5

Numerical Flatness and Stability Criteria

The aim of this chapter is to prove several characterizations of stable and locally stable families $f\colon (X, \Delta) \to S$. An earlier result, established in (3.1), has two assumptions:

- every fiber (X_s, Δ_s) is semi-log-canonical, and
- $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier.

In many applications, the first of these is given, but the second one can be quite subtle.

Note that such difficulties arise already for surfaces, even if $\Delta = 0$. Indeed, we saw in Section 1.4 that there are flat, projective families $g\colon X \to C$ of surfaces with quotient singularities that are not locally stable. In these cases every fiber is log terminal, but $K_{X/C}$ is not $\mathbb{Q}$-Cartier, although its restriction to every fiber $K_{X/C}|_{X_c} = K_{X_c}$ is $\mathbb{Q}$-Cartier.

In all the examples in Section 1.4, this unexpected behavior coincides with a jump in the self-intersection number of the canonical class of the fiber. Our aim is to prove that this is always the case, as shown by the following simplified version of (5.4). The main part of its proof is in Section 5.4.

Theorem 5.1 (Numerical criteria of stability) *Let S be a connected, reduced scheme over a field of characteristic 0, and $f\colon X \to S$ a flat, proper morphism of pure relative dimension n. Assume that all fibers are semi-log-canonical with ample canonical class K_{X_s}. The following are equivalent:*

(5.1.1) *f is stable.*

(5.1.2) *$K_{X/S}$ is $\mathbb{Q}$-Cartier and f-ample.*

(5.1.3) *$h^0(X_s, \omega_{X_s}^{[m]})$ is independent of $s \in S$ for every $m > 0$.*

(5.1.4) *$f_*(\omega_{X/S}^{[m]})$ is locally free for every $m > 0$.*

(5.1.5) *$(K_{X_s}^n)$ is independent of $s \in S$.*

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Proof Note that once $K_{X/S}$ is $\mathbb{Q}$-Cartier, it is f-ample since the K_{X_s} are ample. Then (5.1.1) ⇒ (5.1.2) follows from (3.1.1).

The $m = 1$ case of (5.1.3) is proved in (2.69); the $m \geq 2$ cases follow from (3.1.3) and the vanishing of higher cohomologies (11.34). Next (5.1.3) ⇒ (5.1.4) by Grauert's theorem. By Riemann–Roch (3.33), $(K^n_{X_s})$ is the leading term of $h^0(X_s, \omega^{[m]}_{X_s})$, thus (5.1.3) ⇒ (5.1.5). Finally (5.1.5) ⇒ (5.1.1) is a special case of (5.4). □

If $f: X \to S$ is stable, then $K_{X/S}$ is $\mathbb{Q}$-Cartier, hence $(K^n_{X_s})$ is clearly independent of $s \in S$, but the converse is surprising. General theory says that stability holds iff the Hilbert function $\chi(X_s, \mathscr{O}_{X_s}(mK_{X_s}))$ is independent of $s \in S$. Thus (5.1.2) asserts that if the leading coefficient of the Hilbert function is independent of s, then the same holds for the whole Hilbert function. We collect many similar results in this chapter; see Kollár (2015) for other such examples.

The main theorems are stated in Section 5.1. Related results on simultaneous canonical models and modifications are discussed in Section 5.2. The key claim is that, for families of slc pairs, local stability can fail only in relative codimension 2, and it can be characterized by the constancy of just one intersection number. A similar numerical condition characterizes Cartier divisors on flat families.

A series of examples in Section 5.3 shows that the assumptions of the theorems are likely to be optimal in characteristic 0.

Numerical criteria for stability in codimension ≤ 1 are discussed in Section 5.5. For all of the main theorems the key step is to establish them for families over smooth curves. This is done in Section 5.6. The numerical criterion of global stability, and a weaker version of local stability are derived in Section 5.6. The existence of simultaneous canonical models is studied in Section 5.7, and we treat simultaneous canonical modifications in Section 5.8.

Going from families over smooth curves to families over higher dimensional singular bases turns out to be quite quick, but several of the arguments, presented in Section 5.9, rely heavily on the techniques and results of Chapter 9.

Assumptions For all the main theorems of this chapter we work with varieties over a field of characteristic 0, but the background results worked out in Section 5.4 are established for excellent schemes.

5.1 Statements of the Main Theorems

We develop a series of criteria to characterize stable and locally stable (4.7) morphisms using a few, simple, numerical invariants of the fibers.

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We follow the general set-up of (5.1), but we strengthen it in three ways:
• We add a boundary divisor Δ.
• We assume only that f is flat in codimension 1 on each fiber. The reason for this is that many natural constructions (for instance flips, taking cones or ramified covers) do not preserve flatness. Thus we frequently end up with morphisms that are not known to be flat everywhere.
• We deal with local stability as well. A weak variant, involving several intersection numbers, is quite similar to the global case, but the sharper form requires different considerations.

For the main results of this chapter we work with the following set-up, which is a slight generalization of (3.28) and (4.2).

Notation 5.2 Let $f\colon X \to S$ be a proper morphism of pure relative dimension n (2.71), and $Z \subset X$ a closed subset with complement $U := X \setminus Z$ such that
(5.2.1) $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 2$ for every $s \in S$,
(5.2.2) $f|_U\colon U \to S$ is flat, and
(5.2.3) $\operatorname{depth}_Z X \geq 2$.
We also consider effective $\mathbb{R}$-divisors $\Delta = \sum b_i B_i$ on X, where the B_i are generically Cartier divisors (4.24). (Sheaf versions are studied in Section 5.4.)

We are mainly interested in cases where each fiber (X_s, Δ_s) is slc, but it turns out to be very useful to work with the following more general set-up.

Assumption 5.3 Given $f\colon (X, \Delta) \to S$ as in (5.2), we assume the following:
(5.3.1) $f|_U\colon (U, \Delta|_U) \to S$ is locally stable,
(5.3.2) (X_g, Δ_g) is slc for all generic points $g \in S$, and
(5.3.3) every fiber has lc normalization $\pi_s\colon (\bar{X}_s, \bar{D}_s + \bar{\Delta}_s) \to (X_s, \Delta_s)$.
Note that $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is defined over U_s by (1), and this determines $\bar{D}_s + \bar{\Delta}_s$ since $X_s \setminus U_s$ has codimension ≥ 2. Thus it makes sense to ask whether $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is lc or not.

The next result generalizes (5.1) to pairs. Its proof is given in (5.42).

Theorem 5.4 (Numerical criterion of stability) *We use the notation of (5.2). In addition to (5.3.1–3) assume that S is a reduced scheme over a field of* char 0*, and $K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s$ is ample for every $s \in S$. Then*
(5.4.1) $v(s) := \big((K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^n\big)$ *is an upper semi-continuous function, and*
(5.4.2) $f\colon (X, \Delta) \to S$ *is stable iff $v(s)$ is locally constant on S.*

The local version is the following, to be proved in (5.27) and (5.54).

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Theorem 5.5 (Numerical criterion of local stability) *We use the notation of (5.2). In addition to (5.3.1–3) assume that S is a reduced scheme over a field of* char 0*, and H is a relatively ample Cartier divisor class on X. Then*

(5.5.1) $v_2(s) := (\pi_s^* H^{n-2} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^2)$ *is upper semi-continuous, and*

(5.5.2) $f\colon (X, \Delta) \to S$ *is locally stable iff $v_2(s)$ is locally constant on S.*

Note that the functions $(\pi_s^* H^n)$ and $(\pi_s^* H^{n-1} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s))$ are always locally constant, but $(\pi_s^* H^{n-i} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^i)$ are neither upper nor lower semi-continuous for $i \geq 3$.

A key part of the proof of (5.5) is to show that local stability is essentially a two-dimensional question. The following, proved in (5.54), generalizes (2.7).

Theorem 5.6 (Local stability in codimension ≥ 3) (Kollár, 2013a, Thm.18) *Using the notation and assumptions of (5.2–5.3), let S be a reduced scheme of* char 0*. Assume also that* $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 3$ *for every $s \in S$.*

Then $f\colon (X, \Delta) \to S$ is locally stable.

One can also restate this as a converse of the Bertini-type result (2.13).

Corollary 5.7 *Notation and assumptions as in (5.2–5.3), let S be a reduced scheme of* char 0*. Assume in addition that the relative dimension is $n \geq 3$ and $f|_H\colon (H, \Delta|_H) \to S$ is locally stable, where $H \subset X$ is a relatively ample Cartier divisor. Then $f\colon (X, \Delta) \to S$ is also locally stable.* □

Comment As we noted in (2.14), (11.17) implies that $f\colon (X, H + \Delta) \to S$, and hence also $f\colon (X, \Delta) \to S$, are locally stable in a neighborhood of H. The unexpected new claim is that local stability holds everywhere.

A variant of (5.4) holds for arbitrary divisors and for non-slc fibers, but we have to assume that f is flat with S_2 fibers. On the other hand, this holds over any base scheme.

Theorem 5.8 (Numerical criterion for relative line bundles) Kollár (2016a) *Let S be a reduced scheme, $f\colon X \to S$ a flat, proper morphism of pure relative dimension n with S_2 fibers, and $Z \subset X$ a closed subset such that* $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 2$ *for every $s \in S$. Let A be an f-ample line bundle on X.*

Let L_U be a line bundle on $U := X \setminus Z$ and assume that, for every $s \in S$, the restriction $L_U|_{U_s}$ extends to a line bundle L_s on X_s. Then

(5.8.1) $d_2(s) := (A_s^{n-2} \cdot L_s^2)$ *is an upper semi-continuous function on S, and*

(5.8.2) *L_U extends to a line bundle on X iff $d_2(s)$ is locally constant on S.*

Furthermore, if L_s is ample for every s, then

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(5.8.3) $d(s) := (L_s^n)$ *is an upper semi-continuous function on* S*, and*

(5.8.4) L_U *extends to a line bundle on* X *iff* $d(s)$ *is locally constant on* S.

5.2 Simultaneous Canonical Models and Modifications

Given a morphism $f\colon X \to S$, we would like to know when the canonical models (or the canonical modifications) of the fibers form a flat family; see (5.9) and (5.15) for the precise definitions.

As we discussed in Section 1.5, the canonical models of the fibers need not form a flat family, not even for locally stable morphisms. We develop numerical criteria, after some definitions.

Definition 5.9 (Simultaneous canonical model) Let $f\colon (X, \Delta) \to S$ be a morphism as in (5.2). Assume that (5.3.1) holds, and every fiber has log canonical normalization $\pi_s\colon (\bar{X}_s, \bar{\Delta}_s) \to (X_s, \Delta_s)$. Its *simultaneous canonical model* is a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^{\rm sc} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^{\rm sc}} & \\ & S & \end{array} \qquad (5.9.1)$$

where $f^{\rm sc}\colon (X^{\rm sc}, \Delta^{\rm sc}) \to S$ is stable, and $\phi_s \circ \pi_s\colon (\bar{X}_s, \bar{\Delta}_s) \dashrightarrow (X_s^{\rm sc}, \Delta_s^{\rm sc})$ is the canonical model (over s), as in (11.26), for every $s \in S$.

Warning A "simultaneous" canonical model is not the same as a "relative" canonical model (11.26). For both notions $K + \Delta$ is relatively ample, but the former requires the singularities of the fibers to be lc, the latter the singularities of the total space to be lc. Neither implies the other.

For a pure dimensional, proper morphism $f\colon X \to S$, the *simultaneous canonical model of resolutions* $f^{\rm scr}\colon X^{\rm scr} \to S$ is defined analogously. Here we require that each $\phi_s\colon X_s \dashrightarrow X_s^{\rm scr}$ be obtained by first taking a resolution $X_s^{\rm r} \to \operatorname{red} X_s$, and then the disjoint union of the canonical models of those irreducible components that are of general type.

5.9.2 (Some known cases) Let $f\colon X \to S$ be flat, projective with S reduced. Assume that X_s is of general type and has canonical singularities for some $s \in S$. Then a simultaneous canonical model exists over an open neighborhood $s \in S^\circ \subset S$; see (1.37). The $\Delta \neq 0$ case is more subtle, see (5.20) and (5.48).

$^{0}*$ See Comment 5.9.3.

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5.9.3 (Comment on the conductor) Note that we do not add the conductor of π_s to $\bar{\Delta}_s$. If the fibers are normal in codimension 1 then D_s (the divisorial part of the conductor) is 0, hence our notion is the only sensible one. In general, however, one has a choice, and the simultaneous slc model, to be defined in (5.51), seems a better concept when $D_s \neq 0$.

We give criteria for the existence of simultaneous canonical models in terms of the volume (10.31) of the canonical class of the fibers. Note that if Y is a proper scheme of dimension n then $\mathrm{vol}(K_{Y^{\mathrm{r}}})$ is independent of the choice of the resolution $Y^{\mathrm{r}} \to Y$, and it equals the self-intersection number $\big((K_{Y^{\mathrm{cr}}})^n\big)$. Similarly, if (Y, Δ) is log canonical then $\mathrm{vol}(K_Y + \Delta) = \big((K_{Y^{\mathrm{c}}} + \Delta^{\mathrm{c}})^n\big)$.

Theorem 5.10 (Numerical criterion for simultaneous canonical models I) *Let S be a seminormal scheme of* char 0*, and $f\colon X \to S$ a proper morphism of pure relative dimension n. Then*

(5.10.1) $v(s) := \mathrm{vol}(K_{(X_s)^r})$ *is a lower semicontinuous function on S, and*

(5.10.2) $f\colon X \to S$ *has a simultaneous canonical model of resolutions iff $v(s)$ is locally constant (and positive).*

The key case, when S is a smooth curve, is settled in (5.44); the general case is in (5.55). This is a surprising result on two accounts. First, cohomology groups almost always vary *upper* semicontinuously; the *lower* semicontinuity in this setting was first observed and proved by Nakayama (1986, 1987). Second, usually it is easy to generalize similar proofs from smooth varieties to klt or lc pairs, but here adding any boundary can ruin the argument and the conclusion. Example 5.19 shows that S needs to be seminormal.

The following is a similar result for normal lc pairs, but the lower semicontinuity of (5.10) changes to upper semicontinuity.

Theorem 5.11 (Numerical criterion for simultaneous canonical models II) *Let S be a seminormal scheme of* char 0*, and $f\colon (X, \Delta) \to S$ as in (5.2). Assume furthermore that*

(5.11.1) *$f|_U\colon U \to S$ is smooth with irreducible fibers,*

(5.11.2) *every fiber has lc normalization $\pi_s\colon (\bar{X}_s, \bar{\Delta}_s) \to (X_s, \Delta_s)$, and*

(5.11.3) *the canonical models $\phi_s\colon (\bar{X}_s, \bar{\Delta}_s) \dashrightarrow (\bar{X}_s^c, \bar{\Delta}_s^c)$ exist.*

Then

(5.11.4) $v(s) := \mathrm{vol}(K_{\bar{X}_s} + \bar{\Delta}_s)$ *is an upper semicontinuous function on S, and*

(5.11.5) $f\colon (X, \Delta) \to S$ *has a simultaneous canonical model iff $v(s)$ is locally constant.*

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The proof is given in (5.46), and (5.55).

One should think of (5.11) as a generalization of (5.4), but there are differences. In (5.11) we allow only fibers that are smooth in codimension 1, and S is assumed seminormal. (The extra assumption (3) is expected to hold always.) However, the key difference is in the proofs given in Section 5.9. While the proof of (5.4) uses only the basic theory of hulls and husks, we rely on the existence of moduli spaces of pairs in order to establish (5.11).

Both (5.10) and (5.11) apply to $f\colon X \to S$ iff the normalizations of the fibers have canonical singularities. In this case, f is locally stable (2.8), and the plurigenera – and hence the volume – are locally constant (1.37).

A key ingredient of the proofs of (5.10–5.11) is the following characterization of canonical models. We prove a more general version of it in (10.36).

Proposition 5.12 *Let X be a smooth proper variety of dimension n. Let Y be a normal, proper variety birational to X, and D an effective $\mathbb{R}$-divisor on Y such that $K_Y + D$ is $\mathbb{R}$-Cartier, nef and big. Then*

(5.12.1) $\operatorname{vol}(K_X) \le \operatorname{vol}(K_Y + D) = ((K_Y + D)^n)$, *and*

(5.12.2) *equality holds iff $D = 0$ and Y has canonical singularities.*

For surfaces, the existence criterion of simultaneous canonical modifications is proved in Kollár and Shepherd-Barron (1988, Sec.2). In higher dimensions, we need to work with a sequence of intersection numbers and with their lexicographic ordering.

Definition 5.13 Let X be a proper scheme of dimension n, and A, B $\mathbb{R}$-Cartier divisors on X. Their *sequence of intersection numbers* is

$$I(A, B) := ((A^n), \dots, (A^{n-i} \cdot B^i), \dots, (B^n)) \in \mathbb{R}^{n+1}.$$

Definition 5.14 The *lexicographic* ordering of length $n + 1$ real sequences is

$$(a_0, \dots, a_n) \le (b_0, \dots, b_n).$$

This holds if either $a_i = b_i$ for every i, or there is an $r \le n$ such that $a_i = b_i$ for $i < r$, but $a_r < b_r$. For polynomials we define an ordering

$$f(t) \le g(t) \;\Leftrightarrow\; f(t) \le g(t)\ \forall t \gg 0.$$

We use $\equiv$ to denote identity of sequences or polynomials. Note that

$$\textstyle\sum_i a_i t^{n-i} \le \sum_i b_i t^{n-i} \Leftrightarrow (a_0, \dots, a_n) \le (b_0, \dots, b_n).$$

If we have proper schemes X, X' of dimension n, $\mathbb{R}$-Cartier divisors A, B on X and A', B' on X', then

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$$I(A,B) \preceq I(A',B') \Leftrightarrow (tA+B)^n \preceq (tA'+B')^n.$$

We will consider functions that associate a sequence or a polynomial to all points of a scheme X. Using the above definitions, it makes sense to ask if such a function is *upper/lower semicontinuous* for $\preceq$ or not.

Definition 5.15 (Simultaneous canonical modification) Let $f\colon X \to S$ be a morphism of pure relative dimension n, and $\Delta = \sum a_i D_i$ a generically $\mathbb{Q}$-Cartier effective divisor on X. A *simultaneous canonical modification* is a proper morphism $p^{\mathrm{scm}}\colon (X^{\mathrm{scm}}, \Delta^{\mathrm{scm}}) \to (X, \Delta)$ such that $f \circ p^{\mathrm{scm}}\colon (X^{\mathrm{scm}}, \Delta^{\mathrm{scm}}) \to S$ is locally stable, and

$$p_s^{\mathrm{scm}}\colon ((X^{\mathrm{scm}})_s, (\Delta^{\mathrm{scm}})_s) \to (X_s, \Delta_s)$$

is the canonical modification (11.29) for every $s \in S$.

A *simultaneous log canonical modification* $p^{\mathrm{slcm}}\colon (X^{\mathrm{slcm}}, \Delta^{\mathrm{slcm}}) \to (X, \Delta)$ is defined analogously.

In the following result we need to assume that the base scheme is seminormal; see (5.21) for some examples.

Theorem 5.16 (Numerical criterion for simultaneous canonical modification) *We use the notation of (5.2). In addition to (5.3.1), assume that S is a seminormal scheme of* char 0*, and H is a relatively ample Cartier divisor class on X. For $s \in S$ let $p_s^{cm}\colon (X_s^{cm}, \Delta_s^{cm}) \to (X_s, \Delta_s)$ denote the canonical modification of the fiber (X_s, Δ_s). Then*

(5.16.1) *$I(s) := I(\pi_s^* H_s, K_{X_s^{cm}} + \Delta_s^{cm})$ is lower semi-continuous for $\preceq$, and*

(5.16.2) *$f\colon (X, \Delta) \to S$ has a simultaneous canonical modification iff $I(s)$ is locally constant.*

There is also a similar condition for simultaneous log canonical and semi-log-canonical modifications (5.52), but these only apply when $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier.

5.3 Examples

Here we present a series of examples that show that the assumptions of the Theorems in Sections 5.1–5.2 are close to being optimal, except that the characteristic 0 assumption is probably superfluous.

The following is the simplest example illustrating the difference between being Cartier and fiber-wise Cartier.

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Example 5.17 Consider the family of quadrics

$$X = (x^2 - y^2 + z^2 - t^2w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}_t \quad \text{and} \quad D = (x - y = z - tw = 0).$$

Here X_0 is a quadric cone, and X_t is a smooth quadric for $t \neq 0$. The divisor D is Cartier, except at the origin, where it is not even $\mathbb{Q}$-Cartier. However D_0 is a line on a quadric cone, hence $2D_0 = (x - y = t = 0)$ is Cartier. It is easy to compute that

$$L = \mathscr{O}_X(-2D) = (x - y, z - tw)^2 \cdot \mathscr{O}_X$$

is locally free outside the origin and not locally free at the origin, but the reflexive hull of its restriction

$$L_0^H := \mathscr{O}_{X_0}(-2D_0) = (x - y) \cdot \mathscr{O}_{X_0}$$

is locally free. The natural restriction map gives an identification

$$\mathscr{O}_X(-2D)|_{X_0} = (x, y, z) \cdot \mathscr{O}_{X_0}(-2D_0) \subset \mathscr{O}_{X_0}(-2D_0).$$

Note that the self-intersection number of the fibers of D also jumps. For $t \neq 0$ we have $(D_t^2) = 0$, but $(D_0^2) = 1/2$.

It is harder to get examples where the self-intersections in (5.8) are locally constant, yet the divisor is not Cartier, but, as we see next, this can happen even for the canonical class. Thus in (5.8) one needs to assume that the fibers of f are S_2, and in (5.4) that the fibers are slc.

Example 5.18 (See (2.35) or Kollár (2013b, 3.8–14) for the notation and basic results on cones.) Let $X \subset \mathbb{P}^N$ be a smooth, projective variety of dimension n and $L_X = \mathscr{O}_X(1)$. Let $C(X) := C_p(X, L_X)$ denote the projective cone over X with vertex v and natural ample line bundle $L_{C(X)}$. Let $H \subset X$ be a smooth hyperplane section, and $C(H) := C_p(H, L_H)$ the projective cone over H. Note that

$$(L_X^n) = (L_{C(X)}^{n+1}) = (L_H^{n-1}) = (L_{C(H)}^n).$$

The canonical class of $C(X)$ is Cartier iff $K_X \sim mc_1(L_X)$ for some $m \in \mathbb{Z}$. In this case $K_{C(X)} \sim (m - 1)c_1(L_{C(X)})$.

We can think of H as sitting in $X \subset C(X)$. The pencil of hyperplanes containing $H \subset C(X)$ gives a morphism of the blow-up $p\colon Y := B_H C(X) \to \mathbb{P}^1$ such that $Y_t \simeq X$ for $t \neq 0$, and the normalization $\bar{Y}_0$ of Y_0 is isomorphic to $C(H)$. However, if $H^1(X, \mathscr{O}_X) \neq 0$ then Y_0 is not normal. For instance, this happens if X is the product of nonhyperelliptic curves of genus ≥ 2 with its canonical embedding. Thus, if these hold, then

(5.18.1) Y_t is smooth and K_{Y_t} is ample for $t \neq 0$,

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(5.18.2) Y_0 is not normal, the normalization $\bar{Y}_0 \to Y_0$ is an isomorphism except at v, $K_{\bar{Y}_0}$ is locally free and ample and
(5.18.3) $(K_{Y_t}^n) = (K_{\bar{Y}_0}^n)$ (where $n = \dim X$).

The next example shows that (5.10) fails if S is not seminormal.

Example 5.19 Let S be a local, reduced, nonseminormal scheme with seminormalization $S' \to S$. Choose an embedding of S' into the moduli space of automorphism-free curves of genus g for some g. Let $p' \colon X' \to S'$ be the resulting smooth family. This induces a family $p \colon X' \to S' \to S$ that satisfies the assumptions of (5.10). However, there is no simultaneous canonical model since $p' \colon X' \to S'$ does not descend to $p \colon X \to S$.

The next examples show that there does not seem to be a log version of (5.10) for families with reducible fibers, not even for families of curves.

Example 5.20 Let $g \colon S \to C$ be a smooth family of curves, and $D_i \subset S$ a set of n disjoint sections. Set $\Delta := \sum d_i D_i$. Pick a point $0 \in C$, the fiber over it is $(S_0, \sum d_i[p_i])$ where $p_i = S_0 \cap D_i$. The "log volume" is $2g(S_0) - 2 + \sum d_i$.

Let $\pi \colon S^1 \to S$ be the blow up of all the points p_i with exceptional curves E_i and set $\Delta^1 := \pi_*^{-1}\Delta$. The central fiber of $g^1 \colon (S,^1, \Delta^1) \to C$ is $(S_0^1, 0) + \sum_i (E_i, d_i[p_i'])$. Its normalization consists of S_0 (with no boundary points) and $E_i \simeq \mathbb{P}^1$, each with one marked point of multiplicity d_i. Thus the "log volume" of the central fiber is now $2g(S_0) - 2$; the effect of the boundary vanished.

One can try to compensate for this by adding the double point divisor $\bar{D}_0$. This variant of the "log volume" is now $2g(S_0)-2+n$. This formula remembers only the number of the sections, not their coefficients. Even worse, we can blow up m other points on S_0, then the "log volume" formula gives $2g(S_0)-2+n+m$.

In general, there does not seem to be a sensible and birationally invariant way do define the "log volume" of degenerations.

In (5.16), the base scheme is assumed to be seminormal. The reason for this is that canonical modifications do have unexpected infinitesimal deformations.

Example 5.21 (Deformation of canonical modifications) We give an example of a normal, projective variety with isolated singularities and canonical modification $X^{\mathrm{cm}} \to X$ such that the trivial deformation of X can be lifted to a nontrivial deformation of X^{cm}.

Consider the isolated hypersurface singularity

$$X := X_{n,r} := (x_1^r + \cdots + x_n^r + x_{n+1}^{r+1} = 0) \subset \mathbb{A}_k^{n+1}.$$

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Let $p\colon Y := B_0X \to X$ denote the blow-up of the origin. Then Y is smooth, the exceptional divisor is the cone $E \simeq (x_1^r + \cdots + x_n^r = 0) \subset \mathbb{P}^n$ and $N_{E|Y} \simeq \mathscr{O}_E(-1)$. We compute that $a(E, X_{n,r}, 0) = n - r$. Thus $X_{n,r}$ is canonical iff $r \leq n$, and Y is the canonical modification for $r > n$.

We claim that $p\colon Y \to X$ has a nontrivial deformation over $X \times_k \operatorname{Spec} k[\varepsilon]$. The trivial deformation is obtained by blowing up

$$(x_1 = \cdots = x_{n+1} = 0) \subset X \times_k \operatorname{Spec} k[\varepsilon].$$

The nontrivial deformation is obtained by blowing up

$$Z := (x_1 = \cdots = x_n = x_{n+1} - \varepsilon = 0) \subset X \times_k \operatorname{Spec} k[\varepsilon].$$

We need to check that X is equimultiple along the blow-up center. Introducing a new coordinate $y := x_{n+1} - \varepsilon$, the equations become

$$Z := (x_1 = \cdots = x_n = y = 0) \subset (x_1^r + \cdots + x_n^r + y^{r+1} + (r+1)\varepsilon y^r = 0),$$

thus $X \times_k \operatorname{Spec} k[\varepsilon]$ is clearly equimultiple along Z.

Note that $E \subset Y$ has a unique extension E_ε to a deformation Y_ε of Y since $H^1(E, N_{E|Y}) = 0$. The blow-up ideal is then the push-forward of the ideal sheaf of E_ε. Thus different blow-up ideals give different deformations of Y.

The following examples show that the existence of simultaneous canonical modifications is more complicated for pairs.

Example 5.22 In $\mathbb{P}^2$ consider a line $L \subset \mathbb{P}^2$ and a family of degree 8 curves C_t such that C_0 has four nodes on L plus an ordinary 6-fold point outside L, and C_t is smooth and tangent to L at four points for $t \neq 0$.

Let $\pi_t\colon S_t \to \mathbb{P}^2$ denote the double cover of $\mathbb{P}^2$ ramified along C_t. Note that $K_{S_t} = \pi_t^*\mathscr{O}(1)$, thus $(K_{S_t}^2) = 2$. For each t, the preimage $\pi_t^{-1}(L)$ is a union of two curves $D_t + D_t'$. Our example is the family of pairs (S_t, D_t). We claim that,

(5.22.1) there is a log canonical modification $(S_t^{\mathrm{lcm}}, D_t^{\mathrm{lcm}}) \to (S_t, D_t)$, and

(5.22.2) $((K_{S_t^{\mathrm{lcm}}} + D_t^{\mathrm{lcm}})^2) = 1$, yet

(5.22.3) there is no simultaneous log canonical modification.

If $t \neq 0$ then S_t and D_t are smooth. Furthermore D_t, D_t' meet transversally at four points, thus $(D_t \cdot D_t') = 4$. Using $((D_t + D_t')^2) = 2$, we obtain that $(D_t^2) = -3$. Thus $((K_{S_t} + D_t)^2) = 1$.

If $t = 0$ then S_0 is singular at 5 points. D_0, D_0' meet transversally at four singular points of type A_1, thus $(D_0 \cdot D_0') = 2$. This gives that $(D_0^2) = -1$. Thus $((K_{S_0} + D_0)^2) = 3$. The pair (S_0, D_0) is lc away from the preimage of the 6-fold point. Let $q\colon T_0 \to S_0$ denote the minimal resolution of this point. The exceptional curve E is smooth, has genus 2 and

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$(E^2) = -2$. Thus $K_{T_0} = q^*K_{S_0} - 2E$ hence $(T_0, E + D_0)$ is the log canonical modification of (S_0, D_0), and

$$((K_{T_0} + E + D_0)^2) = ((q^*K_{S_0} - E + D_0)^2) = ((K_{S_0} + D_0)^2) + (E^2) = 1.$$

Thus $((K_{S_t^{\mathrm{lcm}}} + D_t^{\mathrm{lcm}})^2) = 1$ for every t.

Nonetheless, the log canonical modifications do not form a flat family. Indeed, such a family would be a family of surfaces with ordinary nodes, so the relative canonical class would be a Cartier divisor. However, $(K_{S_t}^2) = 2$ for $t \neq 0$, but $(K_{T_0}^2) = ((q^*K_{S_0} - 2E)^2) = -6$.

Example 5.23 We start with a family of quadric surfaces $Q_t \subset \mathbb{P}^3$ where Q_0 is a cone, and Q_t is smooth for $t \neq 0$. We take six families of lines L_t^i such that for $t = 0$ we have six distinct lines, and for $t \neq 0$ two of them – L_t^1, L_t^2 – are from one ruling of the quadric, the other four from the other ruling. S_t denotes the double cover of Q_t ramified along the six lines $L_t^1 + \cdots + L_t^6$.

For $t \neq 0$ the surface S_t has ordinary nodes and $(K_{S_t}^2) = 0$. For $t = 0$ the surface S_0 has a unique singular point. Its minimal resolution $q\colon T_0 \to S_0$ is a double cover of $\mathbb{F}_2$ ramified along six fibers. Thus $(K_{T_0}^2) = -4$. Thus the canonical modifications do not form a flat family. The log canonical modification of S_0 is (T_0, E_0) where E_0 is the q-exceptional curve. Thus $((K_{T_0} + E_0)^2) = 0$.

The numerical condition is satisfied, but the log canonical modifications do not form a flat family since $T_0 = S_0^{\mathrm{lcm}}$ is smooth, but $S_t^{\mathrm{lcm}} = S_t$ is singular for $t \neq 0$. However, there is a flat family that is a weaker variant of a simultaneous log canonical modification.

This is obtained by replacing the singular quadric Q_0 with its resolution $Q_0' \simeq \mathbb{F}_2$. Let $E \subset \mathbb{F}_2$ denote the -2-section, and $|F|$ the ruling. One can arrange that L_t^1, L_t^2 degenerate to $F^i + E$ for $F^i \in |F|$, and the others degenerate to fibers F^j. This way a flat limit of the double cover S_t is obtained as the double cover of $\mathbb{F}_2$ ramified along $F^1 + \cdots + F^6 + 2E$. This is a semi-log-canonical surface whose normalization is the log canonical modification of S_0.

5.4 Mostly Flat Families of Line Bundles

We investigate sheaves that are known to be invertible in codimension 1; a topic we already encountered in Section 2.6. This leads to the proofs of (5.5) and (5.8). Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

Definition 5.24 (Mostly flat family of line bundles) Let $f\colon X \to S$ be a morphism and L a mostly flat divisorial sheaf (3.28). We say that L is a *mostly flat family of line bundles* if the hull L_s^H of L_s (3.27.1) is locally free over the hull $\mathscr{O}_{X_s}^H$ of $\mathscr{O}_{X_s}$. (In most cases of interest f has S_2 fibers, and then $\mathscr{O}_{X_s}^H = \mathscr{O}_{X_s}$.)

A mostly flat family of line bundles L on X is called *fiber-wise ample* if L_s^H is ample for every $s \in S$. See (5.17) for typical examples.

Our aim is to find conditions to ensure that a mostly flat family of line bundles is a flat family of line bundles.

Lemma 5.25 *Let $f\colon X \to S$ be a proper morphism of pure relative dimension n, A a relatively ample line bundle on X, and L a mostly flat family of fiber-wise ample line bundles. Then*

(5.25.1) *$s \mapsto (A_s^i \cdot (L_s^H)^{n-i})$ is upper semi-continuous for every i, and*

(5.25.2) *if $s \mapsto ((L_s^H)^n)$ is constant, then so is every $(A_s^i \cdot (L_s^H)^{n-i})$.*

Proof As we noted in (5.24), there is a dense open subset $S^\circ \subset \operatorname{red} S$ such that $L|_{X^\circ}$ is a line bundle. Thus the functions $s \mapsto (A_s^i \cdot (L_s^H)^{n-i})$ are locally constant on S°, hence constructible on S by Noetherian induction.

It remains to check upper semicontinuity when $(0 \in S)$ is the spectrum of a DVR. We may assume that X is S_2.

L_0 is also S_1, hence $L_0 \to L_0^H$ is an injection. By semicontinuity we have $h^0(X_0, L_0^H) \geq h^0(X_0, L_0) \geq h^0(X_g, L_g)$. Applying this to powers of L and taking the limit, we obtain that $\operatorname{vol}(L_0^H) \geq \operatorname{vol}(L_g)$ by (10.31). If L is fiber-wise ample, then volume equals the self-intersection number, so $((L_0^H)^n) \geq ((L_g^H)^n)$. This shows upper semicontinuity for $i = 0$.

For $i > 0$, we prove (1) by induction on n. We may assume that S is local, and A is relatively very ample. Let $Y \subset X$ be a hypersurface cut out by a general section of A. By (4.26), the restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \to S$. Furthermore

$$(A_s^i \cdot (L_s^H)^{n-i}) = (Y_s \cdot A_s^{i-1} \cdot (L_s^H)^{n-i}) = ((A|_Y)_s^{i-1} \cdot ((L|_Y)_s^H)^{n-i}); \qquad (5.25.3)$$

the latter is constructible and upper semi continuous by induction.

In order to see (2), note that $L^m \otimes A^{-1}$ is also a mostly flat family of fiber-wise ample line bundles for $m \gg 1$, and

$$m^n((L_s^H)^n) = \textstyle\sum_i \binom{n}{i} (A_s^i \cdot ((L^m \otimes A^{-1})_s^H)^{n-i}). \qquad (5.25.4)$$

By (1), all summands on the right are constructible and upper semi continuous. Therefore, if the sum is constant as a function of s, then so is every summand.

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Finally note that

$$(((L^m \otimes A^{-1})_s^H)^n) = \sum_i (-1)^i m^{n-i} \binom{n}{i} (A_s^i \cdot (L_s^H)^{n-i}). \tag{5.25.5}$$

If the left side is constant for $m \gg 1$, as a function of s, then every summand on the right is constant. □

Remark 5.25.6 Let $f\colon X \to S$ be a proper morphism of pure relative dimension n, and L a line bundle on X. It is not well understood when the function $s \mapsto \operatorname{vol}(L_s)$ is constructible; see Lesieutre (2014); Pan and Shen (2013).

5.26 (Proof of 5.8) The assertions (5.8.1) and (5.8.3) are proved in (5.25.1). Furthermore, (5.25.2) shows that (5.8.2) implies (5.8.4).

Thus it remains to prove (5.8.2). We start with the case when S is the spectrum of a DVR; this implies the general case by (4.34).

Our argument has three parts. The first step, when the relative dimension is 2, is done in (5.28).

The next step is induction on the dimension. We may assume that S is local and A is relatively very ample. Let $Y \subset X$ be a general hypersurface cut out by a general section of A. Then (4.26) ensures that $L^H|_Y = (L|_Y)^H$. The restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \to S$ and, as we noted in (5.25.3),

$$(A_s^{n-2} \cdot (L_s^H)^2) = ((A|_Y)_s^{n-3} \cdot ((L|_Y)_s^H)^2).$$

Thus, by induction, $L^H|_Y$ is a line bundle. This implies that L^H is a line bundle along Y. So L^H is a line bundle, except possibly at finitely many points $Z \subset X$.

Finally we need to exclude this finite set Z when the fiber dimension is at least 3. This follows from (2.91). □

5.27 (Start of the proof of 5.5) Note that (5.5.1) follows from (5.25.1). For (5.5.2), the general setting is postponed to (5.54). Here we consider the case when $S = C$ is one-dimensional and regular.

As a first step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion unchanged. By (2.54), a demi-normal pair $(X, \Delta) \to C$ with slc generic fibers is slc iff its normalization is lc. Thus the conclusion is also unchanged.

It would seem that we should use (5.8). However, a key assumption of (5.8) is that every fiber is S_2; this is true, but not obvious in our case. Thus we consider two separate cases.

If $n = 2$, then the weak numerical criterion (5.43) implies (5.5). For $n \geq 3$, the weak numerical criterion involves the terms $(\pi_c^* H^{n-i} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^i)$ for $i \geq 3$; these are unknown to us.

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Instead, using the already established $n = 2$ case and (4.26) as in (5.26), we may assume that $f: (X, \Delta) \to C$ is locally stable outside a subset of codimension ≥ 3. We can now apply (2.7) to complete the argument. □

Proposition 5.28 *Let T be an irreducible, regular, one-dimensional scheme, and $f: X \to T$ a flat, proper morphism of relative dimension 2 with S_2 fibers. Let L be a mostly flat family of line bundles on X. Then*

(5.28.1) *$d(t) := (L_t^H \cdot L_t^H)$ is upper semicontinuous, and*

(5.28.2) *L is locally free on X iff $d(t)$ is constant on T.*

Proof If L is locally free then $(L_t^H \cdot L_t^H) = (L \cdot L \cdot [X_t])$ is independent of $t \in T$. To see the converse we may assume that T is local with closed point $0 \in T$ and generic point $g \in T$. Note that L is locally free, except possibly at a finite set $Z_0 \subset X_0$, and $L_g^H \simeq L_g$.

For each $t \in T$, the Euler characteristic is a quadratic polynomial

$$\chi(X_t, (L_t^H)^{\otimes m}) = a_t m^2 + b_t m + c_t,$$

and we know from Riemann–Roch that $a_t = \frac{1}{2}(L_t^H \cdot L_t^H)$ and $c_t = \chi(X_t, \mathscr{O}_{X_t})$. Furthermore, (9.36.4) implies that

$$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \tag{5.28.3}$$

For $m \gg 1$, the quadratic terms dominate, which gives that

$$(L_0^H \cdot L_0^H) = 2a_0 \geq 2a_g = (L_g \cdot L_g). \tag{5.28.4}$$

Assume now that $(L_0^H \cdot L_0^H) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus (5.28.3) implies that

$$b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \tag{5.28.5}$$

For $m \gg 1$, this implies that $b_0 \geq b_g$, and for $m \ll -1$ that $-b_0 \geq -b_g$. Thus $b_0 = b_g$ and $c_0 = c_g$ also holds since f is flat. Therefore we have equality in (5.28.3). Thus L is a flat family of locally free sheaves by (3.32). □

The following turns out to be quite elementary; see Stacks (2022, tag 0F29) for a subtle local version.

Proposition 5.29 *Let T be the spectrum of a DVR with closed point $0 \in T$ and generic point $g \in T$. Let $f: X \to T$ be a projective morphism with S_2 fibers, and L a mostly flat family of line bundles such that $L^{[m]}$ is locally free for some $m > 0$. Then L is locally free.*

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Proof We claim an equality of the Hilbert polynomials

$$\chi(X_0, (L_0^H)^{\otimes r}) = \chi(X_g, L_g^{\otimes r}). \tag{5.29.1}$$

Since both sides are polynomials in r, it is sufficient to prove that they are equal for all multiples of m.

Note that $L^{[m]}|_{X_0}$ and $(L_0^H)^{\otimes m}$ are both locally free sheaves that agree outside a codimension 2 subset, hence they are isomorphic. Thus

$$\begin{aligned}\chi(X_0, (L_0^H)^{\otimes rm}) &= \chi(X_0, (L^{[m]}|_{X_0})^{\otimes r}) \\ &= \chi(X_g, (L^{[m]}|_{X_g})^{\otimes r}) = \chi(X_g, L_g^{\otimes rm}),\end{aligned} \tag{5.29.2}$$

where the last equality holds since L_g is a line bundle (5.24). In particular we conclude that $\chi(X_0, L_0^H) = \chi(X_g, L_g)$. Let $\mathscr{O}_{X/S}(1)$ be an f-ample invertible sheaf. We can apply the same argument to any $L(t)$ to obtain that $\chi(X_0, L_0^H(t)) = \chi(X_g, L_g(t))$ for every t. By (3.32) this implies that L is locally free. □

5.5 Flatness Criteria in Codimension 1

Let $f\colon X \to S$ be a projective morphism with f-ample $\mathscr{O}_X(1)$, and F a coherent sheaf on X. Assume that S is reduced. By (3.20), the polynomial valued function $s \mapsto \chi(X_s, F_s(*))$ is

- upper semicontinuous on S, and
- it is locally constant iff F is flat over S.

In Sections 5.1–5.2, we discussed numerous situations where we first associate some other object to each (X_s, F_s), and then compute a numerical invariant. Usually these objects cannot be realized as fibers of some morphism. However, we still would like to show that the numerical invariant is an upper or lower semicontinuous function on S. Furthermore, if the numerical invariant is locally constant on S, then we would like to prove that the objects fit together into a flat family over S.

As a typical example – generalizing (5.8) – consider the Hilbert polynomial of the reflexive hulls $\chi(X_s, (F_s)^{[**]}(*))$. Assume that X, S are normal, and so are the fibers of f. Note that (3.20) does not apply, since usually there is no coherent sheaf on X whose fibers are $(F_s)^{[**]}$. There is a natural map

$$r_s\colon (F^{[**]})_s \to (F_s)^{[**]},$$

but frequently it is neither injective nor surjective. So we do not get any comparison between the Hilbert polynomials of $(F^{[**]})_s$ and $(F_s)^{[**]}$. It is also not clear what should happen if $s \mapsto \chi(X_s, (F_s)^{[**]}(*))$ is locally constant.

Next we outline a method to study such problems in three steps.

- Show that the numerical function is upper or lower semicontinuous.
- If the function is locally constant, construct a candidate $f' \colon (X', F') \to S'$ for the flat model.
- Prove that, under suitable assumptions, $S' \simeq S$ and (X', F') as expected.

The details are simple, but at the end they lead to interesting consequences.

5.30 (How to prove semicontinuity?) Let $\phi(\ )$ be a function that associates to certain pairs (X, F) (consisting of a proper scheme over a field k, and a coherent sheaf on it, plus possibly some other data) an element in a partially ordered set. Typical examples for us are $\phi_1(X) := (K_X^n)$, $\phi_2(X, F) := \chi(X, F)$, or $\phi_3(X, F) := \chi(X, \text{pure}\, F(*))$ (if we also have an ample line bundle $\mathscr{O}_X(1)$). We always assume that $\phi(\ )$ is invariant under base field extensions.

Let $f \colon X \to S$ be a morphism, and F a coherent sheaf on X. We would like to prove that $s \mapsto \phi(X_s, F_s)$ is upper or lower semicontinuous. In many cases, this can be done in two stages.

(5.30.1) Prove that $s \mapsto \phi(X_s, F_s)$ is constant on a nonempty open subset $S^\circ \subset S$. If this works inductively for closed subsets of S, then Noetherian induction shows that $s \mapsto \phi(X_s, F_s)$ is constructible.

A constructible function is upper (resp. lower) semicontinuous iff it is upper (resp. lower) semicontinuous after base change to any DVR $T \to S$. Thus it remains to do:

(5.30.2) Let T be the spectrum of a DVR with closed point 0_T, generic point g_T, and $\pi \colon T \to S$ a morphism. Prove that

$$\phi(X_{0_T}, F_{0_T}) \geq \ (\text{resp.}\ \leq)\ \phi(X_{g_T}, F_{g_T}).$$

(Frequently $k(0_T) \neq k(\pi(0_T))$, this is why $\phi(\ )$ should be invariant under base field extensions.)

If we want to prove that $s \mapsto \phi(X_s, F_s)$ is locally constant, then we need only the following.

(5.30.3) Let T be the spectrum of a DVR with closed point 0_T, generic point g_T, and $\pi \colon T \to S$ a morphism that maps g_T to a generic point of S. Prove that

$$\phi(X_{0_T}, F_{0_T}) = \phi(X_{g_T}, F_{g_T}).$$

The generic point property of π is helpful if we have extra information about the generic fibers of $X \to S$.

5.31 (How to construct a candidate?) This is usually the hard part. If our objects $\phi(\ )$ are subvarieties, then $\phi(X_s, F_s)$ is a point in the Hilbert scheme or the Chow variety. Thus we have a set-theoretic map

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$$\sigma^{\mathrm{set}}\colon (\text{points of } S) \to \begin{cases} (\text{points of Hilb}(X/S)), & \text{or} \\ (\text{points of Chow}(X/S)). \end{cases}$$

If the objects $\phi(\ )$ are non-embedded varieties, we get a point in some moduli space. For the case of reflexive hulls we have considered, we need the moduli space of husks, which we discuss in Section 9.5.

Usually there are several choices for the moduli theory, and the proofs need the "correct" one to work. At the end we have a set-theoretic map

$$\sigma^{\mathrm{set}}\colon (\text{points of } S) \to (\text{points of some moduli space } \mathbf{M}).$$

Then the only sensible thing is to let S' be the closure of the image if σ^{set}; it comes with a natural map $\pi\colon S' \to S$.

If $\mathbf{M}$ is a coarse moduli space, then we have to make sure that there is a universal family over S', which usually means that we have to eliminate possible automorphisms (1.71); see (5.55–5.56) for such examples.

If this works out, then we have our candidate family $f'\colon (X', F') \to S'$, and a natural morphism $\pi\colon S' \to S$.

Then we need to show that $S' \simeq S$, and X', F' are as expected. The key is usually the isomorphism $S' \simeq S$. We typically know that π is proper, and an isomorphism over the generic points of S.

5.32 (How to check isomorphism?) Let $\pi\colon S' \to S$ be a proper morphism, $W \subset S$ a nowhere dense, closed subset, and $W' := \pi^{-1}(W) \subset S'$. Assume that $\pi\colon (S' \setminus W') \to (S \setminus W)$ is an isomorphism, and S' (resp. S) has no associated points in W' (resp. W). Then π is an isomorphism in the following cases:

(5.32.1) $\pi^{-1}(w) \simeq w$ for $w \in W$ (by Nakayama's lemma),

(5.32.2) $k(\mathrm{red}\,\pi^{-1}(w))/k(w)$ is purely inseparable for $w \in W$, and (W, S) is weakly normal (by definition (10.74)),

(5.32.3) $k(\mathrm{red}\,\pi^{-1}(w)) = k(w)$ for $w \in W$, and (W, S) is seminormal (by definition (10.74)),

(5.32.4) $\mathrm{depth}_W S \geq 2$ (by (10.6)),

(5.32.5) S is normal.

We illustrate the method in the simplest case, when we look at the reduced structure of the fibers of a morphism. Being reduced is invariant under separable ground field extensions. Thus working with $X_s \mapsto \mathrm{red}(X_s)$ is sensible in characteristic 0, but in general it is better to work with the reduced structure of the geometric fibers.

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Theorem 5.33 *Let $f: X \to S$ be a projective morphism of pure relative dimension n with f-ample $\mathcal{O}_X(1)$. Assume that X is reduced and S is weakly normal. For $s \in S$, let $X_{\bar{s}}$ denote the corresponding geometric fiber. Then*

(5.33.1) $\chi(s) := \chi(\mathrm{red}(X_{\bar{s}}), \mathcal{O}(*))$ *is lower semicontinuous, and*

(5.33.2) *f is flat with geometrically reduced fibers iff the generic fibers are geometrically reduced, and $\chi(s)$ is locally constant on S.*

We prove this in (5.37), but first some consequences and variants. If we understand only the leading coefficient of $\chi(\mathrm{red}(X_{\bar{s}}), \mathcal{O}(*))$, we still get very useful information about f as in Kollár (1996, I.6.5).

Corollary 5.34 (Smoothness criterion in codimension 0) *Let $f: X \to S$ be a projective morphism of pure relative dimension n, and H an f-ample divisor class. Assume that X is reduced and S is weakly normal. For $s \in S$ let $X_{\bar{s}}$ denote the geometric fiber. Then*

(5.34.1) $s \mapsto \deg_H(\mathrm{red}(X_{\bar{s}}))$ *is lower semicontinuous, and*

(5.34.2) *f is smooth on a dense subset of each fiber iff $s \mapsto \deg_H(\mathrm{red}(X_{\bar{s}}))$ is locally constant and f is generically smooth. (The latter is automatic in characteristic 0.)*

Proof Repeated application of (10.56) reduces the proof to $n = 0$, which is a special case of (5.33). □

It turns out that codimension 0 is the hardest part of (5.33), and we have stronger results in higher codimensions. The following is proved in (5.37).

Theorem 5.35 *Let $f: X \to S$ be a projective morphism of pure relative dimension n with f-ample $\mathcal{O}_X(1)$. Assume that X, S are reduced, and f is smooth at the generic points of each fiber. Then f is flat with geometrically reduced fibers iff $s \mapsto \chi(\mathrm{red}(X_s), \mathcal{O}(*))$ is locally constant.*

As a consequence, we get one part of (3.11) about the Hilbert-to-Chow map.

Corollary 5.36 (Flatness criterion in codimension 1) *Let $f: X \to S$ be a projective morphism of pure relative dimension n, and H an f-ample divisor class. Assume that X, S are reduced, and f is smooth at the generic points of each fiber. Then*

(5.36.1) *the sectional genus (3.10) of the fibers is a lower semi-continuous function on S, and*

(5.36.2) *f is flat with reduced fibers at codimension 1 points of each fiber iff the sectional genus is locally constant.*

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Proof Repeated application of (10.56) reduces the proof to the case $n = 1$, which is a special case of (5.35). □

We can thus expect that, for families that are locally stable in codimension 1, there are results connecting the intersection numbers $((\pi_0^* H)^{n-i} \cdot (K_{\bar{X}_0} + \bar{D}_0)^i)$ with the higher codimension behavior of f. There are two surprising twists.

- The *lower* semicontinuity in (5.34) and (5.36) switches to *upper* semicontinuity for $i = 2$.
- In most cases we need only *one* more intersection number to take care of all codimensions.

5.37 (Proof of 5.33 and 5.35) For (5.33.1), we follow (5.30). After a purely inseparable base change $S' \to S$, the generic fiber of $\operatorname{red}(X \times_S S') \to S$ is geometrically reduced, hence $s \mapsto \chi(\operatorname{red} X_{\bar{s}}, \mathscr{O}(*))$ is locally constant on a dense open set by generic flatness. This gives constructibility as in (5.30.1).

Continuing with (5.30.2), let T be the spectrum of a DVR with closed point t, generic point g, and $\pi\colon T \to S$ a morphism mapping t to $s \in S$. Set $Y := \operatorname{red}(X \times_S T)$, and assume that Y_g is geometrically reduced. Since f has pure relative dimension, $Y \to T$ is flat, hence

$$\chi(\operatorname{red}(Y_{\bar{t}}), \mathscr{O}(*)) \leq \chi(\operatorname{red}(Y_t), \mathscr{O}(*)) \leq \chi(Y_t, \mathscr{O}(*)) = \chi(Y_g, \mathscr{O}(*)). \qquad (5.37.1)$$

By (5.30), this proves (5.33.1) since $\chi(\operatorname{red}(Y_{\bar{t}}), \mathscr{O}(*)) = \chi(\operatorname{red}(X_{\bar{s}}), \mathscr{O}(*))$. We also see that the two sides of (5.37.1) are equal iff Y_t is also geometrically reduced.

If f is flat with geometrically reduced fibers then (5.33.2) is clear. For the converse we may assume that S is connected, so $p(*) := \chi(\operatorname{red} X_{\bar{s}}, \mathscr{O}(*))$ is independent of s.

For both (5.33) and (5.35), the relative Hilbert scheme now gives a clear choice for the candidate as in (5.31). Indeed, $\pi\colon \operatorname{Hilb}_p(X/S) \to S$ parametrizes subschemes of the fibers with Hilbert polynomial $p(*)$.

We claim that $\pi\colon \operatorname{Hilb}_p(X/S) \to S$ is an isomorphism. In both theorems we assume that the generic fibers are generically smooth, hence geometrically generically reduced. They are also S_1, hence geometrically S_1. A generically reduced S_1 scheme is reduced, so the generic fibers are geometrically reduced. The latter is an open property by (10.12). Thus π is an isomorphism over the dense open subset $S^\circ \subset S$ where f is flat with geometrically reduced fibers. The question is, what happens over other points.

The easy case is (5.35). By (5.38.4) $\pi^{-1}(s) = \operatorname{Hilb}_p(X_s) \simeq s$, thus π is an isomorphism by (5.32.1).

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For (5.33) the argument is more circuitous. Let $\mathrm{Hilb}'_p(X/S) \subset \mathrm{Hilb}_p(X/S)$ denote the closure of $\pi^{-1}(S^\circ)$ with projection $\pi'\colon \mathrm{Hilb}'_p(X/S) \to S$. First, we claim that π' is geometrically injective.

To see this pick any $s \in S$, and let $\tau\colon T \to S$ be a DVR that maps the closed point $t \in T$ to s and the generic point $g \in T$ to S°. We have a lifting $\tau'\colon T \to \mathrm{Hilb}'_p(X/S)$, and we check in (5.38.5) that $\tau'(t) = [\mathrm{red}(X_t)] \in \mathrm{Hilb}'_p(X/S)$.

Since S is assumed weakly normal, (5.31.2) implies that $\pi'\colon \mathrm{Hilb}'_p(X/S) \to S$ is an isomorphism.

We have $\mathrm{Univ}'_p(X/S) \to \mathrm{Hilb}'_p(X/S)$, and $u'\colon \mathrm{Univ}'_p(X/S) \to X$, which is a closed embedding on each fiber. Thus u' is a closed embedding by (10.54), hence an isomorphism since X is reduced.

Therefore $X \simeq \mathrm{Univ}'_d(X/S)$ is flat over S. In particular, $\mathrm{Hilb}_p(X/S) = \mathrm{Hilb}'_p(X/S) \simeq S$. □

5.38 (Uniqueness of red X) A scheme X uniquely determines red X, but what about in families? What if we know only the Hilbert polynomial of red X?

We start with two negative examples, followed by two positive results.

Example 5.38.1 Let X be the scheme $\operatorname{Spec} k[x,y]/(x^2, xy, y^2(y-1))$. Then $\mathrm{red}\, X = \operatorname{Spec} k[x,y]/(x, y(y-1))$ has length 2, but so are the subschemes $\operatorname{Spec} k[x,y]/(x^2, xy, y^2, x - cy)$. Thus $\mathrm{Hilb}_2 X \simeq \mathbb{P}^1_k \amalg \operatorname{Spec} k$.

Example 5.38.2 $\operatorname{Spec} k$ is the only subscheme of length 1 of $\operatorname{Spec} k[x]/(x^2)$. However, consider the trivial family $\pi\colon \operatorname{Spec} k[x,t]/(x^2,t^2) \to \operatorname{Spec} k[t]/(t^2)$. Then for every $c \in k$, the subscheme $\operatorname{Spec} k[x,t]/(x^2, t^2, x + ct)$ is flat over $\operatorname{Spec} k[t]/(t^2)$. Thus $\mathrm{Hilb}_1 \operatorname{Spec} k[x]/(x^2) \simeq \operatorname{Spec} k[t]/(t^2)$.

Claim 5.38.3 Let $(0 \in A)$ be the spectrum of a local Artinian k-algebra, and $f\colon Y \to A$ a projective morphism with f-ample $\mathscr{O}_X(1)$. Let F be a coherent sheaf on Y, and set $p(*) := \chi(Y_0, \mathrm{pure}(F_0)(*))$. Then F has at most one quotient $q\colon F \twoheadrightarrow Q$ that is flat over A with Hilbert polynomial $p(*)$. If Q exists then $Q = \mathrm{pure}\, F$.

Proof Set $n = \dim F_0$. If $q'\colon F_0 \to Q'$ is any map that is surjective at n-dimensional points, then $\chi(Y_0, F_0(*))$ and $\chi(Y_0, Q'(*))$ have the same leading coefficient iff $\dim(\ker q') < n$. Also, if $\chi(Y_0, Q'(*)) = \chi(Y_0, \mathrm{pure}(F_0)(*))$, then $Q' = \mathrm{pure}(F_0)$.

Thus, if Q is flat over A with Hilbert polynomial $p(*)$, then $\ker q \subset F$ is the largest subsheaf whose support has dimension $< n$. This shows that $Q = \mathrm{pure}\, F$ is the only possibility. □

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Corollary 5.38.4 Let X be a proper scheme of pure dimension n over a field k. Assume that X is geometrically reduced at its generic points. Set $p(*) := \chi(\operatorname{red} X, \mathscr{O}_{\operatorname{red} X}(*))$. Then $\operatorname{Hilb}_p(X) \simeq \operatorname{Spec} k$.

Proof In this case $\operatorname{red}(X_K) = \operatorname{Spec}_{X_K}(\operatorname{pure} \mathscr{O}_{X_K})$ for any field extension $K \supset k$. The rest follows from (5.38.3). □

Claim 5.38.5 Let T be the spectrum of a DVR, and $f\colon Y \to T$ a projective morphism of pure relative dimension n with f-ample $\mathscr{O}_X(1)$. Assume that Y_g is reduced and set $p(*) := \chi(Y_g, \mathscr{O}(*))$.

If $\chi(\operatorname{red}(Y_0), \mathscr{O}(*)) = p(*)$, then $\operatorname{red} Y \subset Y$ is the unique subscheme that is flat over T with Hilbert polynomial $p(*)$.

Proof Let $Z \subset Y$ be such a subscheme. Then $Z_g = Y_g$. Since f has pure relative dimension, the closure of Y_g contains Y_0, thus $\operatorname{red} Y_0 \subset Z_0$. These have the same Hilbert polynomial $p(*)$, hence they are equal. □

Examples 5.39 The following series of examples show that the assumptions in (5.33–5.35) are necessary.

(5.39.1) Let C be a cuspidal curve with normalization $p\colon \bar{C} \to C$. Then p is not flat, but $\operatorname{red} p^{-1}(c) \simeq c$ for every $c \in C$. Here C is not seminormal. Over imperfect fields, (10.75.3) gives similar examples where C is seminormal, but not weakly normal.

(5.39.2) Set $S := (uv = 0)$ and let $X \subset S \times \mathbb{A}^1_w$ be the union of two curves $(u = v - (w-1)^2 = 0) \cup (v = u - (w+1)^2 = 0)$, with projection $\pi\colon X \to S$. Then $\deg(k[\operatorname{red} X_s]/k(s)) = 2$ for every $s \in S$, but π is not flat, and it does not have pure relative dimension 0.

(5.39.3) A more complicated example of relative dimension 1 is the following. Set $S := (uv = 0)$ and let $X \subset S \times \mathbb{P}^3_{\mathbf{x}}$ be a reduced subscheme with three irreducible components as follows.

Over the u-axis we take a planar smooth cubic E_u degenerating to a cuspidal cubic E_0, for example $X_1 := (x_1 = ux_0^3 + x_2^3 - x_0x_3^2 = 0)$. We also add the line $X_3 := (u; 0{:}1{:}0{:}0)$.

Over the v-axis we take a smooth twisted cubic C_v degenerating to E_0. For example, X_2 can be the image of $(v; s{:}t) \mapsto (s^3{:}vs^2t{:}st^2{:}t^3)$. (The flat limit C_0 has an embedded point at the cusp.)

If $v \neq 0$ then $X_{0,v}$ is a smooth rational cubic, so $\chi(X_{0,v}, \mathscr{O}(m)) = 3m + 1$. If $u \neq 0$ then $X_{u,0}$ is a smooth elliptic cubic plus a disjoint point, so again $\chi(X_{u,v}, \mathscr{O}(m)) = 3m + 1$. Finally, $X_{0,0}$ is nonreduced, but $\operatorname{red} X_{0,0}$ is a singular planar cubic plus a disjoint point, so $\chi(\operatorname{red} X_{u,v}, \mathscr{O}(m)) = 3m + 1$.

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However, the projection $\pi\colon X \to S$ is not flat. Here π is not pure dimensional, and $X_{0,0}$ has two subschemes with Hilbert polynomial $3m + 1$. One is red $X_{0,0}$ the other is the one-dimensional irreducible component of $X_{0,0}$.

It is straightforward to generalize (5.35) from $\mathscr{O}_X$ to an arbitrary coherent sheaf F. The only change is that, instead of the Hilbert scheme $\operatorname{Hilb}(X/S)$, we use the quot-scheme $\operatorname{Quot}(F)$ (9.33). Thus we get the following.

Theorem 5.40 *Let $f\colon X \to S$ be a projective morphism with f-ample $\mathscr{O}_X(1)$ with S is reduced. Let F be a coherent sheaf on X that is generically flat (3.26) over S. Assume that $\operatorname{Supp} F \to S$ has pure relative dimension n, and F does not have embedded points. Then F is flat over S with pure fibers iff $s \mapsto \chi(X_s, \operatorname{pure}(F_s)(*))$ is locally constant.* □

5.6 Deformations of SLC Pairs

So far we have focused on locally stable deformations of slc pairs. The next result, due to Kollár and Shepherd-Barron (1988), connects arbitrary flat deformations (X_t, Δ_t) of an slc pair (X_0, Δ_0) to locally stable deformations of a suitable birational modification $f_0\colon (Y_0, \Delta_0^Y) \to (X_0, \Delta_0)$. We then compare various numerical invariants of (X_0, Δ_0) and of (X_t, Δ_t) by going through (Y_0, Δ_0^Y). This implies a weaker version of (5.5).

Theorem 5.41 *Kollár and Shepherd-Barron (1988) Let $(X, D + \Delta)$ be a normal pair, where D is a reduced, $\mathbb{Q}$-Cartier divisor that is demi-normal in codimension 1, and whose normalization $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is lc. Assume also*[1] *that*

(5.41.1) *either $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is klt,*

(5.41.2) *or $K_X + \Delta$ is $\mathbb{R}$-Cartier on $X \setminus D$.*

Then, in a neighborhood of D, the following hold.

(5.41.3) *The log canonical modification $f\colon (Y, D_Y + \Delta_Y + E) \to (X, D + \Delta)$ exists, and it is small, that is, $E = 0$.*

(5.41.4) *$(Y, D_Y + \Delta_Y)$ is lc.*

(5.41.5) *D_Y is normal at the generic point of every f_0-exceptional divisor $F \subset D_Y$, and $a(F, \bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) < 0$.*

(5.41.6) *$f(\operatorname{Ex}(f))$ is precisely the locus where $K_X + \Delta$ is not $\mathbb{R}$-Cartier.*

[1] Conjecturally, these are not needed; see (11.29).

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Proof Let $h\colon X' \to X$ be a log resolution with exceptional divisor E'. Set $\operatorname{discrep}(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) = -1 + \varepsilon$. Let $f\colon (Y, D_Y + \Delta_Y + (1-\varepsilon)E) \to (X, D + \Delta)$ be the relative canonical model of $(X', h_*^{-1}(D+\Delta) + (1-\varepsilon)E')$. This is the same as the relative canonical model of $(X', h_*^{-1}(D+\Delta) + (1-\varepsilon)E' - \eta h^*D)$ since h^*D is numerically h-trivial.

If $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is klt then $\varepsilon > 0$, hence $(X', h_*^{-1}(D+\Delta) + (1-\varepsilon)E' - \eta h^*D)$ is klt, and the relative canonical model exists by (11.28.2).

If $\varepsilon = 0$ then we note that $(X', h_*^{-1}(D+\Delta) + (1-\varepsilon)E' - \eta h^*D)$ has no lc centers over D for $\eta > 0$, hence the relative canonical model exists by (11.30) and (11.28.2).

Let $\pi_X\colon \bar{D} \to D$ and $\pi_Y\colon \bar{D}_Y \to D_Y$ be the normalizations. Then f_0 lifts to $\bar{f}_0\colon \bar{D}_Y \to \bar{D}$. Write $K_{\bar{D}_Y} + \Delta_{\bar{D}_Y} \sim_{\mathbb{R}} \bar{f}_0^*(K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta)$. By adjunction,

$$\begin{aligned}\pi_Y^*(K_Y + D_Y + \Delta_Y + (1-\varepsilon)E) &\sim_{\mathbb{R}} K_{\bar{D}_Y} + \operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E)\\ &\sim_{\mathbb{R}} \bar{f}_0^*(K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta) + (\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E) - \Delta_{\bar{D}_Y}).\end{aligned}$$

Since D has only nodes at codimension 1 points, X is canonical at codimension 1 points of D (11.35), and f is an isomorphism near these points. Thus $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E) - \Delta_{\bar{D}_Y}$ is $\bar{f}_0$-exceptional, and $\bar{f}_0$-ample. By (11.60) this implies that every $\bar{f}_0$-exceptional divisor appears in $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E) - \Delta_{\bar{D}_Y}$ with strictly negative coefficient.

Every divisor in $D_Y \cap E$ appears in $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E)$ with coefficient $\geq 1-\varepsilon$ by (11.16). On the other hand, every exceptional divisor appears in $\Delta_{\bar{D}_Y}$ with coefficient $\leq 1-\varepsilon$ by our choice of ε. Thus the divisors in $D_Y \cap E$ appear in $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y + (1-\varepsilon)E) - \Delta_{\bar{D}_Y}$ with coefficient $\geq (1-\varepsilon) - (1-\varepsilon) = 0$. We noted above that these coefficients are strictly negative, so $D_Y \cap E = \emptyset$.

Hence, after shrinking X, there are no exceptional divisors in $f\colon Y \to X$, so f is small, $D_Y = f^*D$, and $(Y, \Delta_Y + D_Y)$ is lc.

Let $\bar{F} \subset \bar{D}_Y$ be any $\bar{f}_0$-exceptional divisor. Since it appears in $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y) - \Delta_{\bar{D}_Y}$ with negative coefficient, it must appear in $\Delta_{\bar{D}_Y}$ with positive coefficient, and in $\operatorname{Diff}_{\bar{D}_Y}(\Delta_Y)$ with coefficient < 1. By (11.16), the latter implies that D_Y is smooth at the generic point of $\pi_Y(\bar{F})$, proving (5).

Finally let $x \in X \setminus D$ be a point where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Since f is small, $K_Y + \Delta_Y \sim_{\mathbb{R}} f^*(K_X + \Delta)$ over a neighborhood of x. Since $K_Y + \Delta_Y$ is f-ample, f is an isomorphism over a neighborhood of x. □

Complement 5.41.7 If $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is klt then D is normal. This was used in Kollár and Shepherd-Barron (1988) to get a description of the deformation space of log terminal surface singularities. The cone over an elliptic scroll gives

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examples where D is not normal, but its normalization has a simple elliptic singularity, see Mumford (1978).

See also Sato and Takagi (2022) for closely related results.

5.42 (Proof of 5.4) We prove (5.4) when the base S is the spectrum of a DVR. By (4.7), this implies the case when S is higher dimensional, provided f is assumed to be flat with S_2 fibers.

As a preliminary step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion (5.4.1) unchanged. Then (2.54), shows that the conclusion in (5.4.2) is also unchanged.

Thus assume that X is normal. The conclusions are local on C, so pick a point $0 \in C$, and let $f\colon (Y, \Delta^Y + Y_0) \to (X, X_0 + \Delta)$ be the log canonical modification as in (5.41). Let $\pi_Y\colon \bar{Y}_0 \to Y_0$ be the normalization and $\bar{f}_0\colon \bar{Y}_0 \to \bar{X}_0$ the induced birational morphism. We apply (10.32.3–4) to

$$D_Y := K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y \quad \text{and} \quad D_X := K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta = K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0.$$

The assumptions are satisfied since $(\bar{f}_0)_*(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y) = K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta$, and $K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y$ is $\bar{f}_0$-ample. Using the volume (10.31), this implies that

$$(K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta)^n = \operatorname{vol}(K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta) \geq \operatorname{vol}(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y),$$

and equality holds iff $\bar{f}_0$ is an isomorphism. Since $K_Y + \Delta^Y$ is $\mathbb{Q}$-Cartier,

$$\operatorname{vol}(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y) \geq \operatorname{vol}(K_{\bar{Y}_c} + \Delta^Y|_{\bar{Y}_c}) = ((K_{\bar{Y}_c} + \bar{\Delta}_c)^n)$$

for general $c \neq 0$, and $(\bar{Y}_c, \bar{\Delta}_c) = (X_c, \Delta_c)$ by (5.41.4). Combining the inequalities shows that $((K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0)^n) \geq ((K_{\bar{X}_c} + \Delta_c)^n)$ for general $c \neq 0$, and equality holds iff $\bar{f}_0$, and hence f, are isomorphisms over $0 \in C$. □

The same method can be used to prove a weaker version of the numerical criterion of local stability over smooth curves. This establishes (5.5) for families of surfaces over a smooth curve. It is not clear how to use these methods to complete the proof of (5.5) for higher dimensional families. We will derive (5.5) from (5.8) instead; see (5.27) for the key step.

Proposition 5.43 (Weak numerical criterion of local stability) *Let C be a smooth curve of* char 0*, and $f\colon (X, \Delta) \to C$ a morphism satisfying the assumptions (5.5.1–3). Then*

(5.43.1) *$I(c) := I(\pi_c^* H, K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)$ is upper semi-continuous for $\preceq$, and*

(5.43.2) *$f\colon (X, \Delta) \to C$ is locally stable iff $I(c)$ is locally constant on C.*

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Note that the first two numbers in the sequence $I(\pi_c^* H, K_{\bar X_c} + \bar D_c + \bar\Delta_c)$ equal $(H^n \cdot X_c)$ and $(H^{n-1} \cdot (K_X + \Delta) \cdot X_c)$, hence they are always locally constant. The first interesting number is $(\pi_c^* H^{n-2} \cdot (K_{\bar X_c} + \bar D_c + \bar\Delta_c)^2)$ which is thus an upper semicontinuous function on C by (1).

Proof As in (5.42) we may assume that X is normal. Let $f\colon (Y, \Delta^Y + Y_0) \to (X, X_0 + \Delta)$ be the log canonical modification, and $\bar f_0\colon \bar X_0 \to \bar Y_0$ the induced birational morphism between the normalizations. Here we apply (10.32.1–2) to $K_{\bar Y_0} + \operatorname{Diff}_{\bar Y_0} \Delta^Y$ and $K_{\bar X_0} + \operatorname{Diff}_{\bar X_0} \Delta$ to obtain that

$$I(\pi_0^* H, K_{\bar X_0} + \operatorname{Diff}_{\bar X_0} \Delta) \geq I(\bar f_0^* \pi_0^* H, K_{\bar Y_0} + \operatorname{Diff}_{\bar Y_0} \Delta^Y),$$

and equality holds iff $\bar f_0$ is an isomorphism. Since $K_Y + \Delta^Y$ is a $\mathbb{Q}$-Cartier divisor,

$$I(\bar f_0^* \pi_0^* H, K_{\bar Y_0} + \operatorname{Diff}_{\bar Y_0} \Delta^Y) = I(\pi_c^* H, K_{\bar Y_c} + \Delta^Y|_{\bar Y_c}) = I(\pi_c^* H, K_{\bar X_c} + \bar\Delta_c)$$

for general $c \neq 0$. Thus $I(\pi_0^* H, K_{\bar X_0} + \bar D_0 + \bar\Delta_0) \geq I(\pi_c^* H, K_{\bar X_c} + \bar\Delta_c)$ for general $c \neq 0$, and equality holds iff $\bar f_0$, and hence f, are isomorphisms. □

5.7 Simultaneous Canonical Models

In this section, we consider the existence of simultaneous canonical models.

5.44 (Proof of (5.10) over curves) Let B be a smooth curve of char 0, and $f\colon X \to B$ a morphism of pure relative dimension n.

First, we prove that $b \mapsto \operatorname{vol}(K_{X_b^{\mathrm r}})$ is a lower semicontinuous function on B.

If we replace X by a resolution $X^{\mathrm r} \to X$ then $\operatorname{vol}(K_{X_b^{\mathrm r}})$ is unchanged for general fibers, and it can only increase for special fibers. There are two sources for an increase. First, the resolution may introduce new divisors of general type. Second, if X is not normal, an irreducible component of a fiber may be replaced by a finite cover of it. The latter increases the volume by (10.38).

Thus it is enough to check lower semicontinuity when X is smooth, and all fibers are snc. If the volume of the general fiber is 0, then the volume of every fiber is 0 by (5.45), so assume that general fibers are of general type.

Fix a fiber $F = X_b$. By shrinking B we may assume that all other fibers are smooth. Let $f^{\mathrm c}\colon X^{\mathrm c} \to B$ be the relative canonical model of $(X, \operatorname{red} F) \to B$ as in (2.57.2). An irreducible component $E \subset F$ may get contracted. However, when this happens, then $K_E + (\operatorname{red} F - E)|_E = (K_X + \operatorname{red} F)|_E$ is negative on the fibers of the contraction, and so is K_E. Such divisors contribute 0 to the volume. Thus we can check lower semicontinuity on $f^{\mathrm c}\colon X^{\mathrm c} \to B$.

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Write $F^{\mathrm{c}} = \sum e_i E_i$, and let $\pi_i\colon \bar{E}_i \to E_i$ be the normalizations. As in (11.14), write $\pi_i^*(K_{X^{\mathrm{c}}} + \operatorname{red} F^{\mathrm{c}}) = K_{\bar{E}_i} + \bar{D}_i$, where $\bar{D}_i = \operatorname{Diff}_{\bar{E}_i}(\sum_{j\neq i} E_j)$. Let $g \in B$ be a general point. Then F^{c} is disjoint from X^{c}_g, and we have

$$\begin{aligned} (K_{X^{\mathrm{c}}_g})^n &= ((K_{X^{\mathrm{c}}} + \operatorname{red} F^{\mathrm{c}})^n \cdot X^{\mathrm{c}}_g) = ((K_{X^{\mathrm{c}}} + \operatorname{red} F^{\mathrm{c}})^n \cdot F^{\mathrm{c}}) \\ &= \textstyle\sum_i e_i((K_{\bar{E}_i} + \bar{D}_i)^n) \geq \sum_i((K_{\bar{E}_i} + \bar{D}_i)^n). \end{aligned} \tag{5.44.1}$$

Next we use (5.12) to obtain that $((K_{\bar{E}_i} + \bar{D}_i)^n) \geq \operatorname{vol}(K_{E^{\mathrm{r}}_i})$, hence

$$\operatorname{vol}(K_{X_g}) = ((K_{X^{\mathrm{c}}_g})^n) \geq \textstyle\sum_i \operatorname{vol}(K_{E^{\mathrm{r}}_i}) = \operatorname{vol}(\operatorname{red} F^{\mathrm{r}}),$$

proving the lower semicontinuity assertion. Furthermore, by (5.12), equality holds iff $D_i = 0$, the E_i have canonical singularities and $e_i = 1$ for every i. If $D_i = 0$, then E_i is the only irreducible component of its fiber by (11.16). Thus F^{c} is reduced and irreducible and has canonical singularities. So $f^{\mathrm{c}}\colon X^{\mathrm{c}} \to B$ is the simultaneous canonical model of $f\colon X \to B$. □

Lemma 5.45 *Let $f\colon X \to B$ be a projective morphism to a smooth curve B such that* $\operatorname{vol}(K_{X^{\mathrm{r}}_g})$ *is zero for the generic fiber X_g. Then* $\operatorname{vol}(K_{F^{\mathrm{r}}})$ *is zero for every fiber F of f.*

Proof The proof in (5.44) gives this if a resolution of X has a minimal model over B. This is not fully known, so we have to find a way to go around it.

As in (5.44), we can reduce to the case when X is smooth and F is an snc divisor. Let H be a general, smooth relatively ample divisor such that $K_X + H$ is f-ample. Using the continuity of the volume (Lazarsfeld, 2004, 2.2.44), there is a largest $0 \leq c < 1$ such that $\operatorname{vol}(K_{X_g} + cH_g) = 0$. Fix some $c' > c$ and run the MMP for $(X, \operatorname{red} F + c'H) \to B$. Then $K_{X_g} + c'H_g$ is big, so (11.28) applies, and we get a relative canonical model $(X^{\mathrm{c}}, \operatorname{red} F^{\mathrm{c}} + c'H^{\mathrm{c}}) \to B$. Let $\pi\colon \bar{F}^{\mathrm{c}} \to \operatorname{red} F^{\mathrm{c}}$ denote the normalization, and set $\bar{H}^{\mathrm{c}} = \pi^* H^{\mathrm{c}}$. As in (5.44.1), we get that

$$\operatorname{vol}(K_{X_g} + c'H^{\mathrm{c}}_g) \geq \operatorname{vol}(K_{\bar{F}^{\mathrm{c}}} + c'\bar{H}^{\mathrm{c}}) \geq \operatorname{vol}(K_{\bar{F}^{\mathrm{c}}}).$$

Letting $c' \to c$ gives that $0 = \operatorname{vol}(K_{X^{\mathrm{c}}_g} + cH^{\mathrm{c}}_g) \geq \operatorname{vol}(K_{F^{\mathrm{r}}})$, as required. □

5.46 (Proof of (5.11) over curves) Let B be a smooth curve over a field of char 0, and $f\colon (X, \Delta) \to B$ a flat morphism whose fibers are irreducible and smooth outside a codimension ≥ 2 subset. We may replace X by its normalization. Thus we may assume that X is normal, and the generic fiber is lc.

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Assume first that f is locally stable. We prove that $b \mapsto \mathrm{vol}(K_{X_b} + \Delta_b)$ is an upper semicontinuous function on S, and $f\colon (X, \Delta) \to B$ has a simultaneous canonical model iff this function is locally constant.

To see these, let $f^{\mathrm{c}}\colon (X^{\mathrm{c}}, \Delta^{\mathrm{c}}) \to B$ denote the canonical model of $f\colon (X, \Delta) \to B$ (11.28). For every $b \in B$ we need to understand the difference between

- $((X^{\mathrm{c}})_b, (\Delta^{\mathrm{c}})_b)$, the fiber of f^{c} over b, and
- $((X_b)^{\mathrm{c}}, (\Delta_b)^{\mathrm{c}})$, the canonical model of the fiber (X_b, Δ_b) of f.

These two are the same for general $g \in B$, but they can be different for some special points in B.

Let $\phi\colon X \dashrightarrow X^{\mathrm{c}}$ denote the natural birational map. Since the fibers of f are irreducible, they cannot be contracted, thus ϕ induces birational maps $\phi_b\colon X_b \dashrightarrow (X^{\mathrm{c}})_b$. Let Z_b denote the normalization of the closure of the graph of ϕ_b with projections $X_b \stackrel{g}{\leftarrow} Z_b \stackrel{h}{\to} (X^{\mathrm{c}})_b$. The key computation in (5.47), shows that $g^*(K_{X_b} + \Delta_b) \sim_{\mathbb{R}} h^*(K_{(X^{\mathrm{c}})_b} + (\Delta^{\mathrm{c}})_b) + F_b$, where F_b is effective. Thus

$$\mathrm{vol}(K_{X_b} + \Delta_b) = \mathrm{vol}(g^*(K_{X_b} + \Delta_b)) \geq \mathrm{vol}(h^*(K_{(X^{\mathrm{c}})_b} + (\Delta^{\mathrm{c}})_b)) = \mathrm{vol}(K_{(X^{\mathrm{c}})_b} + (\Delta^{\mathrm{c}})_b).$$

Note further that since $f^{\mathrm{c}}\colon (X^{\mathrm{c}}, \Delta^{\mathrm{c}}) \to B$ is flat, and $K_{X^{\mathrm{c}}} + \Delta^{\mathrm{c}}$ is f^{c}-ample, its restrictions to the various fibers have the same volume. Therefore

$$\mathrm{vol}\big(K_{(X^{\mathrm{c}})_b} + (\Delta^{\mathrm{c}})_b\big) = \mathrm{vol}\big(K_{(X^{\mathrm{c}})_g} + (\Delta^{\mathrm{c}})_g\big) = \mathrm{vol}\big(K_{X_g} + \Delta_g\big)$$

for generic $g \in B$. Thus $\mathrm{vol}(K_{X_b} + \Delta_b) \geq \mathrm{vol}(K_{X_g} + \Delta_g)$, and, by (10.39), equality holds iff F_b is h-exceptional. Then $((X^{\mathrm{c}})_b, (\Delta^{\mathrm{c}})_b)$ is the canonical model of (X_b, Δ_b). This proves both claims.

In the general case, when $f\colon (X, \Delta) \to B$ is not locally stable, we first use (5.41) to construct $h\colon (\bar{X}, \bar{\Delta}) \to (X, \Delta)$ such that the composite $f \circ h\colon (\bar{X}, \bar{\Delta}) \to B$ is locally stable. Thus $\mathrm{vol}(K_{\bar{X}_b} + \bar{\Delta}_b) \geq \mathrm{vol}(K_{X_g} + \Delta_g)$.

Note that $h_b\colon (\bar{X}_b, \bar{\Delta}_b) \to (X_b, \Delta_b)$ is birational by (5.41), and $K_{\bar{X}_b} + \bar{\Delta}_b$ is h_b-ample. Thus $\mathrm{vol}(X_b, \Delta_b) \geq \mathrm{vol}(\bar{X}_b, \bar{\Delta}_b)$ by (10.32.1). Putting these together shows the upper semicontinuity of the volume.

It remains to show that if equality holds then there is a simultaneous canonical model. We already proved that if $\mathrm{vol}(K_{\bar{X}_b} + \bar{\Delta}_b) = \mathrm{vol}(K_{X_g} + \Delta_g)$ then $f \circ h\colon (\bar{X}, \bar{\Delta}) \to B$ has a simultaneous canonical model, which is also the simultaneous canonical model of $f\colon (X, \Delta) \to B$ if $\mathrm{vol}(X_b, \Delta_b) = \mathrm{vol}(\bar{X}_b, \bar{\Delta}_b)$. Then $(\bar{X}_b, \bar{\Delta}_b)$ and (X_b, Δ_b) have isomorphic canonical models. The latter follows from (10.39), but it can also be obtained by applying the simpler (10.32) to the (normalization of the closure of the) graph of $(\bar{X}_b, \bar{\Delta}_b) \dashrightarrow (X^{\mathrm{c}}_b, \Delta^{\mathrm{c}}_b)$. □

Lemma 5.47 *Let $(X, D + \Delta)$ be lc where D is a reduced Weil divisor, and $\Delta = \sum a_i D_i$ is an $\mathbb{R}$-divisor. Let $f: X \to S$ be a proper morphism, and $\phi: (X, D + \Delta) \dashrightarrow (X^c, D^c + \Delta^c)$ the relative canonical model. If none of the irreducible components of D are contracted by ϕ, we get a birational map*

$$\phi_{\bar{D}}: (\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) \dashrightarrow (\bar{D}^c, \operatorname{Diff}_{\bar{D}^c} \Delta^c).$$

Moreover, $a(E, \bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) \leq a(E, \bar{D}^c, \operatorname{Diff}_{\bar{D}^c} \Delta^c)$ for every divisor E over $\bar{D}$ and $(\phi_{\bar{D}})_* \operatorname{Diff}_{\bar{D}} \Delta \geq \operatorname{Diff}_{\bar{D}^c} \Delta^c$.

Proof Let Y be the normalization of the main component of the fiber product $X \times_S X^c$ with projections $X \stackrel{g}{\leftarrow} Y \stackrel{h}{\to} X^c$. By definition,

$$g^*(K_X + D + \Delta) \sim_{\mathbb{R}} h^*(K_{X^c} + D^c + \Delta^c) + F$$

where F is effective. Let D_Y denote the birational transform of D on Y. Restricting to D_Y we get that

$$(g|_{D_Y})^*(K_D + \operatorname{Diff}_D \Delta) \sim_{\mathbb{R}} (h|_{D_Y})^*(K_{D^c} + \operatorname{Diff}_{D^c} \Delta^c) + F|_{D_Y}$$

and $F|_{D_Y}$ is also effective. This proves (1) and (2) is a special case. □

The existence of simultaneous canonical models is part of the following.

Question 5.48 Let $(X, D + \Delta)$ be an lc pair, and $(X^c, D^c + \Delta^c)$ its canonical model. What is the relationship between the canonical model of $(D, \operatorname{Diff}_D \Delta)$ and $(D^c, \operatorname{Diff}_{D^c} \Delta^c)$?

The following smooth example shows that these two are usually different.

Start with a smooth variety X', a smooth divisor $D' \subset X'$, and another smooth divisor $C' \subset D'$. Assume that $K_{X'} + D'$ is ample. Set $X := B_{C'} X'$ with exceptional divisor E, and let $D \subset X$ denote the birational transform of D'. Then $(X, D + E)$ is an lc pair whose canonical model is (X', D'), and $(D', 0)$ is its own canonical model. However, $(D, \operatorname{Diff}_D E) \simeq (D', C') \neq (D', 0)$.

The following is proved in Ambro and Kollár (2019, Thm.7).

Theorem 5.49 *Let $(X, D + \Delta)$ be an lc pair that is projective over a base scheme S with relatively ample divisor H, where all divisors in D appear with coefficient 1. Set $(X^0, D^0 + \Delta^0) := (X, D + \Delta)$, and, for $i = 1, \ldots, m$, let*

$$\phi^i: (X^{i-1}, D^{i-1} + \Delta^{i-1}) \dashrightarrow (X^i, D^i + \Delta^i)$$

be the steps of the $(X, D + \Delta)$-MMP with scaling of H. Assume that the intersection of D with the exceptional locus of $\phi^m \circ \cdots \circ \phi^1 : X \dashrightarrow X^m$ does not contain any log center (11.11) of $(X, D + \Delta)$. Let $\varrho: \bar{D} \to D$ be the normalization.

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Then the induced maps

$$\phi^i_{\bar D}\colon (\bar D^{i-1}, \operatorname{Diff}_{\bar D}\Delta^{i-1}) \dashrightarrow (\bar D^i, \operatorname{Diff}_{\bar D}\Delta^i)$$

form the steps of the MMP starting with $(\bar D^0, \operatorname{Diff}_{\bar D}\Delta^0) := (\bar D, \operatorname{Diff}_{\bar D}\Delta)$ *and with scaling of* $\varrho^* H$. □

5.8 Simultaneous Canonical Modifications

If S is smooth, then the simultaneous canonical modification of $f\colon (X,\Delta) \to S$ is also the canonical modification of (X,Δ) by (4.56). Thus, over a smooth curve, we consider the canonical modification of (X,Δ), and aim to prove that it is a simultaneous canonical modification.

5.50 (Proof of (5.16) over curves) Let C be a smooth curve, and $f\colon (X,\Delta) \to C$ a flat, projective morphism of pure relative dimension n that satisfies the assumptions of (5.16).

Each $c \mapsto (\pi_c^* H_c^{n-i} \cdot (K_{X_c^{\rm cm}} + \Delta_c^{\rm cm})^i)$ is a constructible function on C. Thus, in order to prove (5.16.1) we may assume that C is the spectrum of a DVR with closed point $0 \in C$ and generic point $g \in C$. We may also assume that X is reduced, thus f is flat.

By (5.34), $(\pi_0^* H_0^n) \le (\pi_g^* H_g^n)$, and equality holds iff X_0 is generically reduced. It is thus enough to deal with the latter case. Then X is generically normal along X_0, and we can replace X by its normalization without changing any of the assumptions or conclusions. We may now also assume that X is irreducible.

Let $\pi\colon (Y, \Delta^Y = \pi_*^{-1}\Delta) \to (X,\Delta)$ denote the canonical modification.

Write $Y_0 = \sum_i e_i E_i$ where $e_0 = 1$, and E_0 is the birational transform of X_0. (For now E_0 is allowed to be reducible.) Set $E := \operatorname{red} Y_0 = \sum E_i$. Let $\tau\colon \bar E_0 \to E_0$ denote the normalization, and write $\tau^*(K_Y + E + \Delta^Y) = K_{\bar E_0} + D_0$ where $D_0 = \operatorname{Diff}_{\bar E_0}(E - E_0 + \Delta^Y)$ as in (11.14). Choose $m \ge 0$ such that $K_Y + E + \Delta^Y + m\pi^* H$ is ample over C. We claim that

$$\begin{aligned}
&((K_{X_g^{\rm cm}} + \Delta_g^{\rm cm} + m\pi_g^* H)^n)\\
&\quad = ((K_{Y_g} + \Delta_g^Y + m\pi_g^* H)^n) = ((K_Y + \Delta^Y + m\pi^* H)^n \cdot [Y_g])\\
&\qquad = ((K_Y + E + \Delta^Y + m\pi^* H)^n \cdot [Y_g]) = ((K_Y + E + \Delta^Y + m\pi^* H)^n \cdot [Y_0])\\
&\qquad\quad = \textstyle\sum_i e_i (((K_Y + E + \Delta^Y + m\pi^* H)|_{E_i})^n) \ge ((K_{\bar E_0} + D_0 + m\pi_0^* H)^n)\\
&\qquad\qquad \ge \operatorname{vol}(K_{X_0^{\rm cm}} + \Delta_0^{\rm cm} + m\pi_0^* H) = ((K_{X_0^{\rm cm}} + \Delta_0^{\rm cm} + m\pi_0^* H)^n).
\end{aligned}$$

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The first equality holds since (Y_g, Δ_g^Y) is the canonical modification of (X_g, Δ_g), hence $\Delta_g^{\mathrm{cm}} = \Delta_g^Y$. The second equality is clear. We are allowed to add E in the fourth row since it is disjoint from Y_g. We can then replace Y_g by Y_0 since they are algebraically equivalent, and compute the latter one component at a time. $K_Y+E+\Delta^Y+m\pi^*H$ is ample, thus if we keep only the summands corresponding to E_0, we get the first inequality, which is an equality iff $Y_0 = E_0$.

The second inequality follows from (10.36), once we check that $\sigma_*^{-1}\Delta_0 \leq D_0$ where $\sigma := \pi_0 \circ \tau\colon \bar{E}_0 \to \bar{X}_0$ is the natural map. Since D_0 is effective, this is clear for σ-exceptional divisors. Otherwise, either π is an isomorphism over the generic point of a divisor D_0^i (hence D_0^i has the same coefficients in $\sigma_*^{-1}\Delta_0$ and D_0) or $\sigma_*^{-1}D_0^i$ is contained in another irreducible component of red Y_0. In this case $\sigma_*^{-1}D_0^i$ appears in D_0 with coefficient 1, and in $\sigma_*^{-1}\Delta_0$ with coefficient ≤ 1 by assumption. This proves the second inequality and, by (10.36), if equality holds then $D_0 = \sigma_*^{-1}\Delta_0$. The last equality is a general property of ample divisors.

As we noted in (5.14), the inequality proved in (5.50.1) is equivalent to $I(\pi_g^*H_g, K_{X_g^{\mathrm{cm}}} + \Delta_g^{\mathrm{cm}}) \geq I(\pi_0^*H_0, K_{X_0^{\mathrm{cm}}} + \Delta_0^{\mathrm{cm}})$, which proves (5.16.1).

If equality holds everywhere in (5.50.1) then $Y_0 = E_0$, $D_0 = \sigma_*^{-1}\Delta_0$, and $(\bar{E}_0, D_0)$ is canonical. On the other hand, D_0 is the sum of $\sigma_*^{-1}\Delta_0$ and of the conductor of $\bar{E}_0 \to E_0 = Y_0$. So the conductor is 0, Y_0 is normal in codimension 1, $D_0 = (\pi_0)_*^{-1}\Delta_0$, and $(Y_0, (\pi_0)_*^{-1}\Delta_0)$ is canonical in codimension 1. Thus Y_0 is normal and $(Y_0, (\pi_0)_*^{-1}\Delta_0)$ is canonical by (2.3). Since $K_{Y_0} + D_0$ is ample over X_0, these show that $(Y_0, (\pi_0)_*^{-1}\Delta_0)$ is the canonical modification of (X_0, Δ_0). Thus the canonical modification of (X, Δ) is also the simultaneous canonical modification, proving (5.16.2) over curves. □

In analogy with (5.15), we can define simultaneous slc modifications.

Definition 5.51 Let (X, Δ) be a pair over a field k that is slc in codimension 1. Its *semi-log-canonical modification* is a proper, birational morphism $\pi\colon (X^{\mathrm{slcm}}, \Delta^{\mathrm{slcm}}) \to (X, \Delta)$ such that π is an isomorphism over codimension 1 points of X, $\Delta^{\mathrm{slcm}} = \pi_*^{-1}\Delta + E$ where E contains every π-exceptional divisor with coefficient 1, $K_{X^{\mathrm{slcm}}} + \Delta^{\mathrm{slcm}}$ is π-ample, and $(X^{\mathrm{slcm}}, \Delta^{\mathrm{slcm}})$ is slc.

If X is normal, then the SLC modification is automatically normal, and it agrees with the log canonical modification.

In general, lc modifications are conjectured to exist, but there are slc pairs without slc modification: see Kollár (2013b, 1.40). In both cases, existence is known when $K_X + \Delta$ is $\mathbb{R}$-Cartier; see Odaka and Xu (2012).

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Let $f\colon (X,\Delta)\to S$ be a morphism as in (5.2) that satisfies the condition (5.3.1). A *simultaneous slc modification* is a proper morphism $\pi\colon (Y,\Delta^Y)\to (X,\Delta)$ such that $f\circ\pi\colon (Y,\Delta^Y)\to S$ is locally stable, and $\pi_s\colon (Y_s,\Delta^Y_s)\to (X_s,\Delta_s)$ is the slc modification for every $s\in S$.

We get the following variant of (5.16).

Theorem 5.52 *Let C be a smooth curve, $f\colon (X,\Delta)\to C$ a projective morphism as in (5.2) that satisfies the condition (5.3.1). Assume that $K_X+\Delta$ is $\mathbb{R}$-Cartier, and the slc modification $\pi_c\colon (X_c^{slcm},\Delta_c^{slcm})\to (X_c,\Delta_c)$ exists for every $c\in C$. Then*

(5.52.1) $c\mapsto I(\pi_c^*H_c^{n-2}, K_{X_c^{slcm}}+\Delta_c^{slcm})$ *is lower semi-continuous for $\preceq$, and*

(5.52.2) $f\colon (X,\Delta)\to C$ *has a simultaneous slc modification iff this function is locally constant.*

Proof Using (2.54), we may assume that X is normal. Next we closely follow the proof of (5.50).

Let $\pi\colon (Y,\Delta^Y)\to (X,\Delta)$ denote the log canonical modification; this exists by (11.29). Note that here $\Delta^Y=\pi_*^{-1}\Delta+F$ where F is the sum of all π-exceptional divisors that dominate C.

Write $Y_0=\sum_i e_iE_i$ where $e_0=1$ and E_0 is the birational transform of X_0. Let $\tau\colon \bar E_0\to E_0$ denote the normalization, and write $\tau^*(K_Y+Y_0+\Delta^Y)=K_{\bar E_0}+D_0$. Choose $m\ge 0$ such that $K_Y+Y_0+\Delta^Y+m\pi^*H$ is ample over C. As in the proof of (5.50), we get that

$$\begin{array}{rcl} ((K_{X_g^{\mathrm{lcm}}}+\Delta_g^{\mathrm{lcm}}+m\pi_g^*H)^n) & \ge & ((K_{\bar E_0}+D_0+m\pi_0^*H)^n) \quad \text{and}\\ \operatorname{vol}(K_{X_0^{\mathrm{lcm}}}+\Delta_0^{\mathrm{lcm}}+m\pi_0^*H) & = & (K_{X_0^{\mathrm{lcm}}}+\Delta_0^{\mathrm{lcm}}+m\pi_0^*H)^n. \end{array}$$

It remains to prove that $(K_{\bar E_0}+D_0+m\pi_0^*H)^n\ge \operatorname{vol}(K_{X_0^{\mathrm{lcm}}}+\Delta_0^{\mathrm{lcm}}+m\pi_0^*H)$.

We have $\sigma\colon \bar E_0\to X_0$, and we can apply (10.37) if every σ-exceptional divisor $\bar F_0\subset \bar E_0$ appears in D_0 with coefficient 1.

By the definition of lc modifications, every divisor F_i that is exceptional for $Y\to X$ appears in Δ^Y with coefficient 1. If $K_X+\Delta$ is $\mathbb{R}$-Cartier then the exceptional set of $Y\to X$ has pure codimension 1. In this case, $\tau(\bar F_0)$ is contained in a divisor that is exceptional for $Y\to X$. Thus, by adjunction, $\bar F_0$ appears in D_0 with coefficient 1.

If (X_0,Δ_0) is slc at a point x_0 then (X,Δ) is also slc at x_0 by inversion of adjunction (11.17), hence π is a local isomorphism over x_0. Thus $\pi_0\colon (Y_0,\Delta_0^Y)\to (X_0,\Delta_0)$ is an isomorphism over codimension 1 points of X_0.

The rest of the proof works as before. □

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If $K_X + \Delta$ is not $\mathbb{R}$-Cartier then it can happen that an exceptional divisor $\bar{F}_0 \subset \bar{E}_0$ is not contained in any exceptional divisor of $X^{\mathrm{lcm}} \to X$. In such cases we lose control of the coefficient of $\bar{F}$ in D_0. This occurs in (5.22) over the 4 singular points that lie on D_0.

5.9 Families over Higher Dimensional Bases

Here we complete the proofs of Theorems 5.4–5.16. In all cases, the first part asserts that a certain constructible function on the base scheme S is upper or lower semicontinuous. As in (5.30), for constructible functions, semicontinuity can be checked along spectra of DVRs, and this was already done in all cases.

The remaining part is to show that if our functions are locally constant on S, then certain constructions produce a flat family of varieties or sheaves. In all cases, we have already checked that this holds when the base is a smooth curve.

Going to arbitrary reduced bases is quickest in the following example.

5.53 (Proof of 5.1) We already proved the case when S is the spectrum of a DVR in (5.42). As we noted in (5.30), this implies (5.1.1) in general. Thus it remains to prove that if $s \mapsto (K^n_{X_s})$ is constant then $f\colon X \to S$ is stable.

In view of (5.42), we know that $f_T\colon X_T \to T$ is stable for every $T \to S$ where T is the spectrum of a DVR. Thus $f\colon X \to S$ is stable by (4.7). □

We aim to argue similarly for Theorems 5.4, 5.5 and 5.6. Note that in these cases we cannot apply (5.8) since f is not assumed to be flat, and its fibers are not assumed to be S_2. We follow (5.31). For (5.4–5.6) this needs the theory of hulls and husks, to be explained in Chapter 9.

5.54 (Proof of 5.4–5.6) Let $\pi\colon \operatorname{Hull}(\mathscr{O}_X/S) \to S$ denote the hull (9.39) of $\mathscr{O}_X$. We aim to show that π is an isomorphism.

By (9.40), π is a locally closed decomposition (10.83).

Let T be the spectrum of a DVR, and $g\colon T \to S$ a morphism that maps the generic point of T to a generic point of S. We apply (5.42) or (5.27) to the divisorial pull-back $f_T\colon (X_T, \Delta_T) \to T$ to conclude that it is stable (resp. locally stable). For (5.6) we use (2.88.5).

Thus $g\colon T \to S$ factors uniquely through $\pi\colon \operatorname{Hull}(\mathscr{O}_X/S) \to S$, hence π is proper. $\pi\colon H \to S$ is an isomorphism by (10.83.2). In particular, $f\colon X \to S$ is flat with S_2 fibers. Thus the fibers are slc by assumption and (11.37).

Now we can apply (4.35) to conclude that $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier, hence $f\colon (X, \Delta) \to S$ is stable (resp. locally stable). □

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For the remaining cases, (5.31) needs the moduli space of pairs with an artificial, but efficient, rigidification.

5.55 (Proof of 5.10–5.11) Both claims were already established over the spectrum of a DVR, see (5.44) and (5.46). This implies the semicontinuity assertions in both cases.

It remains to show that if the volume is constant then $f\colon X \to S$ (resp. $f\colon (X,\Delta) \to S$) has a simultaneous canonical model.

Consider the moduli space of marked stable pairs $\pi\colon \mathrm{SP}^{\mathrm{red}} \to S$; since S is reduced, the version in (4.1) is sufficient for our purposes. Set

$$S' := \{(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}}) : s \in S\} \subset \mathrm{SP}^{\mathrm{red}}.$$

In order to prove that S' is a closed subset, first we claim that it is constructible. This is clear since the canonical model over a generic point of S extends to a canonical model over an open subset of S, and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVRs, and the latter follows from (5.44) and (5.46).

Thus S' is a scheme, and the projection π induces a geometric bijection $S' \to S$ which is finite by (5.44) and (5.46). Thus $S' \to S$ is an isomorphism since we assumed that S is seminormal.

If each $(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}})$ is rigid, then $S' \subset \mathrm{SP}^{\mathrm{rigid}}$, and there is a universal family $\mathrm{Univ}^{\mathrm{rigid}} \to \mathrm{SP}^{\mathrm{rigid}}$ by (8.71). Therefore the pull-back of the universal family $\mathrm{Univ}^{\mathrm{rigid}}$ to S' gives the simultaneous canonical model over $S \simeq S'$.

We have no reason to assume that the $(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}})$ are rigid, but we can make the proof work by rigidifying $f : (X,\Delta) \to S$.

The simultaneous canonical model is unique, hence it is enough to construct it étale locally. After replacing S by an étale neighborhood of a point $0 \in S$, we may assume that there are r sections $\sigma_i : S \to X$ such that $(X_0, \Delta_0, \sigma_1(0), \dots, \sigma_r(0))$ is rigid, and the $\sigma_i(0)$ are smooth points of $X_0 \setminus \operatorname{Supp} \Delta_0$ such that $(X_0, \Delta_0) \dashrightarrow (X_0^{\mathrm{c}}, \Delta_0^{\mathrm{c}})$ is a local isomorphism at these points.

By (8.65), after further shrinking S we may assume that the same holds at every point $s \in S$. Using the moduli of marked, pointed stable pairs MpSP (8.44) and (8.71.1), we can run the previous argument for

$$S' := \{(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}}, \sigma_1(s), \dots, \sigma_r(s))\colon s \in S\} \subset \mathrm{MpSP}^{\mathrm{rigid}}$$

to prove that the simultaneous canonical model exists over S. □

5.56 (Proof of 5.16) The proof follows very closely the arguments in (5.55). Both claims were already established over the spectrum of a DVR, see (5.50). This implies the semicontinuity assertion in general.

It remains to show that if $I(s) = I(\pi_s^* H_s, K_{X_s^{\rm cm}})$ is constant, then $f\colon (X, \Delta) \to S$ has a simultaneous canonical modification. Since the simultaneous canonical modification is unique, it is sufficient to construct it étale locally over S. So pick a point $s_0 \in S$, in the sequel we are free to replace S by smaller neighborhoods of s_0.

Choose $m > 0$ such that $K_{X_s^{\rm cm}} + m\pi_s^* H_s$ is ample for every $s \in S$. Next choose a general $D \in |mH|$ such that $(X_{s_0}^{\rm cm}, \Delta_{s_0}^{\rm cm} + \pi_{s_0}^* D_{s_0})$ is log canonical. We claim that, possibly after shrinking S, $(X_s^{\rm cm}, \Delta_s^{\rm cm} + \pi_s^* D_s)$ is log canonical for every $s \in S$. By (4.44) this condition defines a constructible subset of S and, by (5.50), it contains every generalization of s_0. Thus it contains an open neighborhood of s_0. Thus $(X_s^{\rm cm}, \Delta_s^{\rm cm} + \pi_s^* D_s)$ is a stable pair for every $s \in S$.

Consider the moduli space of marked stable pairs $\pi\colon \ \mathrm{SP} \to S$, and set

$$S' := \{(X_s^{\rm cm}, \Delta_s^{\rm cm} + \pi_s^* D_s)\colon\ s \in S\} \subset \mathrm{SP}.$$

In order to prove that S' is a closed subset, first we claim that it is constructible. This is clear since the canonical modification over a generic point of S extends to a canonical modification over an open subset of S, and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVRs, and the latter follows from (5.50).

Thus S' is a scheme, and the projection π induces a geometric bijection $S' \to S$ which is finite by (5.50). Thus $S' \to S$ is an isomorphism since S is assumed seminormal.

For general D, the pairs $(X_s^{\rm cm}, \Delta_s^{\rm cm} + \pi_s^* D_s)$ should be rigid, and then the pull-back of the universal family to S' gives the simultaneous canonical modification over $S \simeq S'$. Technically it may be easier to rigidify using étale-local sections as in (5.55). □