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A BEST POSSIBLE TAUBERIAN THEOREM FOR THE COLLECTIVE CONTINUOUS HAUSDORFF SUMMABILITY METHOD

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1. Introduction. The purpose of this paper is to prove that o(1/x) is the best possible Tauberian condition for the collective continuous Hausdorff method of summation. The analogue of this result for the collective (discrete) Hausdorff method is known [**1,** pp. 229, ff.; **7**, p. 318; **8**, p. 254]. Our method involves generalizing a well-known Abelian theorem of Agnew [**2**] to locally compact spaces and then applying the analogue for integrals of a result Lorentz obtained for series [**6**, Theorem 1].

2. Setting. Let T and X denote locally compact, non compact, σ-compact Hausdorff spaces. Let $T' = T \cup (\infty)$ and $X' = X \cup (\infty)$ denote the one-point compactifications of T and X, respectively. Let $\mathrm{B}(T)$ denote the set of locally bounded, complex valued Borel functions on T and let $\mathrm{B}_\infty(T)$ denote the bounded functions in $\mathrm{B}(T)$.

Let $(K_\alpha)_{\alpha \in A}$ be a family of compact subsets of T such that $(T' - K_\alpha)_{\alpha \in A}$ constitutes a fundamental system of neighbourhoods of ∞ in T'. By defining $\alpha_1 \leqq \alpha_2$ if $K_{\alpha_1} \subset K_{\alpha_2}$, the index set A becomes a directed set. Furthermore, since T is σ-compact, A is of sequential character [**3**, p. 94]. Let μ be a complex Borel measure on T and let $f \in \mathrm{B}(T)$. We will say that $\mu(f) = \int_T f d\mu$ exists, if and only if $\lim_A \int_{K_\alpha} f d\mu$ exists.

For each $x \in X$, let μ_x be a complex Borel measure on T. Suppose, furthermore, that $\mu_x(f) = \int f d\mu_x$, when considered as a function of x, is an element of $\mathrm{B}(X)$ whenever $f \in \mathrm{B}(T)$ and $\mu_x(f)$ exists for every $x \in X$. Under these circumstances we define a summability method M by saying that $f \in \mathrm{B}(T)$ converges (M) to the value L if and only if $\mu_x(f) = (Mf)(x)$ exists for every $x \in X$ and $\lim_{x\to\infty} \mu_x(f) = L$. The domain $D(M)$ of this method is the set of $f \in \mathrm{B}(T)$ which are such that $\int_T f d\mu_x$ exists for every $x \in X$. Note that since each μ_x is bounded, $\mathrm{B}_\infty(T) \subset D(M)$ [**12**, p. 119]. The method M is defined to be *regular* if and only if $\lim_{x\to\infty} \mu_x(f) = L$ whenever $f \in \mathrm{B}(T)$ and $\lim_{t\to\infty} f(t) = L$. Necessary and sufficient conditions for regularity are (see [**9**, p. 16]):

$$\sup_{x \in X} \|\mu_x\| = P < \infty, \tag{2.1}$$

$$\lim_{x\to\infty} \mu_x(f) = 0 \text{ for every } f \in \mathrm{B}(T) \text{ of compact support,} \tag{2.2}$$

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(2.3) $$\mu_x(1) = \int_T d\mu_x \to 1 \text{ as } x \to \infty,$$

where $||\mu_x|| = \int_T d|\mu_x|$.

In addition to regularity, we will need to assume that M satisfies the following condition, which is somewhat stronger than (2.2).

(2.4) $$\int_K d|\mu_x| \to 0 \text{ as } x \to \infty \text{ for each compact } K \subset T.$$

We will say that the method M is *equivalent to convergence on the set* $\mathscr{S} \subset \mathrm{B}(T)$ provided that $f \in \mathscr{S} \cap D(M)$ and $\lim_{x\to\infty}(Mf)(x)$ exists imply that $\lim_{t\to\infty} f(t)$ exists.

3. General theorems.

THEOREM 1. *Suppose $\{E_n\}$ is a disjoint collection of Borel sets in T with compact closures such that if $K \subset T$ is compact then K is a subset of the union of a finite number of the $\{E_n\}$. Suppose $\{x_n\}$ is a sequence of elements of X such that $x_n \to \infty$ and*

(3.1) $$\liminf_{n\to\infty}\left[\left|\int_{E_n} d\mu_{x_n}\right| - \int_{T-E_n} d|\mu_{x_n}|\right] = L > 0.$$

Then a regular M satisfying (2.4) *is equivalent to convergence on functions $f \in \mathrm{B}_\infty(T)$ which satisfy*

(3.2) $$f(t) = b_n \text{ if } t \in E_n,\ n = 1, 2, \ldots .$$

Furthermore, if we have the additional hypothesis

(3.3) $$\mu_{x_n}(E) = 0 \text{ if } E \text{ is a Borel subset of } T - Q_n,$$

where $Q_n = E_1 \cup \ldots \cup E_n$, then we can conclude that M is equivalent to convergence on functions $f \in \mathrm{B}(T)$ which satisfy (3.2).

Proof. For the first part of the theorem, suppose $f \in \mathrm{B}_\infty(T)$ diverges and satisfies (3.2). Let c be any complex number. Then our hypotheses imply that

$$\limsup_{t\to\infty} |f(t) - c| = \limsup_{n\to\infty} |b_n - c| = R > 0.$$

Let $\sigma(x) = \int_T f d\mu_x$. Then for each n

$$\sigma(x) - c = \int_{E_n} (f - c)d\mu_x + \int_{T-E_n} (f - c)d\mu_x + c\left[\int_T d\mu_x - 1\right].$$

Choose $\epsilon > 0$. Choose a compact set $K \subset T$ such that $|f(t) - c| < R + \epsilon$ whenever $t \notin K$. Let $\rho(x) = c[\int_T d\mu_x - 1]$. Then by (2.1),

$$\begin{aligned}|\sigma(x) - c| &\geqq -|\rho(x)| + \left|\int_{E_n} (f - c)d\mu_x\right| - \int_{T-(E_n\cup K)} |f - c|d|\mu_x| \\ &\qquad - \left|\int_{(T-E_n)\cap K} (f - c)d\mu_x\right| \\ &\geqq \delta(x) + |b_n - c|\left|\int_{E_n} d\mu_x\right| - (R + \epsilon)\int_{T-(E_n\cup K)} d|\mu_x| \\ &\geqq \delta(x) + |b_n - c|\left|\int_{E_n} d\mu_x\right| - R\int_{T-E_n} d|\mu_x| - \epsilon P,\end{aligned}$$

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where

$$\delta(x) = -|\rho(x)| - \int_K |f - c| d|\mu_x|.$$

By (2.3) and (2.4), $\delta(x) \to 0$ as $x \to \infty$. For an increasing sequence of values of n, we have $|b_n - c| > R - \epsilon$. For such values of n we have

$$|\sigma(x) - c| \geqq \delta(x) - \epsilon P + R\left[\left|\int_{E_n} d\mu_x\right| - \int_{T-E_n} |d\mu_x|\right] - \epsilon\left|\int_{E_n} d\mu_x\right|.$$

Thus, for all sufficiently large such values of n,

$$|\sigma(x_n) - c| \geqq \delta(x_n) - 2\epsilon P + R(L - \epsilon).$$

Therefore $\sigma(x_n) \not\to c$ as $n \to \infty$, and since c is arbitrary, $\sigma(x)$ diverges.

Now suppose (3.3) is satisfied. To establish the second part of the theorem, we must show that if $f \in \mathrm{B}(T)$ is unbounded then σ is divergent. Suppose $f \in \mathrm{B}(T)$ is unbounded. Then $\limsup_{n\to\infty}|b_n| = \infty$ and there are an infinity of n for which $|f(t)| \leqq |b_n|$ for all $t \in Q_n$.

For all sufficiently large such values of n and a given $\epsilon > 0$,

$$\begin{aligned} |\sigma(x_n)| &= \left|\int_{Q_n} f d\mu_{x_n}\right| \\ &\geqq \left|\int_{E_n} f d\mu_{x_n}\right| - \int_{Q_n - E_n} |f|d|\mu_{x_n}| \\ &\geqq |b_n|\left[\left|\int_{E_n} f d\mu_{x_n}\right| - \int_{X-E_n} d|\mu_{x_n}|\right] \\ &\geqq |b_n|[L - \epsilon] \end{aligned}$$

Thus $\limsup_{x\to\infty}|\sigma(x)| = \infty$ and σ diverges.

If $T = [0, \infty)$, $a(t)$ is Lebesgue integrable and

$$s(t) = \int_0^t a(u)du,$$

then we will say that *$o(h(t))$ is a Tauberian condition for M* if and only if $a(t) = o(h(t))$ and $s(t)$ summable (M) imply $\lim_{t\to\infty} s(t)$ exists.

THEOREM 2. *Suppose $T = [0, \infty)$ and $\{t_n\}$ is a sequence in T such that $0 < t_0 < t_1 < t_2 < \ldots; t_n \to \infty$. Suppose h is a positive integrable function defined on T such that*

$$\int_{t_n}^{t_{n+1}} h(t)dt = 0(1) \text{ as } n \to \infty. \tag{3.4}$$

Finally, suppose M is a regular summability method that is equivalent to convergence on functions $f \in B(T)$ which satisfy

$$f(t) = b_n \text{ if } t_n \leqq t < t_{n+1}. \tag{3.5}$$

Then $o(h(t))$ is a Tauberian condition for M.

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Proof. Suppose $a(t)$ is integrable,

$$s(t) = \int_0^t a(u)du, \; a(t) = o(h(t))$$

and $s(t)$ is summable (M). Define s^* by

$$s^*(t) = s(t_n) \quad t_n \leqq t < t_{n+1}.$$

Then for $t_n \leqq t < t_{n+1}$

$$\begin{aligned} |s(t) - s^*(t)| &= \left| \int_0^t a(u)du - \int_0^{t_n} a(u)du \right| \\ &\leqq \int_{t_n}^{t_n+1} |a(u)|du \\ &= o(1) \int_{t_n}^{t_n+1} |h(u)|du. \end{aligned}$$

Therefore, by (3.4), $s(t) - s^*(t) \to 0$ as $t \to \infty$. Since M is regular, $M(s - s^*) = Ms - Ms^* \to 0$ as $x \to \infty$. But we have assumed that s is summable (M) and hence Ms converges. It follows that Ms^* converges, then by hypothesis, that s^* converges, and finally that s converges.

4. Application to Hausdorff methods. Let $\chi(t)$ be of bounded variation on $[0, 1]$, $\chi(1) = 1$, $\chi(0) = \chi(0+) = 0$. Consider the summability method M_χ from the space $T = [0, \infty)$ to the space $X = (0, \infty)$ defined by the measures

$$\mu_x(t) = \begin{cases} \chi\left(\dfrac{t}{x}\right) & 0 \leqq t \leqq x \\ 0 & x < t. \end{cases} \tag{4.1}$$

Then

$$(M_\chi f)(x) = \int_0^x f(t)d\mu_x(t) = \int_0^1 f(xt)d\chi(t).$$

It is easily shown that each M_χ is a regular summability method in the sense of paragraph 2 and that (2.4) is satisfied. Furthermore these methods are consistent. That is, if $\lim M_{\chi_1}(f) = L_1$ and $\lim M_{\chi_2}(f) = L_2$, then $L_1 = L_2$ [see **13**, p. 345; **5**, p. 262, footnote]. Thus we may define the collective continuous Hausdorff method, $\mathscr{H}$, by saying that f is summable $(\mathscr{H})$ to L if and only if f is summable (M_χ) to L for some χ. (See [**1**, p. 229] where collective (discrete) Hausdorff summability is defined.)

When we say that $o(h(x))$ is the *best possible* Tauberian condition for a method M, we mean that $o(h(x))$ is a Tauberian condition for M, but $O(h(x))$ is not.

THEOREM 3. *The best possible Tauberian condition for the collective continuous Hausdorff method is* $o(1/x)$.

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Proof. Suppose $a(x)$ is integrable, $s(x) = \int_0^x a(u)du$, $a(x) = o(1/x)$ and $s(x)$ is summable $(\mathscr{H})$. Then $s(x)$ is summable (M_χ) for some χ. Since $\chi(0+) = \chi(0) = 0$, we may choose $\delta \in (0, 1)$ such that

$$\int_0^\delta d|\chi| < 1/2.$$

Let $t_0 = 0$, $t_1 = 1$, and $t_n = t_{n-1}/\delta$ if $n > 1$. Let $E_1 = (0, 1)$ and $E_n = [t_{n-1}, t_n)$ if $n > 1$, and let $x_n = t_n$ if $n \geqq 1$. Then, with μ_x as in (4.1),

$$\begin{aligned} \left|\int_{E_n} d\mu_{x_n}\right| - \int_{X-E_n} d|\mu_{x_n}| &= \left|\int_{t_{n-1}}^{t_n} d\chi\left(\frac{t}{x_n}\right)\right| - \int_0^{t_{n-1}} d\left|\chi\left(\frac{t}{x_n}\right)\right| \\ &= \left|\int_\delta^1 d\chi(t)\right| - \int_0^\delta d|\chi(t)| \\ &\geqq \left|\int_0^1 d\chi(t)\right| - 2\int_0^\delta d|\chi(t)| \\ &> 0. \end{aligned}$$

Thus (3.1) is satisfied and, furthermore, (3.3) is satisfied. By Theorem 1, M_χ is equivalent to convergence on functions which satisfy

$$f(t) = b_n \quad \text{if } t_{n-1} \leqq t < t_n.$$

But

$$\int_{t_n}^{t_{n+1}} \frac{1}{t}\, dt = \log_e(t_{n+1}/t_n) = \log_e \delta,$$

thus (3.4) is satisfied, and by Theorem 2, $o(1/x)$ is a Tauberian condition for M_χ. This implies that s converges and hence that $o(1/x)$ is a Tauberian condition for $\mathscr{H}$.

To show that $o(1/x)$ is best possible, we will exhibit a continuous Hausdorff method for which $O(1/x)$ is not a Tauberian condition. Let

$$\chi(t) = \begin{cases} 0 & 0 \leqq t < \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} \leqq t < \frac{2}{3} \\ 1 & \frac{2}{3} \leqq t \leqq 1 \end{cases}$$

and

$$a(t) = \begin{cases} 0 & 0 \leqq t < \frac{1}{3} \\ \dfrac{\pi \cos(\pi \log_2 3t)}{t \log_e 2} & \frac{1}{3} \leqq t. \end{cases}$$

Then if $x \geqq \frac{1}{3}$,

$$s(x) = \int_0^x a(t)dt = \sin(\pi \log_2 3x),$$

so $a(t) = 0(1/t)$ and $s(x)$ diverges. However,

$$\int_0^1 s(xt)d\chi(t) = \tfrac{1}{2}(s(x/3) + s(2x/3)) = 0 \text{ if } x \geqq 1.$$

Thus $M_\chi s$ converges.

Since Theorem 3 improves upon one of the author's previous results [**10**, Application D], it is now possible to use the results in [**11**] to prove the following theorem.

THEOREM 4. *If M_χ is a regular continuous Hausdorff method, if $\chi(t)$ is absolutely continuous, and if*

$$\int_0^1 t^{ix} d\chi(t) \neq 0$$

for all x, then $O(1/x)$ is a Tauberian condition for K.

Proof. This follows from Theorem 3 and [**11**, Theorem 5] as in [**11**, Application D].

It is reasonable to suppose that the 'absolutely continuous' hypothesis in Theorem 4 could be removed by casting the theorems of [**11**] in the setting of locally compact spaces.

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