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ON MAXIMAL SUBSETS OF PAIRWISE NONCOMMUTING ELEMENTS IN FINITE p-GROUPS

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Abstract

A subset X of a finite group G is a set of pairwise noncommuting elements if $xy \neq yx$ for all $x \neq y \in X$. If $|X| \geq |Y|$ for any other subset Y of pairwise noncommuting elements, then X is called a maximal subset of pairwise noncommuting elements and the size of such a set is denoted by $\omega(G)$. In a recent article by Azad *et al.* ['Maximal subsets of pairwise noncommuting elements of some finite p-groups', *Bull. Iran. Math. Soc.* **39**(1) (2013), 187–192], the value of $\omega(G)$ is computed for certain p-groups G. In the present paper, our aim is to generalise these results and find $\omega(G)$ for some more p-groups of interest.

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1. Introduction

Let G be a finite nonabelian group and let $Z(G)$ denote the centre of G. A subset $N \subset G \backslash Z(G)$ is called a *pairwise noncommuting subset* of G if $xy \neq yx$ for all $x \neq y \in N$. If $|N| \geq |M|$ for any other subset M of pairwise noncommuting elements in G, then N is said to be a *maximal subset of pairwise noncommuting elements*. The cardinality of such a subset is denoted by $\omega(G)$. In fact, $\omega(G)$ is the maximal size of a clique in the *noncommuting graph* Γ_G of G whose vertex set $V(\Gamma_G)$ is $G \backslash Z(G)$ and whose edge set $E(\Gamma_G)$ consists of those $\{x, y\}$ with $x \neq y \in G \backslash Z(G)$ such that $[x, y] \neq 1$. The noncommuting graph of a finite group G was first considered by Erdős in 1975 [10]. By a famous result of Neumann [10] answering a question of Erdős, the finiteness of $\omega(G)$ is equivalent to the finiteness of the factor group $G/Z(G)$ in G. Mason [9] gave a bound for $\omega(G)$ by covering the group G by $(\frac{1}{2}|G| + 1)$ abelian subgroups. Pyber [12] proved that there is some constant c such that the index of the centre $Z(G)$ in G satisfies $|G : Z(G)| \leq c^{\omega(G)}$. The value of $\omega(G)$ has been computed for various groups G (see for example [1, 3, 5–7]).

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A finite p-group G is called extraspecial if the centre, the Frattini subgroup and the derived subgroup of G all coincide and are cyclic of order p. Chin [5] has shown that

$$np + 1 \leq \omega(G) \leq \frac{p(p-1)^n - 2}{p-1},$$

for extraspecial p groups of odd order p^{2n+1}. For extraspecial 2-groups of order 2^{2n+1}, Isaacs proved that $\omega(G) = 2n + 1$ (see [4, page 40]). The cardinalities of maximal subsets of pairwise noncommuting elements of extraspecial p-groups are important as they provide combinatorial information which can be used to calculate their cohomology lengths. (The cohomology length of a nonelementary abelian p-group is a cohomology invariant derived from a theorem of Serre [15].)

Azad *et al.* [3] proved that $\omega(G) = p + 1$ for any finite p-group G with central quotient of order p^2, where p is a prime number. Moreover, they also determined $\omega(G)$ for any nonabelian group of order p^4. Orfi [11] determined $\omega(G)$ for p-groups of order p^5. Fouladi and Orfi [7] proved that $\omega(G) = |G'|(p + 1)/p$, where G is a finite nonabelian metacyclic p-group with $p > 2$. Further, Fouladi and Orfi [6] determined $\omega(G)$ for some p-groups G of maximal class.

In this paper, we generalise the results of [3]. In particular, [3, Lemma 3.2] states that if G is a p-group with central quotient of order p^3, then G is an AC-group. In Section 3, we prove the following generalisation of this result.

Theorem 1.1. *Let G be a nonabelian p-group of order p^n. Suppose $|Z(G)| = p^r$ with $n - r \geq 3$ and G has an abelian maximal subgroup. Then there exists an element $x \in G \backslash Z(G)$ such that $|C_G(x)| = p^{n-1}$ and $C_G(x)$ is uniquely determined. Moreover, $\omega(G) = p^{n-r-1} + 1$.*

If G satisfies the assumptions of Theorem 1.1 with $|G/Z(G)| = p^3$, it follows that $\omega(G) = p^2 + 1$. This is [3, Theorem 3.3(ii)]. In Section 4, we calculate $\omega(G)$ for AC p-groups where the cardinality of $G/Z(G)$ is either p^4 or p^5. We also discuss the nature of certain centraliser subgroups.

In Section 5, we generalise [3, Lemma 3.1 and Theorem 3.3] by proving the following theorems.

Theorem 1.2. *Let G be a finite nonabelian group and let p be the smallest prime dividing the order of G. If $|G/Z(G)| = p^2$, then G is an AC-group and $\omega(G) = p + 1$.*

Theorem 1.3. *Let G be a finite nonabelian group and let p be the smallest prime dividing the order of G. If $|G/Z(G)| = p^3$, then G is an AC-group and $\omega(G) = p^2 + (1 - \delta)p + 1$, where δ is a nonnegative integer.*

Throughout the paper, G denotes a finite nonabelian group and $Z(G)$, $C_G(x)$ denote respectively the centre of G and the centraliser of an element $x \in G$. If $x, y \in G$, then $[x, y] = x^{-1}y^{-1}xy$. By G' and $Z_2(G)$ we denote the commutator and the second centre of G respectively.

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2. Preliminaries

In this section, we quote some results that are required in the rest of the paper. We start with the following lemma, which is an easy exercise.

LEMMA 2.1. *Let G be a finite group.*

(1) *For any subgroup H of G,* $\omega(H) \leq \omega(G)$.
(2) *For any normal subgroup N of G,* $\omega(G/N) \leq \omega(G)$.

A group G is called an AC-group, if the centraliser of every noncentral element of G is abelian. The following lemma gives equivalent criteria for a group to be an AC-group.

LEMMA 2.2 [14, Lemma 3.2]. *The following statements are equivalent:*

(1) G *is an* AC*-group.*
(2) *If* $[x, y] = 1$, *then* $C_G(x) = C_G(y)$, *where* $x, y \in G\backslash Z(G)$.
(3) *If* $[x, y] = [x, z] = 1$, *then* $[y, z] = 1$, *where* $x \in G\backslash Z(G)$.
(4) *If* A *and* B *are subgroups of* G *and* $Z(G) < C_G(A) \leq C_G(B) < G$, *then* $C_G(A) = C_G(B)$.

REMARK 2.3. If G is an AC-group, then $\{C_G(x) \mid x \in G\backslash Z(G)\}$ is the set of maximal abelian subgroups.

LEMMA 2.4 [6, Lemma 2.2]. *Let G be an* AC*-group.*

(1) *If* $x, y \in G\backslash Z(G)$ *with distinct centralisers, then* $C_G(x) \cap C_G(y) = Z(G)$.
(2) *If* $G = \bigcup_{i=1}^{k} C_G(x_i)$, *where* $C_G(x_i)$ *and* $C_G(x_j)$ *are distinct for* $1 \leq i < j \leq k$, *then* $\{x_1, \ldots, x_k\}$ *is a maximal set of pairwise noncommuting elements of G.*

LEMMA 2.5 [3, Lemma 2.3]. *Let G be a finite* AC*-group. Then* $G = \bigcup_{i=1}^{k} C_G(x_i)$, *where* $C_G(x_i)$ *are distinct for* $i \neq j$ *and* $\{x_1, \ldots, x_k\}$ *is a maximal set of pairwise noncommuting elements of G.*

LEMMA 2.6. *Let G be an* AC*-group and let X be a set of noncommuting elements in G. Then X can be extended to a maximal set of noncommuting elements in G.*

PROOF. Let $\omega(G) = k$ and $M = \{x_1, x_2, \ldots, x_k\}$ be a maximal set of noncommuting elements in G. Since G is an AC-group, we have $G = \bigcup_{i=1}^{k} C_G(x_i)$, where $C_G(x_i)$ and $C_G(x_j)$ are distinct and $C_G(x_i) \cap C_G(x_j) = Z(G)$ for $1 \leq i < j \leq k$. Since $C_G(x_i)$ is abelian, $|C_G(x_i) \cap X| \leq 1$ for $1 \leq i \leq k$. Set $P := \{i \in \{1, \ldots, k\} \mid C_G(x_i) \cap X = \varnothing\}$. Now choose $a_j \in C_G(x_j)\backslash Z(G)$ for each $j \in P$. Then, $X \cup \{a_j \mid j \in P\}$ is a maximal set of noncommuting elements in G. □

LEMMA 2.7 [12, Lemma 3.4]. *Let* $G = H \times K$, *where H and K are nonabelian subgroups of G. Then,* $\omega(G) \geq \omega(H)\omega(K)$.

LEMMA 2.8. *Let H and K be groups.*

(1) *If K is an* AC*-group and* $H' = 1$, *then* $H \times K$ *is also an* AC*-group.*

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(2) *If H, K and $H \times K$ all are AC-groups, then $\omega(H \times K) = \omega(H)\omega(K)$.*
(3) *If H is a nilpotent AC-group, then H is a metabelian.*

Proof. (1) Follows from the fact that $C_{H\times K}(h, k) = C_H(h) \times C_K(k)$, where $(h, k) \in H \times K$.

(2) Let $X = \{(x_i, y_i) \mid 1 \leq i \leq n\}$ be a maximal set of noncommuting elements in $H \times K$ with $\omega(H \times K) = |X|$. Define $X_H := \{x_i \mid (x_i, -) \in X\}$ and $X_K := \{y_i \mid (-, y_i) \in X\}$. Then, $|X_H| \geq \omega(H)$ and $|X_K| \geq \omega(K)$. Suppose $|X_H| > \omega(H)$. Then there exists $x_i \neq x_j \in X_H$ such that $x_i x_j = x_j x_i$. Since H is an AC-group, $C_H(x_i) = C_H(x_j)$. Choose $(x_i, y_i), (x_j, y_j) \in X$, so that $y_i y_j \neq y_j y_i$. Then,

$$\begin{aligned} C_{H\times K}(x_i, y_i) \cap C_{H\times K}(x_j, y_j) &= (C_H(x_i) \cap C_H(x_j)) \times (C_K(y_i) \cap C_K(y_j)) \\ &= C_H(x_i) \times Z(K) \neq Z(H) \times Z(K), \end{aligned}$$

which is a contradiction. Hence, $|X_H| = \omega(H)$ and, similarly, $|X_K| = \omega(K)$.

(3) Since H is a nilpotent group, $Z(H) < Z_2(H)$. Now, by [13, Theorem 5.1.11], we have $[Z_2(H), H'] = 1$. Let $x \in Z_2(H)\backslash Z(H)$. Then $H' \subset C_H(x)$ and so H' is abelian. This shows that H is a metabelian group. □

Lemma 2.9 [2, Lemma 5.7]. *Let $f(x), g(x) \in \mathbb{Z}[x]$ such that $f(x)/g(x)$ takes integer values for infinitely many values of $x \in \mathbb{Z}$. Then, $f(x)/g(x) \in \mathbb{Q}[x]$. Further, if $g(x)$ is monic, $f(x)/g(x) \in \mathbb{Z}[x]$.*

3. *AC* *p*-groups

We begin this section with the following proposition.

Proposition 3.1 [14, Proposition 3.10]. *Let G be a p-group.*

(1) *If G has an abelian subgroup of index p, then G is an AC group.*
(2) *If G has an abelian subgroup A of index p^2, but no abelian subgroup of index p, then G is an AC group if and only if $C_G(x) \cap A = Z(G)$ for every $x \in G\backslash A$.*

Lemma 3.2. *Let G be an AC p-group. Then, $\omega(G) \equiv 1 \pmod p$.*

Proof. Let $\{x_1, \ldots, x_k\}$ be a maximal set of pairwise noncommuting elements. Write $G = \bigcup_{i=1}^{k} C_G(x_i)$, where $C_G(x_i)$ and $C_G(x_j)$ are distinct for $1 \leq i < j \leq k$. By Lemma 2.4(1),

$$|G| = \sum_{i=1}^{k}(|C_G(x_i)| - |Z(G)|) + |Z(G)| = -(k-1)|Z(G)| + \sum_{i=1}^{k} |C_G(x_i)|,$$

$$|G/Z(G)| = -(k-1) + \sum_{i=1}^{k} |C_G(x_i)/Z(G)|$$

and the desired result follows. □

Proof of Theorem 1.1. Since G has an abelian maximal subgroup, G is an AC group by Proposition 3.1(1). By Remark 2.3, there exists a noncentral element x such that $|C_G(x)| = p^{n-1}$. Suppose for $y \neq x$, we have $|C_G(y)| = p^{n-1}$ and $C_G(y) \neq C_G(x)$. Then, $p^{n-r} = |G/Z(G)| = |G/(C_G(x) \cap C_G(y))| \leq |G/C_G(x)|\,|G/C_G(y)| = p^2$, which is impossible. Hence, $C_G(x)$ is uniquely determined. In fact, it is the unique abelian maximal subgroup of G.

Next, we determine $\omega(G)$ by considering two cases.

Case 1: $n - r > 3$. First, we show that there is no noncentral element z such that $|C_G(z)| \geq p^m$, where $r + 2 \leq m \leq n - 2$. On the contrary, suppose there is $z \in G\backslash Z(G)$ such that $|C_G(z)| \geq p^m$. Then, $p^{n-r} = |G/Z(G)| = |G/(C_G(x) \cap C_G(z))| \leq |G/C_G(x)|\,|G/C_G(y)| = p \cdot p^{n-m}$, which is impossible. Hence, $|C_G(z)| = p^{r+1}$.

Let $\{x_1, \ldots, x_k\}$ be a maximal set of pairwise noncommuting elements. By the above observations we may assume that $|C_G(x_1)| = p^{n-1}$ and $|C_G(x_j)| = p^{r+1}$ for $2 \leq j \leq k$. Now, write $G = \bigcup_{i=1}^{k} C_G(x_i)$. By Lemma 2.4(1),

$$|G| = \sum_{i=1}^{k}(|C_G(x_i)| - |Z(G)|) + |Z(G)|,$$

that is, $p^n = (p^{n-1} - p^r) + (k - 1)(p^{r+1} - p^r) + p^r$, which yields $\omega(G) = p^{n-r-1} + 1$.

Case 2: $n - r = 3$. By a similar argument to that in Case 1, $|C_G(x_1)| = p^{n-1}$ and $|C_G(x_j)| = p^{n-2}$ for $2 \leq j \leq k$. Therefore, by (4.1), $\omega(G) = p^2 + 1 = p^{n-r-1} + 1$.

This completes the proof of the theorem. □

4. $|G/Z(G)| = p^4$ or p^5

In this section, we calculate $\omega(G)$ for certain AC p-groups with $|G/Z(G)| = p^4$ or p^5. We also discuss the uniqueness of certain centraliser subgroups.

THEOREM 4.1. *Let G be an AC p-group of order p^n with $|G/Z(G)| = p^4$ and $p \neq 3$. Suppose G has no abelian maximal subgroup and let $X = \{x_1, x_2, \ldots, x_k\}$ be a maximal set of noncommuting elements in G.*

(1) *If G has no noncentral element x such that $|C_G(x)| = p^{n-2}$, then $\omega(G) = p^3 + p^2 + p + 1$.*

(2) *If $|C_G(x_i)| = p^{n-2}$ for $1 \leq i \leq r$ and $|C_G(x_j)| = p^{n-3}$ for $r + 1 \leq j \leq k$, then $\omega(G) = -rp + (p + 1)(p^2 + 1)$ and $k - r \geq 2$. In particular, if G has no noncentral element x such that $|C_G(x)| = p^{n-3}$, then $\omega(G) = p^2 + 1$.*

PROOF. Since G has no abelian maximal subgroup, the cardinality of the centraliser of any noncentral element is either p^{n-2} or p^{n-3}. Write $G = \bigcup_{i=1}^{k} C_G(x_i)$. Now, we consider two cases.

Case 1. Suppose there is no noncentral element x such that $|C_G(x)| = p^{n-2}$. In this case $|C_G(x)| = p^{n-3}$ for every $x \in G\backslash Z(G)$. By Lemma 2.4(1),

$$|G| = \sum_{i=1}^{k}(|C_G(x_i)| - |Z(G)|) + |Z(G)|, \tag{4.1}$$

yielding $\omega(G) = p^3 + p^2 + p + 1$.

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Case 2. Suppose there is a noncentral element x such that $|C_G(x)| = p^{n-2}$. This case divides into two subcases.

Subcase 1. If $|C_G(x)| = p^{n-2}$ for every $x \in G\backslash Z(G)$, then $\omega(G) = p^2 + 1$.

Subcase 2. Without loss of generality, suppose $|C_G(x_i)| = p^{n-2}$ for $1 \leq i \leq r$ and $|C_G(x_j)| = p^{n-3}$ for $r + 1 \leq j \leq k$. Again, Lemma 2.4(1) gives (4.1), which yields $p^n = r(p^{n-2} - p^{n-4}) + (k - r)(p^{n-3} - p^{n-4}) + p^{n-4}$, that is

$$k = -rp + (p + 1)(p^2 + 1). \tag{4.2}$$

If $k - r = r$, then (4.2) leads to $r = (p + 1)(p^2 + 1)/(2 + p)$. By Lemma 2.9 this is not possible for any prime p, except for $p = 3$. If $k - r = 1$, then (4.2) gives $k = p^3 + p^2 + 2p + 1/(p + 1)$, which is impossible. Thus, if there is a noncentral element x such that $|C_G(x)| = p^{n-3}$, then the number of such elements in the maximal noncommuting set is more than one. Moreover, if $r = 1$, then from (4.2), $k = p^3 + p^2 + 1$. □

Lemma 4.2. *Let G be a p-group of order p^6. Suppose G is not an AC group and $|G/Z(G)| = p^4$.*

(1) *If there is an element x such that $|C_G(x)| = p^3$ or p^4, then $C_G(x)$ is abelian.*
(2) *There exists a noncentral element x such that $|C_G(x)| = p^5$ and $C_G(x)/Z(C_G(x)) \cong C_p \times C_p$.*

Proof. (1) If there exists x such that $|C_G(x)| = p^3$ or p^4, then $C_G(x)/Z(C_G(x))$ is either a trivial group or a cyclic group of order p.

(2) If $x \in G\backslash Z(G)$, then $|C_G(x)| \in \{p^3, p^4, p^5\}$. Suppose there is no $x \in G\backslash Z(G)$ such that $|C_G(x)| = p^5$. Then, by (1), G is an AC group, which is a contradiction. Hence, there is a noncentral element x such that $C_G(x)$ is nonabelian and $|C_G(x)| = p^5$. Since $Z(G) < C_G(x)$ and $C_G(x)/Z(C_G(x))$ is not cyclic, $C_G(x)/Z(C_G(x)) \cong C_p \times C_p$. □

Theorem 4.3. *Let G be an AC p-group of order p^n with $|G/Z(G)| = p^5$, where p is odd. Suppose G has no abelian maximal subgroup and let $X = \{x_1, x_2, \ldots, x_k\}$ be a maximal set of noncommuting elements in G.*

(1) *If G has a noncentral element x such that $|C_G(x)| = p^{n-2}$, then $C_G(x)$ is uniquely determined. Further, suppose $|C_G(x_i)| = p^{n-3}$ with $2 \leq i \leq r + 1$ and $|C_G(x_j)| = p^{n-4}$ for $r + 2 \leq j \leq k$. Then, $\omega(G) = p^4 + p^3 - rp + 1$ and $r \neq k - r - 1$.*
(2) *If G has no noncentral element x such that $|C_G(x)| = p^{n-2}$, then $\omega(G) = p(p + 1)(p^2 + 1) + 1 - r(p + 1) + r$, where $|C_G(x_i)| = p^{n-3}$ for $1 \leq i \leq r$ and $|C_G(x_j)| = p^{n-4}$ for $r + 1 \leq j \leq k$ and $r \neq k$.*

Proof. By the hypothesis, G has no abelian maximal subgroup and hence the cardinality of the centraliser of any noncentral element is p^{n-4} or p^{n-3} or p^{n-2}. Write $G = \bigcup_{i=1}^{k} C_G(x_i)$.

Case 1. Suppose there is a noncentral element x such that $|C_G(x)| = p^{n-2}$.

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We claim that $C_G(x)$ is uniquely determined. On the contrary, suppose there is another y such that $|C_G(y)| = p^{n-2}$ and $C_G(x) \neq C_G(y)$. Then $p^5 = |G/Z(G)| = |G/(C_G(x) \cap C_G(y))| \leq |G/C_G(x)|\,|G/C_G(y)| = p^2$, which is impossible. Now, let $|C_G(x_1)| = p^{n-2}$, $|C_G(x_i)| = p^{n-3}$ with $2 \leq i \leq r + 1$ and $|C_G(x_j)| = p^{n-4}$ for $r + 2 \leq j \leq k$. Lemma 2.4(1) again gives (4.1) which in this case yields

$$p^n = (p^{n-2} - p^{n-5}) + r(p^{n-3} - p^{n-5}) + (k - r - 1)(p^{n-4} - p^{n-5}) + p^{n-5},$$

that is

$$k = p^4 + p^3 - rp + 1. \tag{4.3}$$

Thus, $\omega(G) = p^4 + p^3 - rp + 1$.

If $k - r - 1 = 1$, then from (4.3), $k = (p^4 + p^3 + 2p + 1)/(p + 1)$, which is impossible. Thus, if there is a noncentral element x such that $|C_G(x)| = p^{n-4}$, then the number of such elements in the maximal noncommuting set is more than 1. Next, if $k - r - 1 = 0$, then from (4.3), $\omega(G) = p^3 + 1$. Now, if $r = 1$, then from (4.3), we have $k = p^4 + p^3 - p + 1$. This implies that if there is a unique noncentral element x in the maximal noncommuting set such that $|C_G(x)| = p^{n-3}$, then $\omega(G) = p^4 + p^3 - p + 1$. If $r = 0$, then from (4.3), we have $\omega(G) = p^4 + p^3 + 1$. If $k - r - 1 = r$, then from (4.3), we have $r = p^3(p + 1)/(p + 2)$, which is not possible. Hence, this case will not arise.

Case 2. Suppose there is no noncentral element x such that $|C_G(x)| = p^{n-2}$. In this case, suppose that $|C_G(x_i)| = p^{n-3}$ for $1 \leq i \leq r$ and $|C_G(x_j)| = p^{n-4}$ for $r + 1 \leq j \leq k$. By Lemma 2.4(1), we again have (4.1) which gives

$$p^n = r(p^{n-3} - p^{n-5}) + (k - r)(p^{n-4} - p^{n-5}) + p^{n-5}$$

that is

$$k = p(p + 1)(p^2 + 1) + 1 - r(p + 1) + r. \tag{4.4}$$

If $k - r = 1$, then from (4.4), $\omega(G) = p^3 + p + 1$. If $r = 1$, then from (4.4), $\omega(G) = p^4 + p^3 + p^2 + 1$. If $k = 2r$, then from (4.4), $\omega(G) = (p(p + 1)(p^2 + 1) + 1)/(p + 2)$, which is impossible. Hence, this case will not occur.

This completes the proof of the theorem. □

To prove our next lemma, we will use the classification of groups of order p^5 by James [8, Section 4.5]. James divided the groups of order p^5 into isoclinism families. It is well known that if G is isoclinic to H, then $\omega(G) = \omega(H)$ (see [11, Lemma 2.6]).

Lemma 4.4. *Let G be a p-group of order p^6. Suppose G is not an AC group and $|G/Z(G)| = p^5$.*

(1) *If there is an element x such that $|C_G(x)| = p^2$ or p^3, then $C_G(x)$ is abelian.*
(2) *If there exists an element x such that $|C_G(x)| = p^5$, then $C_G(x)/Z(C_G(x))$ is either isomorphic to an elementary abelian group or a nonabelian group of order p^3.*
(3) *If there is no noncentral element x such that $|C_G(x)| = p^5$, then there exists a noncentral element y such that $|C_G(y)| = p^4$ and $C_G(y)/Z(C_G(y)) \cong C_p \times C_p$.*

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Proof. (1) If there exists x such that $|C_G(x)| = p^2$ or p^3, then $C_G(x)/Z(C_G(x))$ is either a trivial group or a cyclic group of order p.

(2) Let $x \in G \backslash Z(G)$ such that $|C_G(x)| = p^5$. By Proposition 3.1, $C_G(x)$ is nonabelian. Since $Z(G) < C_G(x)$ and $C_G(x)/Z(C_G(x))$ cannot be a cyclic group, $C_G(x)/Z(C_G(x))$ is either isomorphic to an elementary abelian group or a nonabelian group of order p^3 (use [8, Section 4.5]).

(3) Let $x \in G \backslash Z(G)$. By assumption, $|C_G(x)| \in \{p^2, p^3, p^4\}$. Therefore, by part (1), there exists a noncentral element y such that $C_G(y)$ is nonabelian and $|C_G(y)| = p^4$. Since $Z(G) < C_G(y)$, $C_G(y)/Z(C_G(y)) \cong C_p \times C_p$. □

5. *AC*-groups

In this section, we prove Theorems 1.2 and 1.3.

Proposition 5.1. *Let G be a p-group having an abelian subgroup H of index p^3 but no abelian subgroup of index p or p^2. Then, G is an AC group if and only if $C_G(x) \cap H = Z(G)$ for every $x \in G \backslash H$.*

Proof. Since G has no abelian subgroup of index p or p^2, $Z(G) \le H$. If G is an AC group, then clearly $C_G(x) \cap H = Z(G)$ for every $x \in G \backslash H$. Conversely, suppose that $C_G(x) \cap H = Z(G)$ for every $x \in G \backslash H$. If $x \in H \backslash Z(G)$, then $C_G(x) = H$. Now if $x \in G \backslash H$, then $C_G(x) = \langle x \rangle Z(G)$ or $C_G(x) = \langle x, y \rangle Z(G)$. This completes the proof. □

Proof of Theorem 1.2. Let $x \in G \backslash Z(G)$. Then $Z(G) < C_G(x) < G$ and

$$p^2 = |G/Z(G)| = |G/C_G(x)|\,|C_G(x)/Z(G)|.$$

Hence, $|C_G(x)/Z(G)| = p$, which implies that $Z(G)$ is a maximal subgroup of $C_G(x)$. Therefore, $C_G(x) = \langle Z(G), x \rangle$ is abelian. This shows that G is an AC-group. Now, write $G = \bigcup_{i=1}^{k} C_G(x_i)$, where $k = \omega(G)$. By Lemma 2.4(1), we again have (4.1), which yields that $\omega(G) = p + 1$. □

Proof of Theorem 1.3. Repeating the argument of the proof of Theorem 1.2, we have

$$p^3 = |G/Z(G)| = |G/C_G(x)|\,|C_G(x)/Z(G)|,$$

where $x \in G \backslash Z(G)$. Now, we have two cases.

Case 1. If $|C_G(x)/Z(G)| = p$, then $Z(G)$ is a maximal subgroup of $C_G(x)$. Therefore, $C_G(x) = \langle Z(G), x \rangle$ is abelian.

Case 2. If $|C_G(x)/Z(G)| = p^2$, then from the tower $Z(G) < Z(C_G(x)) \le C_G(x)$, we obtain

$$p^2 = |C_G(x)/Z(G)| = |C_G(x)/Z(C_G(x))|\,|Z(C_G(x))/Z(G)|.$$

If $|C_G(x)/Z(C_G(x))| = 1$, then $C_G(x)$ is abelian. On the other hand, if $|C_G(x)/Z(C_G(x))| = p$, then $Z(C_G(x))$ is a maximal subgroup of $C_G(x)$. Thus, $C_G(x) = \langle Z(C_G(x)), y \rangle$ is an abelian subgroup, for any $y \in C_G(x)\backslash Z(C_G(x))$. In both cases $C_G(x)$ is abelian subgroup for $x \in G\backslash Z(G)$. Therefore, G is an AC-group.

By the above arguments, either $|C_G(x)| = p|Z(G)|$ or $|C_G(x)| = p^2|Z(G)|$ for $x \in G\backslash Z(G)$. Let $X = \{x_1, x_2, \ldots, x_k\}$ be a maximal set of noncommuting elements in G. Let δ be the number of x_i such that $|C_G(x_i)| = p^2|Z(G)|$ and hence for the remaining $k - \delta$ many x_i, we have $|C_G(x_i)| = p|Z(G)|$. Now, write $G = \bigcup_{i=1}^{k} C_G(x_i)$. By Lemma 2.4(1), we have (4.1) and

$$|G| = \delta(p^2|Z(G)| - |Z(G)|) + (k - \delta)(p|Z(G)| - |Z(G)|) + |Z(G)|.$$

We conclude that $\omega(G) = p^2 + (1 - \delta)p + 1$. □

Remark 5.2. Regarding the proof of the Theorem 1.3, we observe that if $|C_G(x_i)| = p^2|Z(G)|$, then $|G/C_G(x)| = p$. Hence, $C_G(x)$ is a maximal subgroup of G. If G has no abelian maximal subgroup, then $\delta = 0$ and hence $\omega(G) = p^2 + p + 1$. On the other hand if G has a maximal abelian subgroup A, then $A = C_G(x)$ and $\delta = 1$ for some $x \in G\backslash Z(G)$. In this case we get $\omega(G) = p^2 + 1$. Therefore, the above theorem is a generalisation of [3, Theorem 3.3].

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