b_1, \ldots, b_k \geq 2$ be $k \geq 2$ different integers. For each $i \in \{1, \ldots, k\}$, let $B_i \subset \{0, \ldots, b_i - 1\}$ be a subset of digits in base b_i. Let $n, b \geq 2$ be integers, we write $D_b(n)$ for the set of digits used in representing n in base b. We define the set of integers

$$N_{b_1,\ldots,b_k}^{B_1,\ldots,B_k} = \{n \in \mathbb{N} : \text{ for all } i \in \{1, \ldots, k\}, D_{b_i}(n) \subset B_i\}.$$

Thus, $N_{b_1,\dots,b_k}^{B_1,\dots,B_k}$ contains integers with very special digit expansions simultaneously in many different bases. We will call such numbers to be with restricted digits. The original motivation of this type of problem is to study prime factors of $\binom{2n}{n}$. The connection between prime factors of $\binom{2n}{n}$ and digit expansions of n was established by Kummer in [13].

THEOREM 1.3. (Kummer) *Let p be a prime number. Then $p \not| \binom{2n}{n}$ if and only if the p-ary expansion of n contains only digits $\leq (p-1)/2$.*

Due to Kummer's theorem, we see that for Graham's question, one needs to study the set $N_{3,5,7}^{B_3,B_5,B_7}$, where

$$B_p = \{0, \dots, (p-1)/2\}.$$

It is precisely the set of integers n with $\binom{2n}{n}$ being coprime with 3, 5, 7. Graham's question is open, but there has been some progress. In [4], it was proved that, under Schanuel's conjecture (see Conjecture 3.1 below and [3]),

$$\#N_{3,5,7}^{B_3,B_5,B_7} \cap [1, N] \leq N^{0.026}$$

for all sufficiently large N. So we can say that there are not 'too many' integers n such that $\binom{2n}{n}$ is coprime with 3, 5, 7. Unconditionally, at least one of the results listed below holds for all large enough N:

$$\#N_{3,5,7}^{B_3,B_5,B_7} \cap [1, N] \leq N^{0.026},$$
$$\#N_{3,5,11}^{B_3,B_5,B_{11}} \cap [1, N] \leq N^{0.061},$$
$$\#N_{3,5,13}^{B_3,B_5,B_{13}} \cap [1, N] \leq N^{0.073}.$$

The results in [4] suggest a more quantitative conjecture as follows.

Conjecture 1.4. Let $p_1, \dots, p_k$ be $k \geq 2$ different odd prime numbers. For each $i \in \{1, \dots, k\}$, let

$$B_i = \{0, \dots, (p_i - 1)/2\}.$$

Consider the number

$$s = \sum_{i=1}^{k} \frac{\log \#B_i}{\log p_i} = \sum_{i=1}^{k} \frac{\log(p_i + 1) - \log 2}{\log p_i}.$$

If $s \in (k-1, k)$, then for each $\epsilon > 0$, there is a constant $C_\epsilon > 1$ such that

$$C_\epsilon^{-1} N^{s-(k-1)-\epsilon} \leq \#N_{p_1,\dots,p_k}^{B_1,\dots,B_k} \cap [1, N] \leq C_\epsilon N^{s-(k-1)+\epsilon} \tag{1}$$

for all integers $N \geq 2$. If $s < k-1$, then $N_{p_1,\dots,p_k}^{B_1,\dots,B_k}$ is finite.

Remark 1.5. The rightmost inequality of equation (1) was proved in [4] under Schanuel's conjecture. Thus the open problem is the leftmost inequality of equation (1) and the finiteness statement. This is closely related to a Furstenberg's problem, see [8, 17, 18].

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

2. *Results in this paper*

In this paper, we consider Graham's question and other problems related to digit expansions of numbers in different bases. We relate it to projections of fractal sets, a well-studied topic in geometric measure theory. This connection does not directly provide us with new results. It merely represents some number-theoretic problems using fractals (self-similar sets) and translates the problems into geometric properties (intersections, slices, projections) of those fractals. We will provide a more detailed discussion on this topic in §§3 and 4. Here, we only need to know that Π_x for $x \in \mathbb{R}^d$ stands for the map

$$y \in \mathbb{R}^d \setminus \{x\} \to \Pi_x(y) = \frac{x-y}{|x-y|} \in S^{d-1}.$$

Intuitively speaking, let $A \subset \mathbb{R}^d$. Then $\Pi_x(A)$ is what an observer can see of A at a certain position $x \in \mathbb{R}^d$. In what follows, we say that a list of numbers $a_1, \ldots, a_d$ are multiplicatively independent if they are not 0 or 1, and $1, \log a_2/\log a_1, \ldots, \log a_d/\log a_1$ are linearly independent over the field of rational numbers. See §3.2 for the definition of self-similar sets and the open set condition. See [7, Ch. 2] for the definition of the Hausdorff dimension ($\dim_{\mathrm{H}}$).

Conjecture 2.1. Let $A \subset \mathbb{R}^d$, $d \geq 2$ be a Cartesian product of self-similar sets in $\mathbb{R}$ with the open set condition and uniform contraction ratios. Suppose further that the contraction ratios are multiplicatively independent. If $\dim_{\mathrm{H}} A > d-1$, then $\Pi_x(A)$ has a non-empty interior for all $x \in \mathbb{R}^d$.

In §4, we will provide more details. Conjecture 2.1 turns out to be closely related to Graham's question. We do not need the full strength of this conjecture. A special case will be enough. See Conjecture 4.2.

THEOREM 2.2. *Assuming Conjecture 2.1 and Schanuel's conjecture, there are infinitely many integers n such that $\binom{2n}{n}$ is coprime with $3 \times 5 \times 7$.*

Remark 2.3. We do not need the full strength of Schanuel's conjecture. It is enough only to assume that

$$1, \frac{\log 3}{\log 5}, \frac{\log 3}{\log 7}$$

are $\mathbb{Q}$-linearly independent.

Currently, we are not able to prove Conjecture 2.1. Nonetheless, the strategy for proving Theorem 2.2 can be adapted to prove many other (unconditional) results concerning numbers with restricted digits.

First, we prove a quantitative version of a result in [6].

THEOREM 2.4. *Let p, q be two different odd primes. Consider the set*

$$A = \left\{ n \in \mathbb{N} : \gcd\left(pq, \binom{2n}{n} \right) = 1 \right\}.$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Then there is a constant $c > 0$ depending on p, q such that

$$A \cap [1, N] \geq c \log N$$

holds for all large enough integers N.

Next, we prove a result concerning linear forms of numbers with restricted digits.

THEOREM 2.5. *There are infinitely many integer triples $(x, y, z) \in N_3^{\{0,1\}} \times N_4^{\{0,1\}} \times N_5^{\{0,1\}}$ with*

$$x + y = z.$$

Remark 2.6. This result says that there are infinitely many sums of powers of five that can be written as sums of powers of three and four. We list a few examples:

$$\begin{aligned} 5 &= 4 + 1, \\ 5^2 &= 4^2 + 3^2, \\ 5^3 + 5^2 &= 3^4 + 4^3 + 4 + 1, \\ 5^4 + 5^2 &= 3^5 + 4^4 + 3^4 + 4^3 + 4 + 1 + 1. \end{aligned}$$

It is perhaps possible that all large enough integers can be written as a sum of form $N_3^{\{0,1\}} + N_4^{\{0,1\}}$. If so, this theorem would follow as a direct consequence. We believe that the conclusion does not hold for general triples (b_1, b_2, b_3) in the place of $(3, 4, 5)$. For example, we suspect that there are only finitely many integer triples $(x, y, z) \in N_9^{\{0,1\}} \times N_{10}^{\{0,1\}} \times N_{11}^{\{0,1\}}$ with $x + y = z$. A partial result towards this direction is [**20**, Theorem 1.5] which says that for each $\epsilon > 0$ and large enough integer N, the amount of integer triples in $[1, N]^3$ with the above property is $O(N^\epsilon)$.

Another very natural question to consider is whether there are infinite numbers with missing digits in many different bases at the same time. For example, are there integers $b_1 > \cdots > b_{100} > 2$ such that there are infinitely many integers whose base $b_1, \ldots, b_{100}$ expansions do not have digit zero? Our next result answers this question.

THEOREM 2.7. *Let $k \geq 2$ be an integer. Then there is an integer $M \geq 1$ such that for all k-tuples of multiplicative independent integers $b_1, \ldots, b_k$ that are at least M, there are infinitely many integers whose base $b_1, \ldots, b_k$ expansions all omit the digit zero.*

Remark 2.8. The missing digit zero is not a special choice. In fact, one can choose an integer $m \geq 1$ and consider digit expansions in different bases with an arbitrary choice of m missing digits for each base. Of course, this only makes sense if the bases in consideration are all greater than m.

This brings us closer to Graham's question. However, we are still far away. Kummer's theorem indicates that for considering prime divisors of binomial coefficients, one needs to consider numbers that miss at least half of the digits in many different prime bases.

To prove the above results, we use Newhouse's gap lemma. See §3.3. It is possible to prove Theorems 2.5, 2.7 without having the geometry of self-similar sets in mind.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

In fact, it is possible to prove those results by directly using arithmetic of integers and some well-known facts of irrational rotations just like the original arguments in [6, Lemma on p. 84]. For example, [6, Lemma on p. 84] can be viewed as an integer version of Newhouse's gap lemma applied to 'integer self-similar sets'. The point of using the fractal geometric point of view in this paper is to make the ideas behind the proofs more transparent. In fact, the core of the proof of Theorem 2.4 essentially uses the same ideas as in [6, Lemma on p. 84] but expresses them in a more geometric way.

3. *Preliminaries*

3.1. *A remark for Schanuel's conjecture.* We need Schanuel's conjecture because in some of the proofs, we will use properties of (irrational) rotations on a torus. Let $n \geq 1$ be an integer. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. We define the map $T_\alpha : \mathbb{T}^n \to \mathbb{T}^n$ to be

$$x \in \mathbb{T}^n \to T_\alpha(x) = x + \alpha.$$

We say that T_α is the rotation on $\mathbb{T}^n$ with the rotation angle α. If $1, \alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent, then we say that T_α is an irrational rotation. In this case, it is well known that for each $x \in \mathbb{T}^n$,

$$\overline{\{T_\alpha^k(x)\}_{k\geq 1}} = \mathbb{T}^n.$$

See [5, Corollary 4.15].

Let $b_1, b_2 \ldots, b_n$ be $n \geq 2$ integers. To study digit expansions with respect to these integers, it is often useful to consider the rotation on $\mathbb{T}^{n-1}$ with the rotation angle,

$$\Lambda = (\log b_1/\log b_2, \ldots, \log b_1/\log b_n).$$

To do this, it is useful to know whether the above vector generates an irrational rotation. This is the case if

$$\Lambda' = \left(\frac{\prod_{i=1}^n \log b_i}{\log b_1}, \ldots, \frac{\prod_{i=1}^n \log b_i}{\log b_n}\right)$$

are $\mathbb{Q}$-linearly independent. If $n = 2$, then the situation is simple. For $n \geq 3$, the problem becomes challenging. For example, it is not known whether $1, \log 2/\log 3, \log 2/\log 5$ are $\mathbb{Q}$-linearly independent. Problems of this kind are related to Schanuel's conjecture, see [3].

Conjecture 3.1. (Schanuel) Let $n \geq 2$ be an integer. Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree of

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

is at least n.

In the case when $e^{x_1}, \ldots, e^{x_n}$ are integers, the conjectures reduce to saying that $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Q}$. This implies that Λ' is indeed $\mathbb{Q}$-linearly independent if $1, \log b_1, \ldots, \log b_n$ are $\mathbb{Q}$-linearly independent. This is the reason that whenever we are dealing with digit expansions with more than two bases, Schanuel's

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

conjecture is likely to appear. We are not using the full power of Schanuel's conjecture here. However, this conjecture is so difficult that even the homogeneous quadratic case is not known. The only known result in this direction is Baker's theory on linear forms of logarithms, see also [3] and the references therein for more details.

We should nonetheless remark that although we are dealing with digit expansions with more than two bases in this paper, the $\mathbb{Q}$-linear independence of Λ' is not always involved. In fact, we will only need Schanuel's conjecture for proving Theorem 2.2. For Theorem 2.5, the proof would be much simpler by assuming Schanuel's conjecture. Additional efforts need to be taken to get rid of it. For Theorem 2.7, we simply do not meet the situation where Schanuel's conjecture is needed.

3.2. *Self-similar sets and the open set condition.* Let $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ be a finite collection of linear maps on $\mathbb{R}$. We can write down each linear map explicitly as $f_i(x) = r_i x + a_i$. We assume that $r_i \in (0, 1)$ for all $i \in \Lambda$. We call such a collection of linear maps to be a *linear iterated function system (IFS)*. The parameters $r_i, i \in \Lambda$ are called *contraction ratios* and $a_i, i \in \Lambda$ are called *translations*. In the case when all the contraction ratios are equal to $r \in (0, 1)$, we call r to be the *uniform contraction ratio*.

By [11], there is a unique non-empty compact set F such that

$$F = \bigcup_{i\in\Lambda} f_i(F).$$

We call such a set F to be a *self-similar set* determined by $\mathcal{F}$. We say that $\mathcal{F}$ satisfies the *open set condition* if there is a bounded open set $U \subset \mathbb{R}$ such that $f_i(U) \subset U$ for each $i \in \Lambda$ and

$$f_i(U) \cap f_j(U) = \emptyset$$

as long as $i \neq j$. The open set condition is a condition on $\mathcal{F}$ rather than F. However, we also say that F satisfies the open set condition if there is an IFS $\mathcal{F}$ such that $\mathcal{F}$ determines F and has the open set condition.

Let $b > 1$ be an integer and $B \subset \{0, \ldots, b-1\}$. Let A_b^B be the set

$$\begin{aligned}&\overline{\{x \in [0, 1] \setminus \mathbb{Q} : \text{ the } b\text{-ary expansion of } x \text{ contains only digits in } B\}}\\ &\quad = \{x \in [0, 1] : \text{ some } b\text{-ary expansion of } x \text{ contains only digits in } B\}.\end{aligned}$$

Then A_b^B is self-similar with $\Lambda = B$, and for $z \in B$, the linear map f_z is as follows:

$$x \in \mathbb{R} \to f_z(x) = \frac{1}{b}x + \frac{z}{b}.$$

To verify that $\mathcal{F} = \{f_z\}_{z\in B}$ has the open set condition, we can choose $U = (0, 1)$. A famous example of this kind is the middle third Cantor set $A_3^{\{0,2\}}$.

3.3. *Thickness, intersection, and sums of Cantor sets.* Let $A \subset \mathbb{R}$ be a compact, totally disconnected set without isolated points. We shall call A a Cantor set. First, we assume that $A \subset [0, 1]$ and the convex hull of A is $[0, 1]$. In this case, we see that $\mathbb{R} \setminus A$ is a countable union of disjoint open intervals $\{I_i\}_{i\geq 1}$. Notice that $(-\infty, 0)$, $(1, \infty)$ are the only unbounded intervals in $\{I_i\}_{i\geq 1}$. We call bounded intervals in $\{I_i\}_{i\geq 1}$ to be *bounded*

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

gaps of A. Thus, A can be constructed by iteratively chopping out open intervals from $\mathbb{R}$. Let $I = (a, b)$ be one of those bounded open intervals. We find the interval $I^- \in \{I_i\}_{i\geq 0}$ and $I^- \subset (-\infty, a)$ such that $|I^-| \geq |I|$ and there is no other such interval between I^- and I. Similarly, we can find $I^+ \subset (b, \infty)$ to the right of I. Suppose that $I^- = (c, d)$ and $I^+ = (e, f)$. We see that

$$-\infty \leq c < d < a < b < e < f \leq \infty.$$

Let $b_L = a - d$, $b_R = e - b$ and

$$C(I) = \min\{b_L, b_R\}/|I|.$$

We define $C(A) = \inf_{I\in\{I_i\}_{i\geq 1}, I \text{ bounded}} C(I)$. This number $C(A)$ is called *the thickness* of A. We also define *the normalized thickness* of A to be

$$S(A) = \frac{C(A)}{C(A) + 1}.$$

For a general Cantor set A, we can perform a uniquely determined orientation-preserving affine transformation T which maps the convex hull of A to the unit interval. Then we define

$$C(A) = C(T(A)), S(A) = S(T(A)).$$

We have the following result due to Newhouse. See [14].

THEOREM 3.2. (Newhouse's gap lemma) *Let A, B be two compact, totally disconnected sets. Suppose that A is not contained in any of the gaps of B and vice versa. If $S(A) + S(B) \geq 1$, then $A \cap B \neq \emptyset$.*

Newhouse proved the above result with the condition $S(A) + S(B) > 1$. Astels [1, Theorem 2.2] showed that the above result holds for $S(A) + S(B) = 1$ as well.

In the case when we have two very thick Cantor sets, we expect that their intersection could also be very thick. In this direction, we have [10, Theorem 1, and the discussion at the beginning of p. 882, and the discussion before Corollary 6].

THEOREM 3.3. *Let A, B be two compact, totally disconnected sets. Suppose that A is not contained in any of the gaps of B and vice versa. For each $\delta > 0$, there is an $\epsilon > 0$ such that if $S(A)$, $S(B)$ are greater than $1 - \epsilon$, then there is a compact set $C \subset A \cap B$ such that $S(C)$ is greater than $1 - \delta$. Moreover, suppose that the convex hulls* Conv(A), Conv(B) *are such that* Conv(A) ∩ Conv(B) *contains neither A nor B. Then by making ϵ smaller if necessary, C can be chosen so that* |Conv(C)| *is at least* $(1 - \delta)$|Conv(A) ∩ Conv(B)|.

Recently, there are some further development on the intersecting structures of thick Cantor sets. See [19].

The following result was proved in [1, Theorem 2.4].

THEOREM 3.4. *Let $A_1, \ldots, A_k$ be $k \geq 2$ Cantor sets. Suppose that their convex hulls are $I_1, I_2, \ldots, I_k$ and the size of their largest bounded gaps are $g_1, \ldots, g_k$ respectively. Suppose further that $\sum_{i=1}^{k} S(A_i) \geq 1$ and $\min\{|I_1|, \ldots, |I_k|\} > \max\{g_1, \ldots, g_k\}$. Then $A_1 + \cdots + A_k$ is an interval.*

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

For later use, we reformulate the above result in terms of intersections. Let A_1, A_2, A_3 be three Cantor sets satisfying the hypothesis of the above theorem. Consider the Cartesian product $A = A_1 \times A_2 \times A_3$. Let $v \in S^2$ be a direction vector of $\mathbb{R}^3$ and let H_v be the plane passing through the origin and normal to v. The family of planes parallel to H_v can be parameterized by $\mathbb{R}$, more precisely, $\{H_v(a) = H_v + av\}_{a\in\mathbb{R}}$. Denote $P_v : \mathbb{R}^3 \to \mathbb{R}v$ to be the corresponding orthogonal projection in direction v. The above theorem says that $P_{(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})}(A)$ is an interval. Moreover, notice that the normalized thickness is invariant under affine maps. In addition, there is a non-empty open neighborhood O of $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ in S^2 such that whenever $v = (v_1, v_2, v_3) \in O$, the size of the minimum convex hull of v_1A_1, v_2A_2, v_3A_3 is larger than the size of their maximum gap. This implies that $P_v(A)$ is an interval.

Now if $\Pi_v(A)$ is an interval, then we see that

$$\{a \in \mathbb{R} : H_v(a) \cap A \neq \emptyset\}$$

is an interval. More precisely, if $H_v(a) \cap I_1 \times I_2 \times I_3 \neq \emptyset$, then we have $H_v(a) \cap A \neq \emptyset$. From here, we have the following corollary.

COROLLARY 3.5. *Let A_1, A_2, A_3 be 3 Cantor sets with $S(A_1) + S(A_2) + S(A_3) \geq 1$. Suppose that their convex hulls are I_1, I_2, I_3 and the size of their largest gaps are g_1, g_2, g_3 respectively. If*

$$\min\{|I_1|, |I_2|, |I_3|\} > \max\{g_1, g_2, g_3\},$$

then there is a non-empty open set $O \subset S^2$ such that whenever $v \in O$, we have

$$H_v(a) \cap I_1 \times I_2 \times I_3 \neq \emptyset \implies H_v(a) \cap A_1 \times A_2 \times A_3 \neq \emptyset.$$

We now compute the normalized thickness of some examples of Cantor sets. First, let $b > 2$ be an integer and let $B = \{0, 1, \ldots, l\}$ where $l < b - 1$. We consider the set

$$A_b^B = \{x \in [0, 1] : \text{ some } b\text{-ary expansion of } x \text{ contains only digits in } B\}.$$

LEMMA 3.6. *Let b, B, A_b^B be as above. The normalized thickness of A_b^B is*

$$S(A_b^B) = \frac{l}{b-1}.$$

Proof. The convex hull of A_b^B is $[0, a]$, where

$$a = \sum_{i=1}^{\infty} \frac{l}{b^i} = \frac{l}{b-1}.$$

The largest gaps of A_b^B are of size

$$\frac{b-1-l}{b(b-1)}.$$

Those gaps are located in each of the intervals

$$[0, 1/b], \ldots, [(l-2)/b, (l-1)/b].$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Let I be one of those gaps. Then we see that

$$C(I) = \frac{l}{b-1-l}.$$

From here and the fact that A_b^B is self-similar, we see that $C(A_b^B) = l/(b-1-l)$ and $S(A_b^B) = l/(b-1)$. □

Next, we consider the middle third Cantor set $C_3 = A_3^{\{0,2\}}$. Following the above steps, we see that $C(C_3) = 1$, $S(C_3) = 1/2$. Let $k > 1$. We consider the image of C_3 under the map $x \to x^k$. We write this image as C_3^k. We record here a simple observation which illustrates an application of the thickness of the Cantor set.

THEOREM 3.7. *For all $x \in [0, 4]$, there exist $x_1, x_2, x_3, x_4 \in C_3$ such that*

$$x = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

More generally, for each number $k > 1$, there is an integer $1 < n(k) \leq 2^k$ such that all $x \in [0, n(k)]$ can be written as

$$x = \sum_{i=1}^{n(k)} x_i^k,$$

where $x_1, \ldots, x_{n(k)} \in C_3$.

(We were told by S. Chow that this result (for $k = 2$) was conjectured in [2, Conjecture 13] and answered in [12]. We thank him for providing the references.)

LEMMA 3.8. *Let k, C_3^k be as above. Then*

$$S(C_3^k) = \frac{1}{2^k}.$$

Proof. Suppose that $I = (a, a + \Delta)$ is a bounded gap of C_3. Then we see that the next gap on the right of I which is not smaller than I has a left endpoint $a + 2\Delta$. Similarly, the next gap on the left of I which is not smaller than I has a right endpoint $a - \Delta$. For the middle third Cantor set C_3, we always have $a \geq \Delta$.

After taking the k-th power map, we have points

$$(a-\Delta)^k, a^k, (a+\Delta)^k, (a+2\Delta)^k.$$

The gap I is now transformed into a gap of size

$$|a^k - (a+\Delta)^k|.$$

This length is increasing as a function of a as well as Δ. Thus we see that the number b_L in the definition of thickness is at least $|(a-\Delta)^k - a^k|$. We need to take care of b_R. By the above argument, we see that b_R is at most $(a+2\Delta)^k - (a+\Delta)^k$. However, we need a lower bound for b_R. To do this, we need to study how the gaps of C_3 in $[a+\Delta, a+2\Delta]$ are transformed under the map $x \to x^k$. Notice that a gap inside $[a+\Delta, a+2\Delta]$ might become larger than $|a^k - (a+\Delta)^k|$ after taking the k-th power. Note that by the convexity

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

of $x \to x^k$, there is a $\delta \in (0, \Delta]$ such that

$$(a + \Delta + 2\delta)^k - (a + \Delta + \delta)^k = (a + \Delta)^k - a^k.$$

Let $(a + \Delta + \alpha, a + \Delta + \alpha + \beta)$ be a bounded gap of $C_3 \cap [a + \Delta, a + 2\Delta]$ with $\alpha, \beta > 0$ and $\alpha + \beta \le \Delta$. Then we have $\beta \le \alpha$. This follows by self-similarity, since $[a + \Delta, a + 2\Delta]$ is a similar copy of C_3 with contraction factor Δ. From here, we see that to find gaps of C_3^k on the right side of $(a + \Delta)^k$ with length at least $|a^k - (a + \Delta)^k|$, one has to search the gaps of C_3^k in $[(a + \Delta + \delta)^k, \infty)$. Thus b_R is at least

$$(a + \Delta + \delta)^k - (a + \Delta)^k.$$

Observe that for $x = \delta/(a + \Delta)$,

$$\frac{b_R}{(a + \Delta)^k - a^k} \ge \frac{(a + \Delta + \delta)^k - (a + \Delta)^k}{(a + \Delta + 2\delta)^k - (a + \Delta + \delta)^k} = \frac{(1 + x)^k - 1}{(1 + 2x)^k - (1 + x)^k}.$$

Since $0 < \delta \le \Delta \le a$, we see that the above is at least (the value when $x = 1/2$)

$$c_k = \frac{1.5^k - 1}{2^k - 1.5^k}.$$

It follows that $b_R \ge c_k((a + \Delta)^k - a^k)$. We see that

$$\min\{b_L, b_R\}/|a^k - (a + \Delta)^k| \ge \min\left\{\frac{a^k - (a - \Delta)^k}{(a + \Delta)^k - a^k}, c_k\right\} \ge \frac{1}{2^k - 1}. \tag{2}$$

For the last inequality, notice that

$$\frac{a^k - (a - \Delta)^k}{(a + \Delta)^k - a^k} = \frac{1 - (1 - \Delta/a)^k}{(1 + \Delta/a)^k - 1}.$$

As $(a, a + \Delta)$ is a bounded gap of C_3, we see that $0 < \Delta \le a$. The function

$$x \in [0, 1] \to \frac{1 - (1 - x)^k}{(1 + x)^k - 1}$$

takes the minimum at $x = 1$ with the value $1/(2^k - 1)$. For all $k > 1$, we have

$$\frac{1}{2^k - 1} \le \frac{1.5^k - 1}{2^k - 1.5^k}.$$

From here, we conclude (2). As (2) holds for all bounded gaps of C_3^k, we see that

$$C(C_3^k) \ge \frac{1}{2^k - 1}$$

and

$$S(C_3^k) \ge \frac{1}{2^k}.$$

However, let $n \ge 1$ be an integer. Then $I = (3^{-n}, 2 \times 3^{-n})$ is a bounded gap of C_3. Now, in C_3, the next gap on the left of I with length at least $|I|$ is an infinite gap, that is, $(-\infty, 0)$. Thus in C_3^k, the next gap on the left of $(3^{-kn}, 2^k 3^{-kn})$ with at least the same length is again $(-\infty, 0)$. This shows that the inequality (2) is sharp and the proof concludes. □

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

From here, we see that Theorem 3.7 follows.

Proof of Theorem 3.7. By Lemma 3.8 and Theorem 3.4, it is enough to check the gap conditions stated in Theorem 3.4. As the convex hull of C_3^k is [0, 1] and the largest gap is strictly shorter than 1, the result follows. □

4. *Radial projections of fractal sets: general overview*

Let $d \geq 2$ be an integer. Let $x \in \mathbb{R}^d$ be a point. Recall that Π_x is defined as follows:

$$\Pi_x(y) = \frac{y - x}{|x - y|} \in S^{d-1}$$

for $y \neq x$. The following result was proved in [15].

THEOREM 4.1. *Let $d \geq 2$ be an integer. Let $A \subset \mathbb{R}^d$ be a Borel set with $\dim_{\mathrm{H}} A > d - 1$. Then $\Pi_x(A)$ has positive Lebesgue measure for almost all $x \in \mathbb{R}^d$.*

Intuitively speaking, the above theorem says that if $A \subset \mathbb{R}^d$ is large enough, then $\Pi_x(A)$ should also be large enough, at least generically. If the set A is a Cartesian product of self-similar sets with the open set condition, we believe that a much stronger result should hold. See Conjecture 2.1. In our situation, it is convenient to explicitly state a special case.

Conjecture 4.2. Let $d > 1$ be an integer. Let $b_1, \ldots, b_d > 1$ be integers. For each $i \in \{1, \ldots, d\}$, let $B_1 \in \{0, \ldots, b_i - 1\}$ be a choice of digits in base b_i and

$$A_i = \{x \in [0, 1] : \text{ some } b_i\text{-ary expansion of } x \text{ contains only digits in } B_i\}.$$

Consider the set $A = A_1 \times \cdots \times A_d$. If $\dim_{\mathrm{H}} A > d - 1$, then $\Pi_x(A)$ has positive Lebesgue measure for all $x \in \mathbb{R}^d$. If moreover $b_1, \ldots, b_d$ are multiplicatively independent, then $\Pi_x(A)$ contains a non-empty interior for all $x \in \mathbb{R}^d$.

As $A_1, \ldots, A_d$ are self-similar sets with the open set condition, this conjecture is indeed a special case of Conjecture 2.1 under the multiplicative independence of $b_1, \ldots, b_d$. We believe that for the positivity of the Lebesgue measure, it is not necessary to require the multiplicative independence of $b_1, \ldots, b_d$. Currently, almost nothing is known towards Conjectures 2.1, 4.2. For Conjecture 2.1, see [9] or [17] for related results. Although results in [9, 17] are for linear projections rather than radial projections, it is possible to use the arguments to prove results for linear projections. For example, let A be as in Conjecture 2.1 with $d = 2$. Results in [17] provide estimates of the size $l^\delta \cap A$ (l^δ is the δ-neighborhood of l) uniformly across $\delta > 0$ and lines $l \subset \mathbb{R}^2$ with direction strictly away from being parallel with the coordinate axes. In particular, for each $\epsilon > 0$, for all small enough $\delta > 0$ and all lines l, to cover $l^\delta \cap A$, it is enough to use at most $\delta^{-(\dim_{\mathrm{H}} A - 1 + \epsilon)}$ many δ-balls. Using this estimate, we can conclude that $\Pi_{(0,0)} A$ cannot be small. In fact, suppose that $\Pi_{(0,0)} A$ can be covered by at most $0.0001\delta^{-(1-2\epsilon)}$ many δ-balls. Then we are forced to have a fiber of $\Pi_{(0,0)}(A)$ which cannot be covered by

$$\delta^{-\dim_{\mathrm{H}} A} / \delta^{-(1-2\epsilon)} = \delta^{-(\dim_{\mathrm{H}} A - 1 + 2\epsilon)}$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

many δ-balls. Fibers of $\Pi_{(0,0)}(A)$ are lines passing through the origin. In particular, this contradicts with the results in [17]. From here, we conclude that $\dim_{\mathrm{B}} \Pi_{(0,0)} A = 1$. Here, $\dim_{\mathrm{B}} \Pi_{(0,0)} A$ is the box dimension of $\Pi_{(0,0)} A$. It is possible to upgrade this result to $\dim_{\mathrm{H}} \Pi_{(0,0)} A = 1$ by carefully studying a certain regular measure μ supported on A and its image under $\Pi_{(0,0)}$. Of course, (0, 0) can be replaced with any other point in $\mathbb{R}^2$.

5. *Proofs of Theorems 2.2, 2.4, 2.5, and 2.7*

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Consider the set $N_3^{B_3} \times N_5^{B_5} \times N_7^{B_7}$, where

$$B_3 = \{0, 1\}, \quad B_5 = \{0, 1, 2\}, \quad B_7 = \{0, 1, 2, 3\}.$$

Now we construct the sets

$$A_3 = \{x \in [1, 3] : \text{ some 3-ary expansion of } x \text{ contains only digits in } B_3\},$$

$$A_5 = \{x \in [1, 5] : \text{ some 5-ary expansion of } x \text{ contains only digits in } B_5\},$$

$$A_3 = \{x \in [1, 7] : \text{ some 7-ary expansion of } x \text{ contains only digits in } B_7\}.$$

Then we see that [7, §7.1] $\dim_{\mathrm{H}} A_3 \times A_5 \times A_7 = \log 2/\log 3 + \log 3/\log 5 + \log 4/\log 7 > 2$. Let $\{\}$ be the fractional part symbol, that is, for a real number a, $\{a\} \in [0, 1)$ is the unique number t in $[0, 1)$ so that $t - a \in \mathbb{Z}$. For each integer $k \geq 1$, consider the line l_k passing through the origin with direction vector

$$(1, 5^{\{k \log 3/\log 5\}}, 7^{\{k \log 3/\log 7\}}).$$

If $l_k \cap A_3 \times A_5 \times A_7 \neq \emptyset$, then we take a point $(x, y, z) \in l_k \cap A_3 \times A_5 \times A_7$. Consider the point

$$(x', y', z') = (3^k x, 5^{[k \log 3/\log 5]} y, 7^{[k \log 3/\log 7]} z).$$

Since $y = 5^{\{k \log 3/\log 5\}} x$, $z = 7^{\{k \log 3/\log 7\}} x$, we see that

$$5^{[k \log 3/\log 5]} y = 5^{k \log 3/\log 5} x = 3^k x, \; 7^{[k \log 3/\log 7]} z = 7^{k \log 3/\log 7} x = 3^k x.$$

Thus we see that $x' = y' = z'$. It is straightforward to see that the 3-ary expansion of x' contains only digits in B_3, the 5-ary expansion of y' contains only digits in B_5, and the 7-ary expansion of z' contains only digits in B_7. Taking the integer part, we see that

$$[x'] = [y'] = [z'] \in N_{3,5,7}^{B_3, B_5, B_7}.$$

Under Conjecture 2.1, we see that there are infinitely many integers $k \geq 1$ such that

$$l_k \cap A_3 \times A_5 \times A_7 \neq \emptyset.$$

Indeed, the direction vector of l_k is

$$(1, 5^{\{k \log 3/\log 5\}}, 7^{\{k \log 3/\log 7\}}).$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Under Schanuel's conjecture, 1, $\log 3/\log 5$, $\log 3/\log 7$ are linearly independent over $\mathbb{Q}$, and therefore the sequence

$$(\{k \log 3/\log 5\}, \{k \log 7/\log 5\})_{k\geq 0}$$

equidistributes in $[0, 1]^2$. Thus the closure of $\bigcup_{k\geq 0} l_k$ contains the cone C spanned by the origin and the set

$$\{1\} \times [1, 5] \times [1, 7].$$

Now if $\Pi_{(0,0,0)}(A_3 \times A_5 \times A_7 \cap C)$ has a non-empty interior, then l_k will intersect $A_3 \times A_5 \times A_7 \cap C$ infinitely often. Observe that $A_3 \times A_5 \times A_7$ intersects the interior of C. Thus there is a point $a \in A_3 \times A_5 \times A_7$ and a $r > 0$ such that $B_a(r) \subset C$. Since $A_3 \cap [1, 2]$, $A_5 \cap [1, 2]$, $A_7 \cap [1, 2]$ are self-similar with contraction ratios $1/3$, $1/5$, $1/7$ respectively, it is possible to find linear maps $f_3, f_5, f_7 : \mathbb{R} \to \mathbb{R}$:

$$f_3(x) = 3^{-l_3}x + t_3, \quad f_5(x) = 5^{-l_5}x + t_5, \quad f_7(x) = 7^{-l_7}x + t_7$$

for some positive integers l_3, l_5, l_7 and real numbers t_3, t_5, t_7 such that

$$f_3(A_3 \cap [1, 2]) \subset A_3, \quad f_5(A_5 \cap [1, 2]) \subset A_5, \quad f_7(A_7 \cap [1, 2]) \subset A_7,$$

and

$$f_3 \times f_5 \times f_7(A_3 \times A_5 \times A_7 \cap [1, 2]^3) \subset B_a(r).$$

We denote $A_3' = f_3(A_3 \cap [1, 2])$, $A_5' = f_5(A_5 \cap [1, 2])$, $A_7' = f_7(A_7 \cap [1, 2])$. Then we see that

$$A_3' \times A_5' \times A_7' \subset A_3 \times A_5 \times A_7 \cap C.$$

Moreover, since $f_3 \times f_5 \times f_7$ is linear and invertible and thus bi-Lipschitz, by [7, Corollary 2.4], we have

$$\dim_{\mathrm{H}}(A_3' \times A_5' \times A_7') = \dim_{\mathrm{H}}(A_3 \times A_5 \times A_7 \cap [1, 2]^3).$$

Next, observe that $A_3 \times A_5 \times A_7 \cap [1, 2]^3$ is a translation of $K_3 \times K_5 \times K_7$, where

$$K_3 = \{x \in [0, 1] : \text{ some 3-ary expansion of } x \text{ contains only digits in } B_3\},$$

$$K_5 = \{x \in [0, 1] : \text{ some 5-ary expansion of } x \text{ contains only digits in } B_5\},$$

$$K_7 = \{x \in [0, 1] : \text{ some 7-ary expansion of } x \text{ contains only digits in } B_7\}.$$

By [7, §7.1], we see that $\dim_{\mathrm{H}} K_3 \times K_5 \times K_7 = \dim_{\mathrm{H}} A_3 \times A_5 \times A_7$. Now Conjecture 2.1 tells us that $\Pi_{(0,0,0)}(A_3' \times A_5' \times A_7')$ has a non-empty interior. This proves the result. □

Theorem 2.4 follows by using a similar argument.

Proof of Theorem 2.4. The proof is very similar to the previous one. Let p, q be two distinct odd primes. Then we see that $\log p/\log q$ is irrational. Consider the set $N_p^{B_p} \times N_q^{B_q}$, where

$$B_p = \{0, 1, \ldots, (p-1)/2\}, \quad B_q = \{0, 1, \ldots, (q-1)/2\}.$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

We also construct the sets

$$A_p = \{x \in [1, p] : \text{ some } p\text{-ary expansion of } x \text{ contains only digits in } B_p\},$$

$$A_q = \{x \in [1, q] : \text{ some } q\text{-ary expansion of } x \text{ contains only digits in } B_q\}.$$

Now by Lemma 3.6 (applied to suitable affine copies of A_p, A_q), we see that $S(A_p) = S(A_q) = 1/2$. We now use Theorem 3.4. Since $S(A_p) + S(A_q) = 1$, we see that the difference set $A_p - A_q$ is an interval (for the gap condition, observe that the largest gaps of A_p, A_q are shorter than the unit interval). This says that $P_{(-1/\sqrt{2},1/\sqrt{2})}(A_p \cap A_q)$ is an interval. The convex hull of $A_p \cap A_q$ is a rectangle not contained on only one side of the line $\{x = y\}$. This implies that $P_{(-1/\sqrt{2},1/\sqrt{2})}(A_p \cap A_q)$ contains $(0, 0)$ as an interior point. Since the gap condition is preserved under linear perturbations close to the identity, as a result, we see that there is a non-trivial open set $O \subset S^1$ containing $(-1/\sqrt{2}, 1/\sqrt{2})$ such that if $v \in O$, the linear projection $P_v(A_p \times A_q) \subset \mathbb{R}v$ is an interval containing $(0, 0)$. This implies that $l_{v^\perp} \cap A_p \cap A_q \neq \emptyset$. Thus we see that $\Pi_{(0,0)}(A_p \times A_q)$ contains a non-empty interior around $(1/\sqrt{2}), 1/\sqrt{2})$. We can now apply the same argument as in the proof of Theorem 2.2. For each integer $k \geq 1$, consider the line l_k passing through the origin with direction vector $(1, q^{\{k \log p/ \log q\}})$. If $l_k \cap A_p \times A_q \neq \emptyset$, then we find a point $(x, y) \in l_k \cap A_p \times A_q$. Next, consider the point

$$(x', y') = (p^k x, q^{[k \log p/ \log q]} y).$$

As before, we see that

$$[x'] = [y'] \in N_{p,q}^{B_p,B_q}.$$

After replacing $A_p \times A_q$ with a suitable affine copy $A'_p \times A'_q$ if necessary, as in the previous proof, we see that there is an interval $I \subset [0, 1]$ such that whenever $\{k \log p/ \log q\} \in I$, there is a number $n \in N_{p,q}^{B_p,B_q} \cap [p^k, p^{k+1}]$. Since $\{k \log p/ \log q\} \in I$ happens for k inside a subset of integers with positive density, we see that there is a $c > 0$ and for all large enough integers N, there are at least cN many intervals among

$$[1, p), [p, p^2), \ldots, [p^{N-1}, p^N)$$

intersecting $N_{p,q}^{B_p,B_q}$. Thus the result follows by applying Kummer's theorem (Theorem 1.3). □

At this stage, Theorem 2.5 seems to be clear, at least under Schanuel's conjecture. We will first prove this theorem under Schanuel's conjecture and then explain how to get rid of it.

Proof of Theorem 2.5. Consider the set $N_3^{B_3} \times N_4^{B_4} \times N_5^{B_5}$, where

$$B_3 = \{0, 1\}, B_4 = \{0, 1\}, B_5 = \{0, 1\}.$$

Now we construct the self-similar sets

$$A_3 = \{x \in [1/3, 2/3] : \text{ some 3-ary expansion of } x \text{ contains only digits in } B_3\},$$

$$A_4 = \{x \in [1/4, 1/2] : \text{ some 4-ary expansion of } x \text{ contains only digits in } B_4\},$$

$$A_5 = \{x \in [1/5, 2/5] : \text{ some 5-ary expansion of } x \text{ contains only digits in } B_5\}.$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

By Lemma 3.6 (applied to suitable affine copies of A_3, A_4, A_5), we see that $S(A_3) = 1/2$, $S(A_4) = 1/3$, and $S(A_5) = 1/4$. Thus we see that

$$S(A_3) + S(A_4) + S(A_5) = \frac{13}{12} > 1. \tag{3}$$

The convex hulls of A_3, A_4, A_5 are $[1/3, 1/2]$, $[1/4, 1/3]$, $[1/5, 1/4]$. Let j_1, j_2 be integers. For the time being, we treat j_1, j_2 as being general. Later on, we will see that for the proof of this theorem, it is enough to choose $j_1 = j_2 = 1$. Let $k \geq 1$ be an integer and let $H_{j_1,j_2,k}$ be the plane

$$\{x + 4^{-\{k \log 3/ \log 4\}+j_1} y - 5^{-\{k \log 3/ \log 5\}+j_2} z = 0\}.$$

Suppose that $H_{j_1,j_2,k} \cap A_3 \times A_4 \times A_5 \neq \emptyset$. We take a point (x, y, z) in this intersection. Consider the point

$$(x', y', z') = (3^k x, 4^{[k \log 3/ \log 4]+j_1} y, 5^{[k \log 3/ \log 5]+j_2} z).$$

Since we have

$$x + 4^{-\{k \log 3/ \log 4\}+j_1} y - 5^{-\{k \log 3/ \log 5\}+j_2} z = 0,$$

we see that

$$3^{-k} x' + 4^{-\{k \log 3/ \log 4\}-[k \log 3/ \log 4]} y' - 5^{-\{k \log 3/ \log 5\}-[k \log 3/ \log 5]} z' = 0.$$

Thus we have

$$x' + y' - z' = 0.$$

We assume Schanuel's conjecture for the time being. Later on, we will remove this dependence.

From (3) and the discussion above Corollary 3.5, we claim that there exist j_1, j_2 so that $H_{j_1,j_2,k}$ intersects $A_3 \times A_4 \times A_5$ for infinitely many integers k. More precisely, consider the plane

$$H = \{x + k_1 y - k_2 z = 0\},$$

where $k_1 \in [4^{j_1-1}, 4^{j_1}]$, $k_2 \in [5^{j_2-1}, 5^{j_2}]$. It's normal vector is $(1, k_1, -k_2)$. Then, as long as

$$1/2 + k_1/3 - k_2/5 \geq 0 \geq 1/3 + k_1/4 - k_2/4, \tag{4}$$

the Cartesian product of the convex hulls of A_3, A_4, A_5 intersects the plane H. In this case, we see that $H \cap A_3 \times A_4 \times A_5$ is not empty if the minimal hull/maximal gap condition in Corollary 3.5 is satisfied. The lengths of the convex hulls of A_3, A_4, A_5 are $1/6$, $1/12$, $1/20$. The lengths of the largest gaps are $1/18$, $1/24$, $3/100$. The minimal hull/maximal gap condition can be now written as

$$\min\{1/6, k_1/12, k_2/20\} \geq \max\{1/18, k_1/24, 3k_2/100\}. \tag{5}$$

This is equivalent to the conditions

$$k_1 \in [2/3, 4], \quad k_2 \in [10/9, 50/9], \quad k_1/k_2 \in [9/25, 6/5]. \tag{6}$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

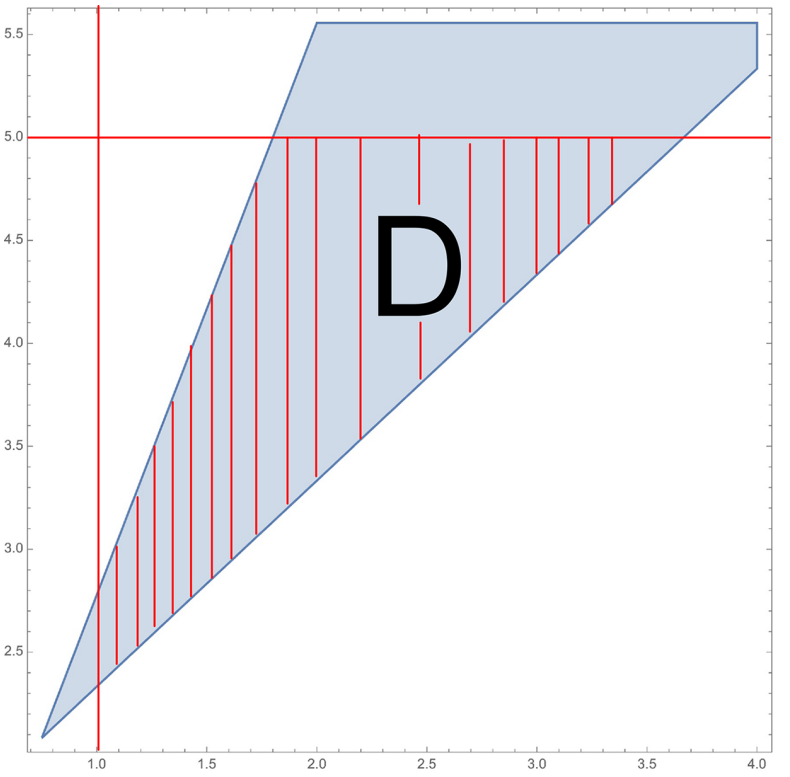

FIGURE 1. Conditions (4), (6) (shaded region), and the set D (covered with vertical lines).

The region determined by (4), (6) is illustrated in Figure 1. From Figure 1, it is possible to see that $j_1 = j_2 = 1$ will satisfy the claim. For later use, we write D for the set of pairs (k_1, k_2) with conditions (4), (6), and $k_1 \in [1, 4]$, $k_2 \in [1, 5]$. See the shaded region covered by vertical lines in Figure 2. Next, observe that the normal vector of $H_{1,1,k}$ is

$$(1, 4^{1-\{k \log 3/ \log 4\}}, -5^{1-\{k \log 3/ \log 5\}}),$$

namely, in terms of k_1, k_2 above,

$$k_1(k) = 4^{1-\{k \log 3/ \log 4\}}, \quad k_2(k) = 5^{1-\{k \log 3/ \log 5\}}.$$

By Schanuel's conjecture, we see that the set

$$\{(k \log 3/ \log 4, k \log 3/ \log 5) \bmod \mathbb{Z}^2\}_{k \geq 1} \subset [0, 1]^2$$

is the orbit of an irrational rotation. Therefore, it is dense in $[0, 1]^2$. From here, we see that there are infinitely many k such that $(k_1(k), k_2(k)) \in D$. For such a k, we have $H_{1,1,k} \cap A_3 \times A_4 \times A_5 \neq \emptyset$. This proves the claim with $j_1 = j_2 = 1$.

Now we know that there are infinitely many integers k so that $H_{1,1,k} \cap A_3 \times A_4 \times A_5 \neq \emptyset$. Now x', y', z' contains only $\{0, 1\}$ in their 3, 4, 5-ary expansions respectively. However, they might not be integers. If we take the integer parts, we see that $[x']$, $[y']$, $[z']$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

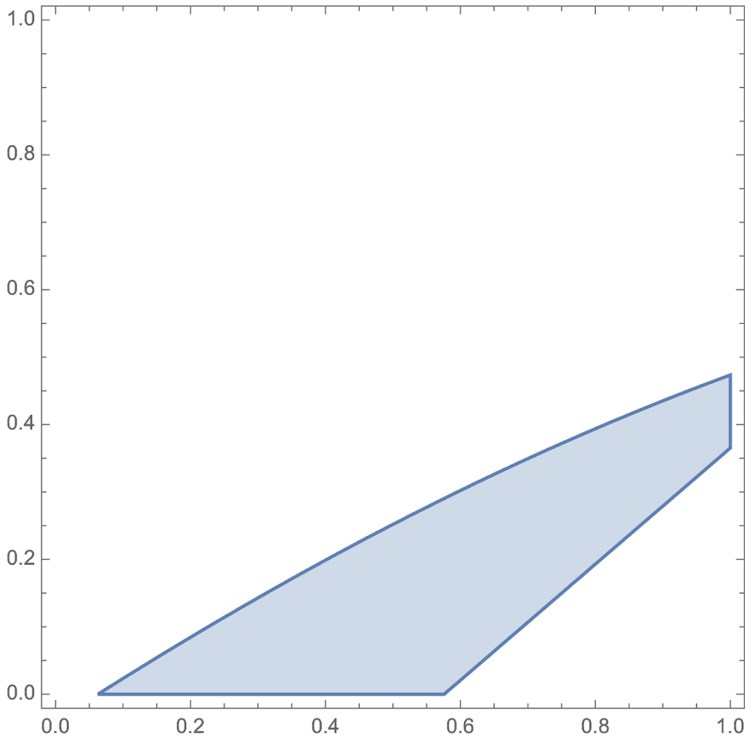

FIGURE 2. The region Log(D).

contains only $\{0, 1\}$ in their 3, 4, 5-ary expansions respectively and

$$[x'] + [y'] = [z'] + \{z'\} - \{x'\} - \{y'\}.$$

The above equation tells us that

$$\{z'\} - \{x'\} - \{y'\}$$

is an integer. Observe that $\{x'\}, \{y'\}, \{z'\}$ are positive numbers whose 3, 4, 5-ary expansions (respectively) contain only digits 0 and 1. Thus we see that

$$\{x'\} \in (0, 1/2], \quad \{y'\} \in (0, 1/3], \quad \{z'\} \in (0, 1/4].$$

This implies that

$$-\tfrac{5}{6} < \{z'\} - \{x'\} - \{y'\} \leq \tfrac{1}{4}.$$

The only integer in this range is 0. Thus, we see that

$$\{z'\} - \{x'\} - \{y'\} = 0$$

and

$$[x'] + [y'] = [z'].$$

From here, the result follows from Schanuel's conjecture.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Now, although we strongly do not believe it, it can be the case that $1, \log 3/\log 4, \log 3/\log 5$ are $\mathbb{Q}$-dependent. Then the rotation on $\mathbb{T}^2$ generated by the translation vector $(\log 3/\log 4, \log 3/\log 5)$ degenerates to an irrational rotation on a subtorus of dimension one. To be more precise, suppose that there are integers k_1, k_2, k_3 such that

$$k_1 \frac{\log 3}{\log 4} + k_2 \frac{\log 3}{\log 5} + k_3 = 0.$$

Neither k_1 nor k_2 is zero. For example, suppose that $k_1 = 0$, then this implies that $1, \log 3, \log 5$ are not $\mathbb{Q}$-linearly independent, which is not possible. For convenience, we write $a = \log 3/\log 4$, $b = \log 3/\log 5$. We see that

$$k_1 a + k_2 b = -k_3.$$

We can find non-zero coprime integers l_1, l_2 such that

$$l_1 a + l_2 b = c \in \mathbb{Q}. \tag{7}$$

Here, $c \neq 0$ since otherwise $\log 5/\log 4 = a/b \in \mathbb{Q}$, which is not the case. Since $\gcd(l_1, l_2) = 1$, it is possible to find $S \in SL_2(\mathbb{Z})$ with entries

$$S = \begin{pmatrix} l_1 & l_2 \\ l_1' & l_2' \end{pmatrix}.$$

Thus, S is a well-defined invertible map $\mathbb{T}^2 \to \mathbb{T}^2$. Let k be an integer and consider the point $(ka, kb) \in \mathbb{T}^2$. Now, we see that

$$\begin{aligned} S(ka, kb) &= (l_1 ka + l_2 kb, l_1' ka + l_2' kb) \bmod \mathbb{Z}^2 \\ &= (kc, k(l_1' a + l_2' b)) \bmod \mathbb{Z}^2. \end{aligned}$$

It is simple to check that $l_1' a + l_2' b \notin \mathbb{Q}$ for otherwise, both a, b are rational, which is impossible.

We identify $\mathbb{T}^2$ with $[0, 1]^2$ by letting

$$(x, y) \bmod \mathbb{Z}^2$$

be the unique point $(x', y') \in [0, 1)^2$ such that $(x', y') - (x, y) \in \mathbb{Z}^2$. Under this identification, the closure $\overline{\{S(ka, kb)\}_{k \geq 0}}$ is a union of vertical line segments in $[0, 1]^2$. In particular, it contains the vertical line $\{x = 0\} \cap [0, 1]^2$. Performing S^{-1}, we see that $\overline{\{(ka, kb)\}_{k \geq 0}}$ contains the line

$$L_0 = S^{-1}(\{x = 0\}) = \{(x, y) \in \mathbb{T}^2 : l_1 x + l_2 y = 0\}.$$

Although, in general, L_0 is a union of line segments in $[0, 1]^2$, we will call it to be a line. In fact, it is actually a line in $\mathbb{T}^2$. Since $c \neq 0$, we write $c = p/q$ or $c = -p/q$ with positive integers p, q so that $\gcd(p, q) = 1$. Then, with similar arguments as above, we see that $\overline{\{(ka, kb)\}_{k \geq 0}}$ also contains

$$L_t = \{(x, y) \in [0, 1]^2 : l_1 x + l_2 y \in \mathbb{Z} + t\}$$

for $t \in \{1/q, 2/q, \ldots, (q-1)/q\}$.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Next, we use our knowledge of the set D in Figure 1. We want to choose an integer k such that

$$V_k = (4^{-\{k \log 3/ \log 4\}+1}, 5^{-\{k \log 3/ \log 5\}+1}) \in D.$$

The closure of V_k, $k \geq 1$ is a union of curves. After performing the logarithmic map $\mathrm{Log} : (x, y) \to (1 - \log x/ \log 4, 1 - \log y/ \log 5)$, those curves become line segments. As we discussed above, $\overline{\{V_k\}_{k\geq 1}}$ contains the line L_0.

Recall that neither l_1 nor l_2 is zero. To find $V_k \in D$, it is enough to find k so that

$$(\{k \log 3/ \log 4\}, \{k \log 3/ \log 5\}) \in \mathrm{Log}(D).$$

To do this, it is enough to show that at least one of the lines

$$L_t, t \in \{0, 1/q, 2/q, \ldots, (q-1)/q\}$$

intersect the interior of $\mathrm{Log}(D)$. If this is done, then we can find infinitely many k so that $V_k \in D$. The region $\mathrm{Log}(D)$ is illustrated in Figure 2.

In order that the line L_0 avoids $\mathrm{Log}(D)$, both l_1, l_2 cannot be too large. First, $\mathrm{Log}(D) \cap \{y = 0\}$ has length bigger than 0.5. Since L_0 contains points $(0, 1), (1/|l_1|, 1), \ldots, ((|l_1| - 1)/|l_1|, 1), (1, 1)$, we see that $|l_1| < 2$. Therefore, $l_1 = \pm 1$. Without loss of generality, we can assume that $l_1 = 1$. Then we can consider possible values for l_2. We see that $l_2 < 0$. Otherwise if $l_2 > 0$, L_0 must contain a line segment starting from $(1, 0)$ to the boundary $\{x = 0\}$ and this line segment must intersect the interior of $\mathrm{Log}(D)$ unless it is horizontal, which is not possible. Similarly, as long as $l_2 \leq -3$, L_0 contains a line segment starting from a point $(1, r)$, $r \in [0, 1/3]$ to the boundary $\{x = 0\}$ and this line segment must intersect the interior of $\mathrm{Log}(D)$. Thus we see that all possible values for l_2 are $-1, -2$.

Now, we consider all possible values for $c = p/q$. We see that $(t, 0) \in L_t$ for $t \in \{1/q, \ldots, (q-1)/q\}$. Then with a similar argument as above, we see that $q < 2$. Therefore, $q = 1$. This implies that $c \in \mathbb{Z}$.

In conclusion, in order that $V_k \in D$ for at most finitely many integers k, we must have

$$l_1 a + l_2 b \in \mathbb{Z}$$

for $l_1 = 1, l_2 = -1$ or -2. It can be checked directly that

$$\frac{\log 3}{\log 4} - \frac{\log 3}{\log 5} = 0.109875 + e_1,$$
$$\frac{\log 3}{\log 4} - 2\frac{\log 3}{\log 5} = -0.572731 + e_2,$$

where $|e_1|, |e_2| < 0.001$. Thus, the numbers on the left-hand side above are not integers. From here, the proof is finished. □

We discuss more on Theorem 2.5. First, we remark that with the same proof, one can also show that the conclusion holds with $(3, 4, 5)$ being replaced with $(3, 4, 7)$. More generally, let $l \geq 1$ be an integer. Let $D = \{0, \ldots, l\}$. We want to consider the set $(N_{b_1}^D + N_{b_2}^D) \cap N_{b_3}^D$ for $b_1, b_2, b_3 \geq l + 1$. In this case, under Schanuel's conjecture, it is

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

possible to show that

$$(N_{b_1}^D + N_{b_2}^D) \cap N_{b_3}^D$$

is infinite as long as

$$\frac{l}{b_1 - 1} + \frac{l}{b_2 - 1} + \frac{l}{b_3 - 1} \geq 1,$$

and $1, \log b_1, \log b_2, \log b_3$ are $\mathbb{Q}$-linearly independent. Again, for specific cases, Schanuel's conjecture may not be required. For example, when $l = 2$, it is possible to check that the above statement holds for $(b_1, b_2, b_3) = (4, 5, 6)$.

Lastly, we are going to prove Theorem 2.7. Before that, we first show the following results.

LEMMA 5.1. *Let $b \geq 3$ be an integer and let $B = \{1, 2, \ldots, b-1\}$. Consider the set*

$$A_b^B = \{x \in [0, 1] : \textit{ some } b\textit{-ary expansion of } x \textit{ contains only digits in } B\}.$$

Then we have $S(A_b^B) = (b-2)/(b-1)$. Moreover, let $k \in \{0, 1, \ldots, b-2\}$. Consider the set

$$\tilde{A}_b^B = A_b^B \cap [0, (b-k)/b].$$

Then, $S(\tilde{A}_b^B) = S(A_b^B)$.

Remark 5.2. The second conclusion is special for our self-similar set A_b^B. We need to cut A_b^B at the 'right places'. For general Cantor sets, cutting out a small portion may decrease the thickness dramatically.

Proof. The first conclusion follows from Lemma 3.6. Indeed, by applying the symmetry $x \in \mathbb{R} \to -x \in \mathbb{R}$ to the set A_b^B, we obtain a set for which Lemma 3.6 can be used.

For the second conclusion, we can use the self-similarity of A_b^B. First, let $[a, 1]$ be the convex hull of A_b^B (this determines the value for $a \in (0, 1/b)$). Then we see that $[a, (b-k)/b]$ is the convex hull of $\tilde{A}_b^B$. Observe that $A_b^B \cap [1/b, 2/b], \ldots, A_b^B \cap [(b-k-1)/b, (b-k)/b]$ are translated copies of each other. The largest gaps of $\tilde{A}_b^B$ are located inside each of those copies. From here, we see that

$$S(\tilde{A}_b^B) = (b-2)/(b-1).$$

□

LEMMA 5.3. *Let $\delta_1, \delta_2 \in (0, 1)$ numbers so that $\delta_1 < \delta_2$. Let $C \subset [0, 1]$ be a Cantor set with* Conv(C) $\neq [0, 1]$, *$S(C) > 1 - \delta_1$, and* $|\text{Conv(C)}| > 1 - \delta_1$. *Consider the set A_b^B as in Lemma 5.1. Then for each $\delta_2 \in (0, 1)$, as long as δ_1 is small enough, for all sufficiently large $b > 1$, there is a compact set $C' \subset C \cap A_b^B$ satisfying*

$$|\text{Conv(C}')| > 1 - \delta_2,$$
$$S(C') > 1 - \delta_2.$$

Moreover, there is a number $c > 0$ such that for linear maps $T : x \to rx + t$, $T' : x \to r'x + t'$ with $r, r' \in (1 - c, 1 + c)$ and $t, t' \in (-c, c)$, there is a compact set $C' \subset T(C) \cap T'(A_b^B) \cap [0, 1]$ satisfying the above properties as well.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

Proof. We want to use Theorem 3.3. The issue is that $\mathrm{Conv(C)} \cap \mathrm{Conv}(\mathrm{A_b^B})$ may contain C or A_b^B. Observe that $\mathrm{Conv}(\mathrm{A_b^B}) = [a_b, 1]$ with $\lim_{b\to\infty} a_b \to 0$, and that $\lim_{b\to\infty} S(A_b^B) \to 1$. So as long as b is large enough, C, A_b^B are not contained in each other's gaps. Let $\mathrm{Conv(C)} = [c, d] \subset [0, 1]$.

If $c = 0$, then $d < 1$ and we see that $\mathrm{Conv(C)} \cap \mathrm{Conv}(\mathrm{A_b^B})$ contains neither C nor A_b^B. Using Theorem 3.3, we conclude that as long as δ_1 is small enough and b is large enough, there is a compact set $C' \subset C \cap A_b^B$ such that

$$S(C'),\ |\mathrm{Conv(C')}|$$

are both greater than $1 - \delta_2$.

If $c > 0$, by making b sufficiently large, we can then assume that $a_b < c$. As $|\mathrm{Conv(C)}| > 1 - \delta_1$, we see that $d > 1 - \delta_1$. Let k be the smallest integer with $(b - k)/b < d$. Consider the set $\tilde{A}_b^B$ as in Lemma 5.1. We see that

$$|\mathrm{Conv}(\tilde{\mathrm{A}}_\mathrm{b}^\mathrm{B})| = \frac{b-k}{b} - a_b > 1 - \delta_1 - \frac{1}{b} - a_b.$$

This number can be made arbitrarily close to $1 - \delta_1$ by making b sufficiently large (in a manner that depends on δ_1). Now since $\mathrm{Conv(C)} \cap \mathrm{Conv}(\tilde{\mathrm{A}}_\mathrm{b}^\mathrm{B})$ contains neither C nor $\tilde{A}_b^B$, by Theorem 3.3, we see that (for small enough δ_1 and large enough b) there is a compact set $C' \subset C \cap \tilde{A}_b^B \subset C \cap A_b^B$ such that

$$S(C'),\ |\mathrm{Conv(C')}|$$

are both greater than $1 - \delta_2$.

For the second conclusion, notice that if $\mathrm{Conv(C)} \cap \mathrm{Conv}(\mathrm{A_b^B})$ does not contain C, A_b^B, then as long as c is small enough, $\mathrm{Conv(T(C))} \cap \mathrm{Conv}(\mathrm{T'}(\mathrm{A_b^B}))$ does not contain $T(C)$, $T'(A_b^B)$. The same conclusion holds with A_b^B being replaced with $\tilde{A}_b^B$. Thus it is possible to find a compact subset $C' \subset T(C) \cap T'(A_b^B)$ satisfying the desired properties. However, later on, it is more convenient to consider $T(C) \cap T'(A_b^B) \cap [0, 1]$. We cannot simply take $C' \cap [0, 1]$ as the thickness can decrease significantly. Instead, we consider the set $T'(A_b^B) \cap [0, 1]$. The set $T'(A_b^B)$ is a self-similar set. The scaling ratio is still $1/b$. The first level branches are intervals of length r'/b. It can happen that some of the first level branches are not contained in $[0, 1]$. If this is the case, we simply ignore them. We can choose c to be so small that we only need to ignore at most one first level branch. As a result, as long as c is small enough, we obtain a subset $\tilde{\tilde{A}}_b^B \subset T'(A_b^B) \cap [0, 1]$ whose thickness is the same as A_b^B and whose diameter is at least $1 - 3r'/b$. We can then consider the intersection between $T(C)$ and $\tilde{\tilde{A}}_b^B$ as in the case when T, T' are the identity map. From here, the result follows. □

Proof of Theorem 2.7. Let $3 \le b_1 \le \cdots \le b_k$ be (not necessarily distinct) integers. We need them to be sufficiently large in a manner that will be discussed later in the proof.

For each $i \in \{1, \ldots, k\}$, we construct the set

$$A_i = \{x \in (0, \infty) :\ \text{some } b_i\text{-ary expansion of } x \text{ does not contain digit } 0\}.$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

If x has two possible b_i-ary expansions, we will take the finite expansion. Thus the b_i-ary expansion for $x > 0$ is well defined. It is sufficient to consider the set

$$A = A_1 \cap A_2 \cdots \cap A_k.$$

We claim that it is enough to prove that A is unbounded. Indeed, if A is unbounded, then we can find arbitrarily large x whose base $b_1, \ldots, b_k$ expansions do not have digit zero. We simply take the integer part of x to obtain an integer $[x]$ whose base $b_1, \ldots, b_k$ expansions do not have digit zero. Since A is unbounded, the proof finishes.

Now it is enough to prove that A is unbounded. We define the integer sequence

$$\mathbf{a}_j = (a_{1,j}, \ldots, a_{k,j}),\ j \geq 0$$

by starting with

$$\mathbf{a}_0 = (1, \ldots, 1).$$

For $j \geq 1$, define $a_{1,j} = b_1 a_{1,j-1}$. For $i \in \{2, \ldots, k\}$, we define $a_{i,j} = b_i a_{i,j-1}$ if $a_{i,j-1}/a_{1,j} < b_i^{-1}$ and $a_{i,j} = a_{i,j-1}$ otherwise. In this way, we have

$$\mathbf{a}_j = (b_1^j, b_2^{[j \log_{b_2} b_1]}, \ldots, b_k^{[j \log_{b_k} b_1]}) = b_1^j (1, b_2^{-\{j \log_{b_2} b_1\}}, \ldots, b_k^{-\{j \log_{b_k} b_1\}}).$$

The rotation on $\mathbb{T}^{k-1}$ with the rotation angle

$$v = \left(\frac{\log b_1}{\log b_2}, \ldots, \frac{\log b_1}{\log b_k} \right)$$

may not be an irrational rotation on $\mathbb{T}^{k-1}$. Nonetheless, we claim that the orbit

$$nv \bmod \mathbb{Z}^{k-1}, n \geq 0$$

can be close to the origin, that is, $d(nv, \mathbb{Z}^{k-1})$ can be arbitrarily small. In fact, $\overline{\{nv \bmod \mathbb{Z}^{k-1}\}_{k \geq 0}}$ is a set of form $T' + P$, where T' is a (possibly trivial) subtorus and P is a finite set of rational points containing $(0, \ldots, 0)$. It is possible that T' is a singleton in which case, $nv \bmod \mathbb{Z}^{k-1}$ is periodic. In any case, T' contains the origin and this proves the claim.

Let $\delta' \in (0, 1/2)$. Let $\epsilon > 0$ be a small number such that

$$b_i^{-\epsilon} > 1 - \delta', i \in \{1, 2, \ldots, k\}. \tag{8}$$

Notice that there are infinitely many n such that $d(nv, \mathbb{Z}^{k-1}) < \epsilon$. Let n be such an integer. Then we see that

$$\mathbf{a}_n = b_1^n (1, v_2, v_3, \ldots, v_k),$$

where $v_i \in (1 - \delta', 1]$. For each $i \in \{1, 2, \ldots, k\}$, let

$$B_{i,n} = A_i \cap [a_{i,j}, b_i a_{i,j}].$$

We claim that if $b_1, b_2, \ldots, b_k$ are sufficiently large, then $B_{1,n} \cap B_{2,n} \cap \cdots \cap B_{k,n} \neq \emptyset$. The most convenient way to look at this is to rescale the whole situation by a factor of b_1^{-n}. After doing this, $B_{1,n}$ fits to the interval $[1, b_1]$ while $B_{2,n}, \ldots, B_{k,n}$ fit to the intervals $[v_2, b_2 v_2], \ldots, [v_k, b_k v_k]$. As one can choose δ' to be arbitrarily small, $v_1, v_2, \ldots, v_k$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

can be arbitrarily close to one. It is enough to consider the situation with $1 = v_1 = v_2 = v_3 = \cdots = v_k$. Indeed, both the thickness and the gap conditions in Theorems 3.2, 3.3 are preserved if one rescales and translates each Cantor set slightly. We will come back to this point shortly. If $v_1, v_2, \ldots, v_k$ are exactly 1, we see that it is enough to consider the intersection

$$C_1 \cap C_2 \cdots \cap C_k,$$

where $C_i = A_i \cap [1, b_i]$. The idea is to use Theorem 3.3 and Lemma 5.3 inductively. We find large (in diameter) and thick (in normalized thickness) Cantor sets contained in $C_1 \cap C_2, C_1 \cap C_2 \cap C_3, \ldots, C_1 \cap \cdots \cap C_k$. After that, we show that this procedure still goes through if one changes $C_1, \ldots, C_k$ by applying affine maps. This helps us to restore the information when $v_1, \ldots, v_k$ are almost but not exactly one.

Now, C_i is a self-similar set with contraction ratio $1/b_i$. Indeed,

$$C_i \cap [1, 2], C_i \cap [2, 3], \ldots, C_i \cap [b_i - 1, b_i]$$

are all scaled copies of C_i. The largest bounded gaps of C_i are of length $1/(b_i - 1)$ and $S(C_i) = (b_i - 2)/(b_i - 1)$.

We now start to show the base step for the inductive argument. It is simple to check that $C_1 \cap [1, 2], C_2 \cap [1, 2]$ satisfy the gap condition of Theorem 3.3. Thus for each $\delta > 0$, we can find a compact subset $C_{1,2} \subset C_1 \cap C_2 \cap [1, 2]$ with $S(C_{1,2}) > 1 - \delta$ as long as b_1, b_2 are sufficiently large.

We now show that the diameter of $C_{1,2}$ can be arbitrarily close to one at the cost of making b_1, b_2 sufficiently large. By Lemma 5.3, we see that as long as b_1, b_2 are large enough, there is a compact set $C_{1,2} \subset C_1 \cap C_2 \cap [1, 2]$ with thickness and diameter both at least $1 - \delta$.

We can apply the same argument inside the intervals $[2, 3], [3, 4], \ldots, [b_1 - 1, b_1]$. We now replace $C_{1,2}$ with the union of all such compact sets (in $[1, 2], [2, 3], \ldots$).

To summarize, for each $\delta > 0$, as long as b_1, b_2 are large enough, we can find a compact set $C_{1,2} \subset C_1 \cap C_2$ so that for each $l \in \{1, \ldots, b_1 - 1\}$, $S(C_{1,2} \cap [l, l + 1])$, $|\mathrm{Conv}(\mathrm{C}_{1,2} \cap [\mathrm{l}, \mathrm{l} + 1])|$ are at least $1 - \delta$.

From the second part of Lemma 5.3, we conclude that for each $\delta > 0$, as long as b_1, b_2 are large enough, there is a number $c > 0$ such that for all affine maps T_1, T_2 with scaling ratios c-close to one and translations c-close to zero, there is a compact set $C_{1,2} \subset T_1(C_1) \cap T_2(C_2)$ such that for each $l \in \{1, \ldots, b_1 - 1\}$, $S(C_{1,2} \cap [l, l + 1])$ and $|\mathrm{Conv}(\mathrm{C}_{1,2} \cap [\mathrm{l}, \mathrm{l} + 1])|$ are both larger than $1 - \delta$. Moreover, it is possible to see that $\mathrm{Conv}(\mathrm{C}_{1,2} \cap [\mathrm{l}, \mathrm{l} + 1]) \neq [l, l + 1]$. This is because if c is small enough, then $\mathrm{Conv}(\mathrm{T}_1(\mathrm{C}_1) \cap [\mathrm{l}, \mathrm{l} + 1]) \neq [l, l + 1]$, $\mathrm{Conv}(\mathrm{T}_2(\mathrm{C}_2) \cap [\mathrm{l}, \mathrm{l} + 1]) \neq [l, l + 1]$. This finishes the base step.

We now perform the induction. Let $j \in \{2, 3 \ldots, k - 1\}$. Suppose that for each $\delta > 0$, as long as $b_1, \ldots, b_j$ are large enough, there is a number $c > 0$ such that as long as $r_1, \ldots, r_j \in (1 - c, 1 + c), t_1, \ldots, t_j \in (-c, c)$, we can find a compact set

$$C_{1,2,\ldots,j} \subset \bigcap_{i=1}^{j} T_i(C_i)$$

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

such that for each $l \in \{1, \ldots, b_1 - 1\}$, we have that $\mathrm{Conv}(\mathrm{C}_{1,2\ldots,\mathrm{j}} \cap [\mathrm{l}, \mathrm{l}+1]) \neq [l, l+1]$, and that

$$S(C_{1,2,\ldots,j} \cap [l, l+1]), |\mathrm{Conv}(\mathrm{C}_{1,2,\ldots,\mathrm{j}} \cap [\mathrm{l}, \mathrm{l}+1])|$$

are greater than $1 - \delta$, where T_i, $i \in \{1, \ldots, j\}$ are linear maps $x \to r_i x + t_1$. We call this statement to be $ST(j)$. Under $ST(j)$, inside each interval $[l, l+1]$, $l \in \{1, \ldots, b_1 - 1\}$, it is possible to use Lemma 5.3 for $C_{1,2\ldots,j}$ and C_{j+1}. From here, we see that $ST(j+1)$ holds.

Finally, by induction, we see that $ST(k)$ holds. Thus there is a small number $c > 0$ such that as long as $r_1, r_2, \ldots, r_k \in (1 - c, 1 + c)$ and $t_1, t_2, \ldots, t_k \in (-c, c)$, we can find a compact set

$$C_{1,2,\ldots,k} \subset \bigcap_{i=1}^{k} T_i(C_i)$$

such that for each $l \in \{1, \ldots, b_1 - 1\}$, $C_{1,2,\ldots,k} \cap [l, l+1]$ has thickness at least $1 - \delta$ and diameter at least $1 - \delta$, where $T_i : x \to r_i x + t_i$. In particular, we see that $\bigcap_{i=1}^{k} T_i(C_i) \neq \emptyset$. Now we can choose δ' in equation (8) to be small enough according to c. We can then rescale this situation by b_1^n and conclude that

$$B_{1,n} \cap B_{2,n} \cap \cdots \cap B_{k,n} \neq \emptyset.$$

We proved that for sufficiently large $b_1, \ldots, b_k$, there are infinitely many $n \geq 0$ such that

$$B_{1,n} \cap B_{2,n} \cap \cdots \cap B_{k,n} \neq \emptyset.$$

This implies that $A = A_1 \cap A_2 \cap \cdots \cap A_k$ is unbounded, and the proof of the theorem is finished. □

Acknowledgements. H.Y. was financially supported by the University of Cambridge and the Corpus Christi College, Cambridge. H.Y. has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 803711). H.Y. especially thanks an anonymous referee for many instructive comments, which helped to improve the manuscript significantly.

References

[1] S. Astels. Cantor sets and numbers with restricted partial quotients. *Trans. Amer. Math. Soc.* **352**(1), (1999), 133–170.
[2] J. Athreya, B. Reznick and J. Tyson. Cantor set arithmetic. *Amer. Math. Monthly* **126**(1), 2019, 4–17.
[3] J. Ax. On Schanuel's conjectures. *Ann. of Math. (2)* **93**(2), (1971), 252–268.
[4] S. Burrell and H. Yu. Digit expansions of numbers in different bases. *J. Number Theory* **226** (2021), 284–306.
[5] M. Einsiedler and T. Ward. *Ergodic Theory: With a View Towards Number Theory (Graduate Texts in Mathematics, 259)*. Springer, 2011.
[6] P. Erdős, R. Graham, I. Ruzsa and E. Straus, On the prime factors of $\binom{2n}{n}$. *Math. Comp.* **29**(129), (1975), 83–92.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press

[7] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edn. Wiley, Chichester, 2005.
[8] H. Furstenberg. Intersections of Cantor sets and transversality of semigroups. *Problems in Analysis*. Ed. R. C. Gunning. Princeton University Press, NJ, 1970, pp. 41–59.
[9] M. Hochman and P. Shmerkin. Local entropy averages and projections of fractal measures. *Ann. Math.* **175** (2012), 1001–1059.
[10] B. Hunt, I. Kan and J. Yorke. When Cantor sets intersect thickly. *Trans. Amer. Math. Soc.* **339**(2) (1993), 869–888.
[11] J. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. J.* **30**(5) (1981), 713–747.
[12] K. Jiang, W. Li, Z. Wang and B. Zhao. On the sum of squares of middle-third Cantor set. *J. Number Theory* **218** (2021), 209–222.
[13] E. Kummer. Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. *J. Reine Angew. Math.* **44** (1852), 93–146.
[14] S. Newhouse. The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 101–151.
[15] T. Orponen. On the dimension and smoothness of radial projections. *Anal. PDE* **12**(5) (2019), 1273–1294.
[16] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences.* Available at http://oeis.org/A030979.
[17] P. Shmerkin. On Furstenberg's intersection conjecture, self-similar measures, and the L^q norms of convolutions. *Ann. of Math.* **189**(2) (2019), 319–391.
[18] M. Wu. A proof of Furstenberg's conjecture on the intersections of $\times p$ and $\times q$-invariant sets. *Ann. of Math.* **189**(3) (2019), 707–751.
[19] A. Yavicoli. Patterns in thick compact sets. *Israel J. Math.* **244**(1) (2021), 95–126.
[20] H. Yu. Additive properties of numbers with restricted digits. *Algebra Number Theory* **15**(5) (2021), 1283–1301.

https://doi.org/10.1017/etds.2022.2 Published online by Cambridge University Press