

Soluble groups satisfying the minimal condition for normal subgroups

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In this thesis we investigate the structural properties of certain classes of soluble groups satisfying the minimal condition for normal subgroups (for which we use the usual abbreviation $\text{min-}n$), and we describe a technique for constructing a number of examples to illustrate some of the difficulties of obtaining a general theory for such groups. The impetus for these investigations was provided by work on metabelian groups satisfying $\text{min-}n$ published recently in a paper by McDougall [4] and in a subsequent paper by Hartley and McDougall [2].

Our first results are concerned with the class of metanilpotent groups satisfying $\text{min-}n$. We show that much of the structure theory for metabelian groups satisfying $\text{min-}n$ developed in McDougall's paper can be carried over to the groups in this class. In particular we prove analogues of McDougall's results concerning Sylow subgroups and we investigate conditions under which metanilpotent groups satisfying $\text{min-}n$ will split over their derived groups. We also describe a number of specific examples of metanilpotent groups satisfying $\text{min-}n$ to indicate the scope of these results.

In the next chapter we describe a connection between certain classes of soluble groups satisfying $\text{min-}n$ and the better known class of soluble groups satisfying min (the minimal condition for all subgroups). Using the concept of a twisted wreath product we give a new description for some important types of metabelian groups satisfying $\text{min-}n$ which shows how the structure of these groups is determined by the structure of certain subgroups satisfying min . Among the groups to which this description applies are the well-known examples due to Čarin [1], which are the simplest

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examples of soluble groups satisfying $\text{min-}n$ but not min .

In our final chapter we use a particular type of twisted wreath product introduced recently by Heineken and Wilson [3] for the construction of a number of examples of groups satisfying $\text{min-}n$. These examples demonstrate, among other things, that the methods used for establishing our results on metanilpotent groups satisfying $\text{min-}n$ cannot be extended to give analogous results for the more general class of soluble groups of derived length three satisfying $\text{min-}n$. In fact the groups we construct satisfy a rather stronger condition than $\text{min-}n$, since in each one the normal subgroups form a well-ordered chain. We also apply the same construction to exhibit examples of perfect locally soluble groups whose normal subgroups form well-ordered chains. The first example of a group of this type is due to McLain [5]; however, our method enables us to construct a large class of groups with these properties, containing in particular uncountably many pairwise non-isomorphic periodic hopfian groups.

References

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