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# WEAK HAAGERUP PROPERTY OF $W^*$-CROSSED PRODUCTS

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**Abstract**

We show that if $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property, then both $M$ and $\Gamma$ have the weak Haagerup property, and if $\Gamma$ is an amenable group, then the weak Haagerup property of $M$ implies that of $M \bar{\rtimes}_\alpha \Gamma$. We also give a condition under which the weak Haagerup property for $M$ and $\Gamma$ implies that of $M \bar{\rtimes}_\alpha \Gamma$.

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## 1. Introduction

The crossed product of a noncommutative dynamical system is one of the most important constructions in the theory of operator algebras. Approximation theory is also central and there are many different approximation properties such as nuclearity, the completely bounded approximation property, the Haagerup property, property $T$ and so on. Many authors have explored different approximation properties of crossed products (see [1–6, 9, 11, 12] and [14]).

In order to study the relation between weak amenability and the Haagerup property, Knudby [7] introduced the weak Haagerup property for discrete groups as a weakened version of weak amenability and the Haagerup property.

Let $\Gamma$ be a discrete group, $C_0(\Gamma)$ the space of functions vanishing at infinity and $B_2(\Gamma)$ the algebra of Herz–Schur multipliers. In fact, $B_2(\Gamma)$ is a unital Banach algebra when equipped with the Herz–Schur norm $\|\cdot\|_{B_2}$. (See [2, Appendix D] and [8, Section 3] for more information on Herz–Schur multipliers.)

DEFINITION 1.1. Let $\Gamma$ be a discrete group. Then $\Gamma$ has the weak Haagerup property if there are a constant $C > 0$ and a net $\{u_\alpha\}_{\alpha\in I}$ in $B_2(\Gamma) \cap C_0(\Gamma)$ such that:

(1) $\|u_\alpha\|_{B_2} \leq C$ for every $\alpha \in I$;
(2) $u_\alpha(g) \to 1$ as $\alpha \to \infty$ for every $g \in \Gamma$.

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The weak Haagerup constant $\Lambda_{WH}(\Gamma)$ for $\Gamma$ is defined as the infimum of those $C$ for which such a net $\{u_\alpha\}_{\alpha\in I}$ exists. If no such net exists we write $\Lambda_{WH}(\Gamma) = \infty$.

In [8], Knudby studied the weak Haagerup property for von Neumann algebras and proved that a discrete group has the weak Haagerup property if and only if its group von Neumann algebra does.

Definition 1.2. Let $M$ be a von Neumenn algebra with a faithful normal tracial state $\tau$. Then $M$ has the weak Haagerup property if there is a constant $C > 0$ and a net $\{T_\alpha\}_{\alpha\in I}$ of normal completely bounded maps on $M$ such that:

(1) $\|T_\alpha\|_{cb} \le C$ for every $\alpha \in I$;
(2) $\langle T_\alpha(x), y\rangle_\tau = \langle x, T_\alpha(y)\rangle_\tau$ for every $x, y \in M$;
(3) each $T_\alpha$ can be extended to a compact operator on $L^2(M, \tau)$;
(4) $\|T_\alpha(x) - x\|_{2,\tau} \to 0$ for every $x \in M$.

The weak Haagerup constant $\Lambda_{WH}(M)$ for $M$ is defined as the infimum of those $C$ for which such a net $\{T_\alpha\}_{\alpha\in I}$ exists, and if no such net exists we write $\Lambda_{WH}(M) = \infty$. From [8, Proposition 8.4], the weak Haagerup property does not depend on the choice of the faithful normal tracial state.

Motivated by these results, we study the weak Haagerup property of $W^*$-crossed products. First, we obtain the following result in Theorem 2.2.

Theorem 1. *If $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property, then both $M$ and $\Gamma$ have the weak Haagerup property.*

Conversely, if $\Gamma$ has some stronger properties than the weak Haagerup property, the weak Haagerup property of $M$ may imply that of $M \bar{\rtimes}_\alpha \Gamma$. In particular, we show in Theorem 2.3 a more elaborate version of the following important result.

Theorem 2. *Let $\Gamma$ be an amenable group. If $M$ has the weak Haagerup property, then $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property.*

Moreover, in Theorem 2.4, we give a condition under which the weak Haagerup property of $M$ and $\Gamma$ implies that of $M \bar{\rtimes}_\alpha \Gamma$.

## 2. Main results

Throughout this paper, $M \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, where $\mathcal{H}$ is a separable Hilbert space, $\Gamma$ is a discrete group that acts on $M$ through an action $\alpha$, $\tau$ is a faithful $\alpha$-invariant normal tracial state on $M$ and $e$ is the identity element of $\Gamma$. We denote by $M \bar{\rtimes}_\alpha \Gamma$ the $W^*$-crossed product of $(M, \Gamma, \alpha)$, and identify $M \subseteq M \bar{\rtimes}_\alpha \Gamma$ as well as $\Gamma \subseteq M \bar{\rtimes}_\alpha \Gamma$ through their canonical embeddings. Let $C_c(\Gamma, M)$ be the linear space of finitely supported functions on $\Gamma$ with values in $M$. We write $\tau' = \tau \circ \mathcal{E}$ for the induced faithful normal tracial state of $M \bar{\rtimes}_\alpha \Gamma$, where $\mathcal{E} : M \bar{\rtimes}_\alpha \Gamma \to M$ is the canonical faithful normal conditional expectation.

Since $\tau$ is a faithful normal tracial state on $M$, by the Gelfand–Naimark–Segal construction, $\tau$ defines a Hilbert space, denoted by $L^2(M, \tau)$. We also denote by $\|\cdot\|_{2,\tau}$

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the associated Hilbert norm, and by $\langle\, ,\rangle_\tau$ the associated inner product. Suppose that $T : M \to M$ is a normal completely bounded map such that $\langle T(x), y\rangle_\tau = \langle x, T(y)\rangle_\tau$ for every $x, y \in M$. It follows from [8, Proposition 7.1] that $T$ can be extended to a bounded operator on $L^2(M,\tau)$ with norm at most $\|T\|$. It is easy to see that a normal completely bounded map $T$ on $M$ extends to a compact operator on $L^2(M,\tau)$ if and only if, for any $\varepsilon > 0$, there exists a finite rank map $Q : M \to M$ such that $Q$ is bounded with respect to the norm $\|\cdot\|_{2,\tau}$ and such that

$$\|T(x) - Q(x)\|_{2,\tau} \le \varepsilon\|x\|_{2,\tau}$$

for all $x \in M$.

Lemma 2.1. *If $x \in M$, then the following statements hold:*

(1) $\tau'(x^*y) = \tau(x^*\mathcal{E}(y))$ *for all* $y \in M \bar{\rtimes}_\alpha \Gamma$;
(2) $\tau'(y^*x) = \tau(\mathcal{E}(y)^*x)$ *for all* $y \in M \bar{\rtimes}_\alpha \Gamma$.

Proof. We just need to prove statement (1). For any $\sum_{s\in\Gamma} b_s s \in C_c(\Gamma, M)$,

$$\tau'\Big(x^* \sum_{s\in\Gamma} b_s s\Big) = \tau'\Big(\sum_{s\in\Gamma}(x^* b_s)s\Big) = \tau(x^* b_e) = \tau\Big(x^*\mathcal{E}\Big(\sum_{s\in\Gamma} b_s s\Big)\Big).$$

Hence, $\tau'(x^*y) = \tau(x^*\mathcal{E}(y))$ for all $y \in M \bar{\rtimes}_\alpha \Gamma$. □

Theorem 2.2. *If $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property, then both $M$ and $\Gamma$ have the weak Haagerup property and*

$$\Lambda_{WH}(M) \le \Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma), \quad \Lambda_{WH}(\Gamma) \le \Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma).$$

Proof. Suppose there is a net $\{\Phi_i\}_{i\in I}$ of normal completely bounded maps from $M \bar{\rtimes}_\alpha \Gamma$ to itself witnessing the weak Haagerup property of $M \bar{\rtimes}_\alpha \Gamma$ with $\|\Phi_i\|_{cb} \le C$ for all $i \in I$. For each $i \in I$, let

$$T_i(x) = \mathcal{E} \circ \Phi_i(x)$$

for all $x \in M$. Then $T_i$ is a normal completely bounded map from $M$ to itself and

$$\|T_i\|_{cb} = \|\mathcal{E} \circ \Phi_i|_M\|_{cb} \le \|\Phi_i\|_{cb} \le C.$$

As $i$ tends to infinity,

$$\|T_i(x) - x\|_{2,\tau}^2 \le \tau \circ \mathcal{E}((\Phi_i(x) - x)^*(\Phi_i(x) - x)) = \|\Phi_i(x) - x\|_{2,\tau'}^2 \to 0$$

for all $x \in M$. Moreover,

$$\langle T_i(x), y\rangle_\tau = \tau(y^*\mathcal{E} \circ \Phi_i(x)) = \tau'(y^*\Phi_i(x)) = \langle \Phi_i(x), y\rangle_{\tau'}$$

and

$$\langle x, T_i(y)\rangle_\tau = \tau(\mathcal{E} \circ \Phi_i(y)^*x) = \tau'(\Phi_i(y)^*x) = \langle x, \Phi_i(y)\rangle_{\tau'}$$

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for all $x, y \in M$. Hence, $\langle T_i(x), y\rangle_\tau = \langle x, T_i(y)\rangle_\tau$ for all $x, y \in M$. For any $\varepsilon > 0$, there exists a finite rank map $Q : M \bar{\rtimes}_\alpha \Gamma \to M \bar{\rtimes}_\alpha \Gamma$ such that $Q$ is bounded with respect to the norm $\|\cdot\|_{2,\tau'}$ and such that

$$\|\Phi_i(x) - Q(x)\|_{2,\tau'} \le \varepsilon \|x\|_{2,\tau'}$$

for all $x \in M \bar{\rtimes}_\alpha \Gamma$. Hence, for all $x \in M$,

$$\begin{aligned}\|T_i(x) - \mathcal{E} \circ Q(x)\|_{2,\tau} &= \|\mathcal{E} \circ \Phi_n(x) - \mathcal{E} \circ Q(x)\|_{2,\tau} \\ &\le \|\Phi_n(x) - Q(x)\|_{2,\tau'} \\ &\le \varepsilon \|x\|_{2,\tau'} = \varepsilon \|x\|_{2,\tau}.\end{aligned}$$

Thus, $M$ has the weak Haagerup property and $\Lambda_{WH}(M) \le \Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma)$.

For each $i \in I$ and $g \in \Gamma$, let

$$\varphi_i(g) = \tau'(g^{-1}\Phi_i(g)).$$

Then

$$|\varphi_i(g) - 1| = |\tau'(g^{-1}(\Phi_i(g) - g))| \le \|\Phi_i(g) - g\|_{2,\tau'} \to 0$$

for all $g \in \Gamma$. It follows from the compactness of $\Phi_i$ that

$$\limsup_{g\to\infty} |\varphi_i(g)| = \limsup_{g\to\infty} |\tau'(g^{-1}\Phi_i(g))| \le \limsup_{g\to\infty} \|\Phi_i(g)\|_{2,\tau'} \to 0.$$

We identify $M \bar{\rtimes}_\alpha \Gamma \subseteq \mathcal{B}(L^2(M \bar{\rtimes}_\alpha \Gamma, \tau'))$. Then there exists a unique unit vector $\xi \in L^2(M \bar{\rtimes}_\alpha \Gamma, \tau')$ such that $\tau'(x) = \langle x\xi, \xi\rangle_{\tau'}$ for all $x \in M \bar{\rtimes}_\alpha \Gamma$. From the Fundamental Factorisation theorem of completely bounded maps (see [13, Theorem 1.6]), there is a Hilbert space $\mathcal{K}$, a representation $\pi : \mathcal{B}(L^2(M \bar{\rtimes}_\alpha \Gamma, \tau')) \to \mathcal{B}(\mathcal{K})$ and operators $V_1 : L^2(M \bar{\rtimes}_\alpha \Gamma, \tau') \to \mathcal{K}$, $V_2 : \mathcal{K} \to L^2(M \bar{\rtimes}_\alpha \Gamma, \tau')$ such that $\|V_1\|\,\|V_2\| = \|\Phi_i\|_{cb}$ and $\Phi_i(x) = V_2\pi(x)V_1$. Hence,

$$\varphi_i(h^{-1}g) = \tau'((h^{-1}g)^{-1}\Phi_i(h^{-1}g)) = \langle \pi(g)V_1 g^{-1}\xi, \pi(h)V_2^* h^{-1}\xi\rangle$$

for all $g, h \in \Gamma$. It follows from [8, Proposition 3.1] that $\varphi_i \in B_2(\Gamma)$ and $\|\varphi_i\|_{B_2} \le C$. This proves the weak Haagerup property of $\Gamma$ and $\Lambda_{WH}(\Gamma) \le \Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma)$. □

Next, we show that if $\Gamma$ is an amenable group, then the weak Haagerup property of $M$ implies that of $M \bar{\rtimes}_\alpha \Gamma$. We denote by $\rho$ the right regular representation of $\Gamma$ on $\ell^2(\Gamma)$. For every bounded, complex-valued function $f$ on $\Gamma$, we denote by $m_f$ the associated multiplication operator on $\ell^2(\Gamma)$, and by $m(f)$ the operator $1 \otimes m_f$ on $L^2(M, \tau) \otimes \ell^2(\Gamma)$. Let $\theta$ be the action of $\Gamma$ on $M \bar{\otimes} \mathcal{B}(\ell^2(\Gamma))$ defined by $\theta_t = \alpha_t \otimes \mathrm{Ad}\rho_t$.

Theorem 2.3. *Let $\Gamma$ be an amenable group. If $M$ has the weak Haagerup property, then $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property and*

$$\Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma) = \Lambda_{WH}(M).$$

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Proof. Let $f$ be a finitely supported, nonnegative-valued function on $\Gamma$ such that $\sum_{t\in\Gamma} f(t)^2 = 1$. It follows from the proof of [6, Proposition 3.1] that the mapping $\Psi_f : M \bar{\otimes} \mathcal{B}(\ell^2(\Gamma)) \to M \bar{\rtimes}_\alpha \Gamma$ defined by

$$\Psi_f(x) = \sum_{t\in\Gamma} \theta_t(m(f)xm(f))$$

is well defined, normal, and unital completely positive. Now, suppose there is a normal completely bounded map $T$ on $M$ such that $\langle T(x), y\rangle_\tau = \langle x, T(y)\rangle_\tau$ for all $x, y \in M$. We define $\Phi_f$ on $M \bar{\rtimes}_\alpha \Gamma$ by

$$\Phi_f(x) = \Psi_f \circ T \otimes \mathrm{id}_{\mathcal{B}(\ell^2(\Gamma))}(x)$$

for all $x \in M \bar{\rtimes}_\alpha \Gamma$. Clearly, $\Phi_f$ is normal and $\|\Phi_f\|_{cb} \le \|T\|_{cb}$. Moreover, for $x \in M$ and $g \in \Gamma$, it follows from the proof of [6, Proposition 3.1] that

$$\Phi_f(xg) = \sum_{t\in S(f)} f(t)f(g^{-1}t)\alpha_t \circ T \circ \alpha_{t^{-1}}(x)g$$

where $S(f)$ is the support of $f$. Hence,

$$\begin{aligned}\langle \Phi_f(xg), yh\rangle_{\tau'} &= \tau'\Big(\alpha_{h^{-1}}(y^*)h^{-1} \sum_{t\in S(f)} f(t)f(g^{-1}t)\alpha_t \circ T \circ \alpha_{t^{-1}}(x)g\Big)\\ &= \begin{cases} 0 & \text{if } h \neq g,\\ \tau\Big(\sum_{t\in S(f)} f(t)f(g^{-1}t)y^*\alpha_t \circ T \circ \alpha_{t^{-1}}(x)\Big) & \text{if } h = g,\end{cases}\end{aligned}$$

and

$$\begin{aligned}\langle xg, \Phi_f(yh)\rangle_{\tau'} &= \tau'\Big(\Big(\sum_{t\in S(f)} f(t)f(h^{-1}t)\alpha_t \circ T \circ \alpha_{t^{-1}}(y)h\Big)^* xg\Big)\\ &= \begin{cases} 0 & \text{if } h \neq g,\\ \tau\Big(\sum_{t\in S(f)} f(t)f(g^{-1}t)\alpha_t \circ T \circ \alpha_{t^{-1}}(y)^* x\Big) & \text{if } h = g,\end{cases}\end{aligned}$$

for all $x, y \in M$ and $g, h \in \Gamma$. Also, since $\tau(y^*T(x)) = \tau(T(y)^*x)$,

$$\tau(\alpha_{t^{-1}}(y)^*T \circ \alpha_{t^{-1}}(x)) = \tau(T \circ \alpha_{t^{-1}}(y)^*\alpha_{t^{-1}}(x))$$

for all $x, y \in M$ and $t \in \Gamma$. Consequently, $\langle \Phi_f(xg), yh\rangle_{\tau'} = \langle xg, \Phi_f(yh)\rangle_{\tau'}$ for all $x, y \in M$ and $g, h \in \Gamma$. Thus, $\langle \Phi_f(x), y\rangle_{\tau'} = \langle x, \Phi_f(y)\rangle_{\tau'}$ for all $x, y \in M \bar{\rtimes}_\alpha \Gamma$. The rest of the proof is similar to that of [6, Proposition 3.1]. Moreover, $\Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma) \le \Lambda_{WH}(M)$. It follows from Theorem 2.2 that $\Lambda_{WH}(M \bar{\rtimes}_\alpha \Gamma) = \Lambda_{WH}(M)$. □

Finally, we give a condition under which the weak Haagerup property of $M$ and $\Gamma$ implies that of $M \bar{\rtimes}_\alpha \Gamma$.

Theorem 2.4. *If $\Gamma$ is a countable group, then the following statements are equivalent:*

(1) $\Gamma$ *has the weak Haagerup property and* $M$ *has the weak Haagerup property with the approximating maps* $T_i : M \to M$ *satisfying* $T_i \circ \alpha_t = \alpha_t \circ T_i$ *for all* $t \in \Gamma$.

(2) $M \bar{\rtimes}_\alpha \Gamma$ *has the weak Haagerup property and, for all* $t \in \Gamma$ *and* $x \in M$, *the approximating maps* $\Phi_i : M \bar{\rtimes}_\alpha \Gamma \to M \bar{\rtimes}_\alpha \Gamma$ *satisfy* $\mathcal{E} \circ \Phi_i \circ \alpha_t(x) = \alpha_t \circ \mathcal{E} \circ \Phi_i(x)$.

Proof. (1) ⇒ (2). Suppose that we are given a net $\{u_\gamma\}_{\gamma \in \mathfrak{I}}$ in $B_2(\Gamma) \cap C_0(\Gamma)$ witnessing the weak Haagerup property of $\Gamma$ and a net $\{T_i\}_{i \in I}$ of normal completely bounded maps witnessing the weak Haagerup property of $M$. Replacing $u_\gamma$ with $\frac{1}{2}(u_\gamma + \overline{u}_\gamma)$, we may assume that $u_\gamma$ is real. The covariance condition on the map $T_i$ implies that the map

$$\widetilde{T}_i : M \bar{\rtimes}_\alpha \Gamma \to M \bar{\rtimes}_\alpha \Gamma; \quad \sum_{t \in \Gamma} x_t t \mapsto \sum_{t \in \Gamma} T_i(x_t) t \quad \text{for } x_t \in M,\ t \in \Gamma,$$

can be identified with the restriction of $T_i \otimes \mathrm{id}_{\mathcal{B}(\ell^2(\Gamma))}$ on $M \bar{\otimes} \mathcal{B}(\ell^2(\Gamma))$ to $M \bar{\rtimes}_\alpha \Gamma$. Hence, $\widetilde{T}_i$ is completely bounded, normal and $\|\widetilde{T}_i\|_{cb} \le \|T_i\|_{cb}$. Define $S_\gamma : M \bar{\rtimes}_\alpha \Gamma \to M \bar{\rtimes}_\alpha \Gamma$ such that

$$S_\gamma\Big(\sum_{t \in \Gamma} x_t t\Big) = \sum_{t \in \Gamma} u_\gamma(t) x_t t$$

for all $\sum_{t \in \Gamma} x_t t \in C_c(\Gamma, M)$. It follows from [10, Proposition 4.1] that $S_\gamma$ is well defined, normal and completely bounded. Let $\Phi_{\gamma,i} = S_\gamma \circ \widetilde{T}_i$. Then $\Phi_{\gamma,i}$ is completely bounded and normal. Moreover,

$$\|\Phi_{\gamma,i}\|_{cb} = \|S_\gamma \circ \widetilde{T}_i\|_{cb} \le \Lambda_{WH}(\Gamma)\Lambda_{WH}(M).$$

For any $\sum_{t \in \Gamma} x_t t, \sum_{s \in \Gamma} y_s s \in C_c(\Gamma, M)$,

$$\begin{aligned}
\Big\langle \Phi_{\gamma,i}\Big(\sum_{t \in \Gamma} x_t t\Big), \sum_{s \in \Gamma} y_s s \Big\rangle_{\tau'} &= \tau'\Big(\Big(\sum_{s \in \Gamma} y_s s\Big)^* \sum_{t \in \Gamma} u_\gamma(t) T_i(x_t) t\Big) \\
&= \tau\Big(\sum_{s \in \Gamma} y_s^* u_\gamma(s) T_i(x_s)\Big) \\
&= \tau\Big(\sum_{s \in \Gamma} T_i(y_s)^* u_\gamma(s) x_s\Big) \\
&= \tau'\Big(\sum_{s \in \Gamma} \alpha_{s^{-1}}(u_\gamma(s) T_i(y_s)^*) s^{-1} \sum_{t \in \Gamma} x_t t\Big) \\
&= \Big\langle \sum_{t \in \Gamma} x_t t, \sum_{s \in \Gamma} u_\gamma(s) T_i(y_s) s \Big\rangle_{\tau'} \\
&= \Big\langle \sum_{t \in \Gamma} x_t t, \Phi_{\gamma,i}\Big(\sum_{s \in \Gamma} y_s s\Big) \Big\rangle_{\tau'}.
\end{aligned}$$

For any $\varepsilon > 0$, since $u_\gamma \in C_0(\Gamma)$, there exists a finite subset $F_\gamma \subseteq \Gamma$ such that $u_\gamma(t) < \varepsilon$ for all $t \in \Gamma \setminus F_\gamma$. Moreover, there exists a finite rank map $Q_i : M \to M$ such that $Q_i$ is bounded with respect to the norm $\|\cdot\|_{2,\tau}$ and such that

$$\|T_i(x) - Q_i(x)\|_{2,\tau} \le \varepsilon \|x\|_{2,\tau}$$

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for all $x \in M$. We define a finite rank linear map such that

$$Q_{\gamma,i}\Big(\sum_{t\in\Gamma} x_t t\Big) = \sum_{t\in F_\gamma} u_\gamma(t) Q_i(x_t) t$$

for all $\sum_{t\in\Gamma} x_t t \in C_c(\Gamma, M)$. Then

$$\Big\| Q_{\gamma,i}\Big(\sum_{t\in\Gamma} x_t t\Big)\Big\|_{2,\tau'} = \Big\|\sum_{t\in F_\gamma} u_\gamma(t) Q_i(x_t) t\Big\|_{2,\tau'} \le \Lambda_{WH}(\Gamma)\|Q_i\| \Big\|\sum_{t\in\Gamma} x_t t\Big\|_{2,\tau'}$$

for all $\sum_{t\in\Gamma} x_t t \in C_c(\Gamma, M)$. Moreover,

$$\begin{aligned}
&\Big\|\Phi_{\gamma,i}\Big(\sum_{t\in\Gamma} x_t t\Big) - Q_{\gamma,i}\Big(\sum_{t\in\Gamma} x_t t\Big)\Big\|_{2,\tau'} \\
&\quad \le \Big\|\sum_{t\in\Gamma\setminus F_\gamma} u_\gamma(t) T_i(x_t) t\Big\|_{2,\tau'} + \Big\|\sum_{t\in F_\gamma} u_\gamma(t) T_i(x_t) t - \sum_{t\in F_\gamma} u_\gamma(t) Q_i(x_t) t\Big\|_{2,\tau'} \\
&\quad \le \varepsilon(\Lambda_{WH}(M) + \Lambda_{WH}(\Gamma))\Big\|\sum_{t\in\Gamma} x_t t\Big\|_{2,\tau'}
\end{aligned}$$

for all $\sum_{t\in\Gamma} x_t t \in C_c(\Gamma, M)$. On the other hand, it is easy to see that

$$\Big\|\Phi_{\gamma,i}\Big(\sum_{t\in\Gamma} x_t t\Big) - \sum_{t\in\Gamma} x_t t\Big\|_{2,\tau'} = \Big\|\sum_{t\in\Gamma} u_\gamma(t) T_i(x_t) t - \sum_{t\in\Gamma} x_t t\Big\|_{2,\tau'} \to 0$$

for all $\sum_{t\in\Gamma} x_t t \in C_c(\Gamma, M)$. In a word, $M \bar{\rtimes}_\alpha \Gamma$ has the weak Haagerup property. For any $x \in M$ and for all $t \in \Gamma$,

$$\mathcal{E} \circ \Phi_{\gamma,i} \circ \alpha_t(x) = u_\gamma(e) T_i(\alpha_t(x)) = u_\gamma(e)\alpha_t(T_i(x)) = \alpha_t \circ \mathcal{E} \circ \Phi_{\gamma,i}(x).$$

(2) ⇒ (1). This follows from Theorem 2.2. □

REMARK 2.5. The commutation conditions in Theorem 2.4 seem too strong. It is interesting to investigate whether we can replace the commutation conditions with some sort of asymptotic commutation condition. However, without the help of the commutation conditions, it seems rather difficult to construct completely bounded maps on $M \bar{\rtimes}_\alpha \Gamma$. So it is not as simple as requiring the approximating maps $T_i$ to satisfy $T_i \circ \alpha_t = \alpha_t \circ T_i$ for all $t \in \Gamma$.

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