LOWER RADICALS IN ASSOCIATIVE RINGS

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Introduction. Given a homomorphically closed class of (not necessarily associative) rings $\mathscr{M}$, the lower radical property determined by $\mathscr{M}$ is the least radical property for which all rings in $\mathscr{M}$ are radical. Recently (**7**) a process of constructing the lower radical property from a class $\mathscr{M}$ of associative rings has been given which terminates after a countable number of steps. In this process, an ascending chain of classes

$$\mathscr{M} = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \ldots \subseteq \mathscr{M}_{\omega_0}$$

is obtained and the property of being a ring in the class $\mathscr{M}_{\omega_0}$ is the lower radical property determined by $\mathscr{M}$. In Theorem 1 we give another characterization of the rings in the class $\mathscr{M}_\lambda$, $\lambda \in \{1, 2, \ldots, \omega_0\}$, and a procedure for constructing the lower radical determined by $\mathscr{M}$ in an arbitrary associative ring is given. This procedure generalizes Baer's construction of the lower radical determined by the class of all nilpotent rings.

In (**7**) it was shown that for a hereditary class of rings $\mathscr{M}$, containing all zero rings, the class $\mathscr{M}_{\omega_0} = \mathscr{M}_1$. In § 4 we generalize this result and show also that for any hereditary class $\mathscr{M}$ the class $\mathscr{M}_{\omega_0} = \mathscr{M}_2$. In addition, the class $\mathscr{M}_2$ is hereditary.

The notation in the present paper differs from that in (**7**) in that the homomorphically closed class of rings from which we start out is denoted here by $\mathscr{M}_0$ and not $\mathscr{M}_1$, and in general, the class $\mathscr{M}_{n+1}$ of (**7**) is $\mathscr{M}_n$ here.

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1. The construction. In this paper we are mainly concerned with associative rings. However, in this section, we proceed as far as possible without assuming associativity.

Given any ring R we say that a subring S is an *n-accessible subring* of R if there is a chain of subrings of R

$$S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n = R$$

with each S_i an ideal in S_{i+1} for $i = 0, 1, \ldots, n-1$. The 1-accessible subrings of R are the ideals of R. A subring S of R is said to be a *σ-accessible* (or just *accessible*) *subring* if it is n-accessible for some integer n.

The mark σ has been introduced here as a notational convenience to allow us to combine results on both n-accessible and accessible subrings. The set

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whose elements are the natural numbers and the mark σ will be written $\{1, 2, \ldots ; \sigma\}$.

If $\mathscr{M}$ is a class of rings, then a ring R is called an *$\mathscr{M}$-ring* if R belongs to the class $\mathscr{M}$. An *$\mathscr{M}$-ideal* (*$\mathscr{M}$-subring*) of an arbitrary ring R is an ideal (subring) of R and an $\mathscr{M}$-ring.

Given a homomorphically closed class of rings $\mathscr{M}$, we define, for each $\lambda \in \{1, 2, \ldots ; \sigma\}$ and for any ring R, a sequence of ideals as follows:

(a) $M_{\lambda,0} = 0$;

(b) If α is not a limit ordinal, $\alpha = \beta + 1$ say, then $M_{\lambda,\alpha}$ is the ideal of R such that $M_{\lambda,\alpha}/M_{\lambda,\beta}$ is the ideal of $R/M_{\lambda,\beta}$ generated by all the λ-accessible $\mathscr{M}$-subrings of $R/M_{\lambda,\beta}$,

(c) If α is a limit ordinal, then

$$M_{\lambda,\alpha} = \bigcup_{\beta<\alpha} M_{\lambda,\beta}.$$

We denote by M_λ, the ideal $M_{\lambda,\gamma}$, where γ is the minimal ordinal for which $M_{\lambda,\gamma} = M_{\lambda,\gamma+1}$. Thus, R/M_λ contains no non-zero λ-accessible $\mathscr{M}$-subrings. Where it is necessary to avoid confusion, we shall denote the ideal $M_{\lambda,\alpha}$ in a ring R by $M_{\lambda,\alpha}(R)$. We shall now establish some elementary properties of the ideals $M_{\lambda,\alpha}$.

LEMMA 1. *If λ is a positive integer and if μ is either a positive integer such that $\lambda \leqq \mu$ or if $\mu = \sigma$, then $M_{\lambda,\alpha} \subseteq M_{\mu,\alpha}$ for every ordinal α.*

Proof. The result is immediate for $\alpha = 0$ and $\alpha = 1$, and we complete the proof by induction on α.

Suppose that $\alpha = \beta + 1$ and $M_{\lambda,\beta} \subseteq M_{\mu,\beta}$; then, consider the quotient ring $\bar{R} = R/M_{\lambda,\beta}$. Let $S/M_{\lambda,\beta}$ be a λ-accessible $\mathscr{M}$-subring of $\bar{R}$. Since

$$\frac{S + M_{\mu,\beta}}{M_{\mu,\beta}} \cong \frac{S}{S \cap M_{\mu,\beta}} \cong \frac{S/M_{\lambda,\beta}}{[S \cap M_{\mu,\beta}]/M_{\lambda,\beta}},$$

we have that $[S + M_{\mu,\beta}]/M_{\mu,\beta}$ is an $\mathscr{M}$-subring of $R/M_{\mu,\beta}$. Since $S/M_{\lambda,\beta}$ is λ-accessible in $\bar{R}$, the ring $[S + M_{\mu,\beta}]/M_{\mu,\beta}$ is λ-accessible in $R/M_{\mu,\beta}$. However, λ-accessible subrings are μ-accessible subrings; therefore $S + M_{\mu,\beta} \subseteq M_{\mu,\alpha}$. But $S/M_{\lambda,\beta}$ is any λ-accessible $\mathscr{M}$-subring of $\bar{R}$ and $M_{\mu,\alpha}$ is an ideal of R, thus $M_{\mu,\alpha} \supseteq M_{\lambda,\alpha}$.

If α is a limit ordinal and the result is proved for all $\beta < \alpha$, then from definition (c), $M_{\lambda,\alpha} \subseteq M_{\mu,\alpha}$.

COROLLARY 1. *In any ring R we have the chain of ideals*

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_\sigma.$$

The following two lemmas are also proved by induction arguments using the isomorphism theorems together with the fact that $\mathscr{M}$ is a homomorphically closed class of rings. We omit the proofs.

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LEMMA 2. *Let* $\lambda \in \{1, 2, \ldots ; \sigma\}$. *If* I *is an ideal in a ring* R, *then, for every ordinal* α,

$$M_{\lambda,\alpha}(R/I) \supseteq [M_{\lambda,\alpha}(R) + I]/I.$$

LEMMA 3. *Let* $\lambda \in \{1, 2, \ldots\}$. *If* I *is an ideal in a ring* R, *then, for every ordinal* α,

$$M_{\lambda,\alpha}(I) \subseteq M_{\lambda+1,\alpha}(R) \cap I.$$

Also, $M_{\sigma,\alpha}(I) \subseteq M_{\sigma,\alpha}(R) \cap I$.

For each $\lambda \in \{1, 2, \ldots ; \sigma\}$ we define a class of rings $\mathscr{M}_\lambda'$ by saying that $R \in \mathscr{M}_\lambda'$ if and only if $R = M_\lambda(R)$. From Corollary 1,

$$\mathscr{M} \subseteq \mathscr{M}_1' \subseteq \mathscr{M}_2' \subseteq \ldots \subseteq \mathscr{M}_\sigma'.$$

From Lemma 2, each class $\mathscr{M}_\lambda'$ is homomorphically closed. If I is an ideal in a ring R and $M_\sigma(I) = I$, then, by Lemma 3, $I \subseteq M_\sigma(R)$. Finally, as we have already noted, the quotient ring $R/M_\sigma(R)$ contains no non-zero accessible $\mathscr{M}$-subrings. Hence, the ring $R/M_\sigma(R)$ contains no non-zero $\mathscr{M}_\sigma'$-ideals. Therefore, the property of being an $\mathscr{M}_\sigma'$-ring would be a radical property if, for every ring R, $M_\sigma(R) \in \mathscr{M}_\sigma'$. This we are unable to prove in general. For associative rings we have an indirect proof which is the subject of the next section.

2. The lower radical determined by $\mathscr{M}$. In the remainder of this paper, all rings are assumed associative.

Given a homomorphically closed class of rings $\mathscr{M}$, we define $\mathscr{M}_1$ to be the class of rings R such that every non-zero homomorphic image of R contains a non-zero $\mathscr{M}$-ideal. It is clear that the class $\mathscr{M}_1$ is homomorphically closed. An ascending chain of classes

$$\mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \ldots$$

is obtained by setting $\mathscr{M} = \mathscr{M}_0$ and defining $\mathscr{M}_{n+1} = (\mathscr{M}_n)_1$, where $n = 0, 1, \ldots$. It is known (**7**) that if $\mathscr{N} = \bigcup_{n=0}^{\infty} \mathscr{M}_n$, then $\mathscr{N}_1 = \mathscr{N}_2 = \ldots$. The class $\mathscr{N}_1$ determines the least radical property for which all $\mathscr{M}$-rings are radical. In (**7**), the class $\mathscr{N}_1$ is denoted by $\mathscr{M}_{\omega_0}$, but here, for convenience, it is denoted by $\mathscr{M}_\sigma$.

Also, the class $\mathscr{M}_n$ of the present paper is denoted by $\mathscr{M}_{n+1}$ in (**7**). The reason for these changes is that in Theorem 1 we show that a ring R is in the class $\mathscr{M}_\lambda$, $\lambda \in \{1, 2, \ldots ; \sigma\}$ if and only if $M_\lambda(R) = R$.

The following Lemma 4 is a restatement of (**7**, Lemma 2) and the proof is omitted. Our Lemma 5 has been obtained independently by Heinicke (**5**), and therefore we omit a proof here.

LEMMA 4. *If* S *is an* n*-accessible* $\mathscr{M}$*-subring of a ring* R, *then* S', *the ideal of* R *generated by* S, *is an* $\mathscr{M}_{n-1}$*-ring.*

LEMMA 5. *A ring R is an $\mathscr{M}_\lambda$-ring, where $\lambda \in \{1, 2, \ldots ; \sigma\}$, if and only if every non-zero homomorphic image of R contains a non-zero λ-accessible $\mathscr{M}$-subring.*

THEOREM 1. *Let $\lambda \in \{1, 2, \ldots ; \sigma\}$. A ring R is an $\mathscr{M}_\lambda$-ring if and only if $M_\lambda(R) = R$.*

Proof. If R belongs to $\mathscr{M}_\lambda$ and $M_\lambda \neq R$, then by Lemma 5 the ring R/M_λ contains a non-zero λ-accessible $\mathscr{M}$-subring. This contradicts the definition of M_λ. Hence, $M_\lambda = R$.

Now suppose that $R = M_\lambda$ and I is an ideal of R, properly contained in R. Then there is an ordinal α such that $M_{\lambda,\alpha} \subseteq I$ but $M_{\lambda,\alpha+1} \not\subseteq I$. Hence, there is a λ-accessible subring S of R such that $S/M_{\lambda,\alpha}$ is a non-zero $\mathscr{M}$-ring and $S \not\subseteq I$. Now $[S + I]/I$ is a homomorphic image of $S/M_{\lambda,\alpha}$ and $\mathscr{M}$ is homomorphically closed; therefore, $[S + I]/I$ is a non-zero λ-accessible $\mathscr{M}$-subring of R/I. Thus, every non-zero homomorphic image of R contains a non-zero λ-accessible $\mathscr{M}$-subring, and hence, by Lemma 5, $R \in \mathscr{M}_\lambda$.

It follows from this theorem that for each $\lambda \in \{1, 2, \ldots ; \sigma\}$ the class $\mathscr{M}_\lambda'$ discussed in the first section of this paper coincides with the class $\mathscr{M}_\lambda$ discussed here. From now on we shall denote this class by $\mathscr{M}_\lambda$. Now the property of being an $\mathscr{M}_\sigma$-ring is the lower radical properly determined by $\mathscr{M}$. We use this fact to prove the following theorem.

THEOREM 2. *The ideal M_σ in a ring R is the $\mathscr{M}_\sigma$-radical of R.*

Proof. As we noted at the end of § 1, this theorem is proved once we have shown that $M_\sigma \in \mathscr{M}_\sigma$.

Let K denote the $\mathscr{M}_\sigma$-radical of M_σ and suppose that $K \neq M_\sigma$. Since $K \neq M_\sigma$, there is an ordinal α such that $M_{\sigma,\alpha}(R) \subseteq K$ but $M_{\sigma,\alpha+1}(R) \not\subseteq K$. Let $S/M_{\sigma,\alpha}(R)$ be an accessible $\mathscr{M}$-subring of $R/M_{\sigma,\alpha}(R)$. Then $T = [S + K]/K$ is an accessible $\mathscr{M}$-subring of M_σ/K. By Lemma 4 the ideal T' of M_σ/K generated by T is an $\mathscr{M}_\sigma$-ring. However, M_σ/K is $\mathscr{M}_\sigma$-semisimple, thus T' is the zero ideal. This implies that $S \subseteq K$. Therefore, every subring S of R such that $S/M_{\sigma,\alpha}(R)$ is an accessible $\mathscr{M}$-subring of $R/M_{\sigma,\alpha}(R)$ is contained in K. Now the ring K is in fact an ideal in R (**1**, Theorem 1). Hence, $M_{\sigma,\alpha+1}(R) \subseteq K$. This is a contradiction; therefore, $K = M_\sigma$ and $M_\sigma \in \mathscr{M}_\sigma$.

Frequently, the sequence of classes

$$\mathscr{M} = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \mathscr{M}_2 \subseteq \ldots \subseteq \mathscr{M}_\sigma$$

terminates after a finite number of steps.

If $\mathscr{M}$ is the class of all nilpotent rings, then $\mathscr{M}_1 = \mathscr{M}_\sigma$ and the ideal $M_1(R)$ is the Baer lower radical of the ring R. In (**5**), Heinicke gives an example of a class $\mathscr{M}$ for which this chain is strictly increasing.

In the next lemma we relate the termination of the sequence of classes to the termination of the sequence of ideals

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_\sigma$$

in a ring R.

LEMMA 6. *If d is a positive integer such that $\mathscr{M}_d = \mathscr{M}_\sigma$, then, in any ring R, $M_{d+1} = M_\sigma$.*

Proof. We know that $M_\sigma \in \mathscr{M}_\sigma$, and therefore if $\mathscr{M}_\sigma = \mathscr{M}_d$, we have that $M_\sigma \in \mathscr{M}_d$.

The inclusion $M_{d+1} \subseteq M_\sigma$ is given by Corollary 1. Suppose that the inclusion is proper. Then M_σ/M_{d+1} is a non-zero $\mathscr{M}_d$-ring. Hence, M_σ/M_{d+1} contains non-zero d-accessible $\mathscr{M}$-subrings, which implies that R/M_{d+1} contains non-zero $(d+1)$-accessible $\mathscr{M}$-subrings. This is a contradiction, therefore $M_{d+1} = M_\sigma$.

Remark 1. If $M_{d+1} = M_\sigma$ in every ring R, then using Theorem 1 we see that $\mathscr{M}_\sigma = \mathscr{M}_{d+1}$. However, as we shall now show, the classes $\mathscr{M}_\sigma$ and $\mathscr{M}_d$ can be distinct.

Let $\mathscr{M}$ consist of all homomorphic images of the zero ring on the infinite cyclic group. If R is a ring with $M_2 \neq M_\sigma$, then $\bar{R} = R/M_2$ is a ring which contains no non-zero 2-accessible $\mathscr{M}$-subrings but does contain non-zero accessible $\mathscr{M}$-subrings. If S is one such subring, then S', the ideal of $\bar{R}$ generated by S, is nilpotent. This follows from a lemma of Andrunakievič (**2**, Lemma 4). Hence, some power of S' is a zero ring and also an ideal in $\bar{R}$; that is, $\bar{R}$ contains an ideal N with $N^2 = 0$. However, N will contain as an ideal a non-zero $\mathscr{M}$-ring which will then be a non-zero 2-accessible $\mathscr{M}$-subring of $\bar{R}$. This is a contradiction; therefore $M_2 = M_\sigma$ in every ring R. However, the classes $\mathscr{M}_1$ and $\mathscr{M}_\sigma$ $(= \mathscr{M}_2)$ are distinct; see (**7**, p. 421).

Remark 2. The result in Lemma 6 is best possible in the sense that there are classes $\mathscr{M}$ such that $\mathscr{M}_1 = \mathscr{M}_\sigma$ but for some ring R, $M_1 \neq M_\sigma$.

Let $\mathscr{M}$ be the class consisting of the ring $\{0\}$ and the zero ring on the group $Z(p^\infty)$, where p is a prime. Then $\mathscr{M}_1 = \mathscr{M}_\sigma =$ the class of all rings whose additive groups are divisible p-groups. For if $R \in \mathscr{M}_1$ and D is the maximal divisible p-subgroup of R^+, the additive group of R, then D is an ideal in R and $R/D \in \mathscr{M}_1$. If R/D is non-zero, then it contains a non-zero $\mathscr{M}$-ideal, and hence $(R/D)^+$ has a non-zero divisible p-subgroup. This contradicts the maximality of D, therefore $R/D = \{0\}$ and R^+ is a divisible p-group. All such rings are zero rings by a theorem of Szele (**4**, Theorem 71.1).

Let $S = A \oplus B$, where A is the ring given by

$$\{a_0, a_1, \ldots : pa_0 = 0, pa_{i+1} = a_i, a_i^2 = 0, i = 0, 1, \ldots\}$$

and B is the ring given by

$$\{b_0, b_1, \ldots : pb_0 = 0, pb_{i+1} = b_i, b_i^2 = 0, i = 0, 1, \ldots\},$$

where p is a prime. Thus, A and B are isomorphic to the zero ring on the group $Z(p^\infty)$. Let θ be the endomorphism of S^+ defined by

$$a_i\theta = b_i, \quad b_i\theta = a_{i-1} \qquad \text{for } i = 1, 2, \ldots,$$

and

$$a_0\theta = b_0, \qquad b_0\theta = 0.$$

Denote by K the ring of endomorphisms of S^+ generated by θ and I, the identity endomorphism. Every endomorphism in K is uniquely expressible in the form $\alpha I + \beta\theta$, where α and β are arbitrary integers.

Consider the set

$$R = \{(s, \phi): s \in S, \phi \in K\}.$$

The set R can be given a ring structure by defining on R a componentwise addition and multiplication by

$$(s_1, \phi_1)(s_2, \phi_2) = (s_1\phi_2 + s_2\phi_1, \phi_1\phi_2), \qquad s_1, s_2 \in S, \phi_1, \phi_2 \in K.$$

The ring R is commutative and if the subring $\{(s, 0): s \in S\}$ is identified with S, then S is an ideal of R such that $R/S \cong K$.

We assert that the ring R contains no non-zero $\mathscr{M}$-ideals so that $M_1 = 0$. Suppose that C is an $\mathscr{M}$-ideal of R and C is given by

$$\{c_0, c_1, \ldots : pc_0 = 0, pc_{i+1} = c_i, c_i^2 = 0, i = 0, 1, \ldots\}.$$

It is easy to see from the structure of the additive group of R that $C \subseteq S$. Since $pc_0 = 0$ and $c_0 \in S$, we have that

$$c_0 = \gamma a_0 + \delta b_0,$$

where γ and δ are integers, not both zero, such that $0 \leqq \gamma, \delta < p$. Since C is an ideal in R,

$$c_0\theta = \gamma b_0 \in C.$$

Thus, if $\gamma \neq 0$, $b_0 \in C$. On the other hand, if $\gamma = 0$ we have that $\delta \neq 0$ and $c_0 = \delta b_0$. Again $b_0 \in C$, therefore we can choose $c_0 = b_0$ in either case.

Now, $p^2 c_1 = 0$ and $c_1 \in S$, thus

$$c_1 = \gamma_1 a_1 + \delta_1 b_1,$$

where γ_1 and δ_1 are integers, not both zero, such that $0 \leqq \gamma_1, \delta_1 < p^2$. However,

$$b_0 = c_0 = pc_1 = \gamma_1 a_0 + \delta_1 b_0;$$

therefore $\gamma_1 = 0$, and $\delta_1 = 1$. And then

$$c_1\theta = b_1\theta = a_0 \in C.$$

Thus, C contains both a_0 and b_0. This implies that C contains $p^2 - 1$ non-zero elements of additive order p, which is a contradiction. Therefore, R contains no $\mathscr{M}$-ideals and $M_1 = 0$.

It is clear that $M_2 \supseteq S$ since S is an ideal in R and both A and B are ideals in S. But then the additive group of R/S contains no divisible subgroups; thus $M_2 = S$.

Hence, $\mathscr{M}$ is a class of rings with $\mathscr{M}_1 = \mathscr{M}_\sigma$ but R is a ring with $M_1 \neq M_\sigma$.

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3. The degree of a class $\mathscr{M}$. All classes discussed in this section are homomorphically closed. If $\mathscr{M}$ is a given class of rings (not a radical class) such that, for some integer $d \geqq 1, \mathscr{M}_d = \mathscr{M}_{d+1}$, but $\mathscr{M}_{d-1} \neq \mathscr{M}_d$, then we shall call d the *degree* of $\mathscr{M}$. If $\mathscr{M}$ is a radical class, then we shall say that $\mathscr{M}$ has degree zero.

LEMMA 7. *All non-radical classes of idempotent rings are of degree one.*

Proof. Let $\mathscr{M}$ be a non-radical class of idempotent rings and suppose that $R \in \mathscr{M}_2 \backslash \mathscr{M}_1$. Then the non-zero ring $\bar{R} = R/M_1$ contains no non-zero $\mathscr{M}$-ideals but does contain non-zero accessible $\mathscr{M}$-subrings. Let S be one such subring and S' the ideal of $\bar{R}$ generated by S. Again from a lemma of Andrunakievič (**2**, Lemma 4), some power of S', S'^k say, is contained in S. Hence,

$$S^k \subseteq S'^k \subseteq S = S^k.$$

Therefore, S is an ideal in $\bar{R}$ which is a contradiction. Thus, $\mathscr{M}$ is of degree one.

THEOREM 3. *Let I be an index set. Let $\mathscr{L}_i$, $i \in I$, be classes of degree d_i and suppose that* $\sup d_i = d - 1$, *where d is an integer. If the class $\mathscr{M}$ contains each $\mathscr{L}_i$ and if each $\mathscr{M}$-ring is an extension of an $\mathscr{L}_i$-ring by an $\mathscr{L}_j$-ring for some $i, j \in I$, then the degree of $\mathscr{M}$ is at most d.*

Proof. Let R be an element of $\mathscr{M}_\sigma$ and suppose that $M_d \neq R$. Then $\bar{R} = R/M_d$ contains a non-zero accessible $\mathscr{M}$-subring S/M_d but no non-zero d-accessible $\mathscr{M}$-subrings. The ring S/M_d contains an ideal J/M_d such that $J/M_d \in \mathscr{L}_i$ and $S/J \in \mathscr{L}_j$ for some $i, j \in I$.

If J/M_d is non-zero, then $\bar{R}$ contains non-zero accessible $\mathscr{L}_i$-subrings, and therefore the $(\mathscr{L}_i)_\sigma$-radical of $\bar{R}$ is non-zero. Then, by Lemma 6, the ring $\bar{R}$ contains non-zero $(d_i + 1)$-accessible $\mathscr{L}_i$-subrings. Since $d_i + 1 \leqq d$ and $\mathscr{L}_i \subseteq \mathscr{M}$, we have that $\bar{R}$ contains non-zero d-accessible $\mathscr{M}$-subrings. This is a contradiction; hence $J = M_d$.

We now have that $S/M_d \in \mathscr{L}_j$ which means that the $(\mathscr{L}_j)_\sigma$-radical of $\bar{R}$ is non-zero. As in the preceding paragraph, this leads to a contradiction. Therefore, $R = M_d$ and, by Theorem 1, $R \in \mathscr{M}_d$. Hence $\mathscr{M}_\sigma = \mathscr{M}_d$.

COROLLARY 2. *If the classes $\mathscr{L}_i$ are as in Theorem* 3 *and $\mathscr{M} = \bigcup \mathscr{L}_i$, then the degree of $\mathscr{M}$ is at most d.*

Remark 3. The result in Theorem 3 is again best possible.

Let $\mathscr{L}_{i_1}$ be the class consisting of the ring $\{0\}$ and the zero ring on the group $Z(p^\infty)$, p a prime, and let $\mathscr{L}_{i_2}$ be the class of homomorphic images of the ring K of Remark 2. The class $\mathscr{L}_{i_1}$ is of degree one. The ring K has an identity element, thus every ring in $\mathscr{L}_{i_2}$ is idempotent. By Lemma 7 the class $\mathscr{L}_{i_2}$ has degree at most one. Let $\mathscr{M} = \mathscr{L}_{i_1} \cup \mathscr{L}_{i_2}$ and let R be the ring constructed in Remark 2. We have already seen that R contains no $\mathscr{L}_{i_1}$-ideals ($\neq 0$). If R contains a non-zero $\mathscr{L}_{i_2}$-ideal J, then J contains a non-zero idempotent element. However, the only non-zero idempotent element of R is the identity element, therefore $J = R$. Since $R \notin \mathscr{L}_{i_2}$, this is a contradiction, and thus $M_1 = 0$. Now $M_2 \supseteq (L_{i_1})_2 = S$ and $R/S \in \mathscr{L}_{i_2}$, hence $M_2 = R$. Therefore $R \in \mathscr{M}_2 \backslash \mathscr{M}_1$ and $\mathscr{M}$ is of degree two.

4. Hereditary lower radicals. Throughout this section, $\mathscr{M}$ will denote a homomorphically closed class of rings. The class $\mathscr{M}$ is said to be *hereditary* if every ideal of an $\mathscr{M}$-ring is an $\mathscr{M}$-ring. A class $\mathscr{M}$ is called a $P_1(P_2)$*-class* if given any ring R and any $1(2)$-accessible $\mathscr{M}$-subring S of R, then every principal ideal $(x)_R$ of R contained in S is an $\mathscr{M}$-ring. It is clear that every P_2-class is a P_1-class, but I do not know of a P_1-class which is not a P_2-class. The P_2-classes correspond to the locally-hereditary preradicals of Michler (**6**) and if $\mathscr{M}$ is a P_1-class, then the property of being an $\mathscr{M}$-ideal is a property which is hereditary for principal ideals in the sense of (**6**, p. 20). Every hereditary class is a P_2-class but not conversely. The class of all weakly regular rings (that is, rings in which every right ideal is idempotent) form a P_2-class which is not hereditary (**6**, Remark 2.12).

Our main aim in this section is to show that a P_1-class of rings is of degree at most two and that such a class determines a hereditary lower radical.

THEOREM 4. *If $\mathscr{M}$ is a P_1-class of rings, then $\mathscr{M}_1$ is a hereditary class.*

Proof. Let $R \in \mathscr{M}_1$, let I be an ideal of R, and I_1 an ideal of I, properly contained in I. Since $R \in \mathscr{M}_1$, there is a non-zero $\mathscr{M}$-ideal J in R.

If $J \cap I \not\subseteq I_1$, then $[J \cap I + I_1]/I_1$ is a non-zero ideal of I/I_1. Let $x \in J \cap I \backslash I_1$. Then $(x)_R \subseteq J$, and therefore, since $\mathscr{M}$ is a P_1-class of rings, $(x)_R \in \mathscr{M}$. Hence, the image of $(x)_R$, under the natural homomorphism from $J \cap I$ onto $[J \cap I + I_1]/I_1$, is a non-zero $\mathscr{M}$-ideal of I/I_1.

If $J \cap I \subseteq I_1$, then $\mathscr{J}$, the set of ideals of R whose intersection with I is contained in I_1, is non-empty. By Zorn's lemma, the set $\mathscr{J}$ has maximal elements. If N is a maximal element of $\mathscr{J}$, then, since $I_1 \subset I$, the ideal N is properly contained in R. Hence, $\bar{R} = R/N$ is a non-zero homomorphic image of R, and thus contains a non-zero $\mathscr{M}$-ideal, K/N say. Since $I \cap N \subseteq I_1 \cap K$, there is a natural homomorphism from $[I \cap K + N]/N$ onto $[I \cap K + I_1]/I_1$. By the maximality of N, the quotient $[I \cap K + I_1]/I_1$ is non-zero. Let $y \in I \cap K \backslash I_1$ and put $\bar{y} = y + N$. Then $(\bar{y})_{\bar{R}} \subseteq K/N$; therefore, since $\mathscr{M}$ is a P_1-class of rings, $(\bar{y})_{\bar{R}} \in \mathscr{M}$. Hence, the image of $(\bar{y})_{\bar{R}}$ in

$$[I \cap K + I_1]/I_1$$

is a non-zero $\mathscr{M}$-ideal of I/I_1.

Therefore, every non-zero homomorphic image of I contains a non-zero $\mathscr{M}$-ideal, hence I is an $\mathscr{M}_1$-ring. Thus, $\mathscr{M}_1$ is a hereditary class.

Remark 4. Any P_1-class $\mathscr{M}$, which is also a radical class, is a hereditary class. For if $\mathscr{M}$ is a radical class we have that $\mathscr{M} = \mathscr{M}_1$, and therefore, by Theorem 4, $\mathscr{M}$ is hereditary.

The following three results generalize (**6**, Lemmas 2.13 and 2.14 and Theorem 2.20(a)). The results are stated here in the context of homomorphically closed classes of rings rather than that of preradicals.

Let $\mathscr{X}$ be any class of rings. A class of rings $\mathscr{M}$ is said to be *inductive* over $\mathscr{X}$ if for any ring R and any $\mathscr{X}$-ideal X of R, the sum of all the $\mathscr{M}$-ideals of R contained in X is an $\mathscr{M}$-ideal of R. If $\mathscr{M} \supseteq \mathscr{X}$, then $\mathscr{M}$ is inductive over $\mathscr{X}$.

LEMMA 8. *Let $\mathscr{M}$ be inductive over $\mathscr{Z}$, the class of all zero rings. If A is a $\mathscr{Z}$-ideal in a ring R, then the sum of all the $\mathscr{M}$-ideals of A is an $\mathscr{M}$-ideal of R.*

Proof. Let I be an $\mathscr{M}$-ideal of A and $r \in R$. The map $\rho: x \rightarrow xr$, $x \in I$, is a homomorphism from I onto Ir since $Ir \subseteq A$ and $A^2 = 0$. For these same reasons, Ir is an ideal in A and therefore, since $\mathscr{M}$ is homomorphically closed, Ir is an $\mathscr{M}$-ideal of A. Similarly, sI, $s \in R$, is an $\mathscr{M}$-ideal of A. Thus, if S is the sum of all the $\mathscr{M}$-ideals of A, S is an ideal of R.

In the ring A, S is a $\mathscr{Z}$-ideal and, since $\mathscr{M}$ is inductive over $\mathscr{Z}$, the sum of all the $\mathscr{M}$-ideals of A contained in S is an $\mathscr{M}$-ideal of A. However, S is the sum of all the $\mathscr{M}$-ideals of A, thus S is an $\mathscr{M}$-ideal of A and, in particular, an $\mathscr{M}$-ring. We have proved above that S is an ideal of R, hence this shows that S is an $\mathscr{M}$-ideal of R.

COROLLARY 3. *Let $\mathscr{M}$ be any class inductive over the class $\mathscr{Z}$. If A is a $\mathscr{Z}$-ideal in a ring R and A contains no non-zero $\mathscr{M}$-ideal of R, then A contains no non-zero $\mathscr{M}$-ideals.*

THEOREM 5. *If $\mathscr{M}$ is a P_2-class of rings, inductive over $\mathscr{Z}$, then $\mathscr{M}$ is of degree at most one and $\mathscr{M}_1$ is hereditary.*

Proof. That $\mathscr{M}_1$ is hereditary follows from Theorem 4 and the fact that a P_2-class is a P_1-class.

If $\mathscr{M}$ is of degree greater than one, then there is a ring $R \in \mathscr{M}_2 \backslash \mathscr{M}_1$. By Theorem 1, the ideal M_1 is properly contained in R. Hence, $\bar{R} = R/M_1$ is a non-zero $\mathscr{M}_2$-ring containing no non-zero $\mathscr{M}$-ideals.

Since $\bar{R}$ is a non-zero $\mathscr{M}_2$-ring, it has a non-zero 2-accessible $\mathscr{M}$-subring I. Thus, I is an ideal of an ideal I_1 of $\bar{R}$. Denote the ideal of $\bar{R}$ that I generates by I'. Then $I' \subseteq I_1$ and by Andrunakievič's lemma (**2**, Lemma 4), $I'^3 \subseteq I$.

If $I'^3 \neq 0$, then we have a non-zero element x in I'^3 and

$$(x)_{\bar{R}} \subseteq I \subseteq I_1 \subseteq \bar{R}.$$

Since $\mathscr{M}$ is a P_2-class, $(x)_{\bar{R}}$ is an $\mathscr{M}$-ring and $\bar{R}$ contains a non-zero $\mathscr{M}$-ideal. This is a contradiction; therefore $I'^3 = 0$.

Now I' contains the non-zero $\mathscr{M}$-ideal I and $\mathscr{M}$ is inductive over $\mathscr{Z}$; therefore, if $I'^2 = 0$, we have, from Lemma 8, that $\bar{R}$ contains a non-zero $\mathscr{M}$-ideal. This is a contradiction; thus $I'^2 \neq 0$. Therefore I'^2 is a non-zero $\mathscr{Z}$-ideal of $\bar{R}$ and, by Lemma 8, it contains no non-zero $\mathscr{M}$-ideals.

Denote II' by J. The ideal $I'^2 = J + \bar{R}J$, thus $J \neq 0$. The ring J is an ideal in I_1 and contained in I. Let x be a non-zero element of J and consider the principal ideal of I_1, $(x)_{I_1}$. We have that $(x)_{I_1} \subseteq I \subseteq I_1$ and, since $\mathscr{M}$ is a P_1-class, $(x)_{I_1} \in \mathscr{M}$. However, $(x)_{I_1} \subseteq J \subseteq I'^2$ and $(x)_{I_1}$ is an ideal in I'; thus I'^2 contains a non-zero $\mathscr{M}$-ideal. This is a contradiction; therefore $\mathscr{M}_2 = \mathscr{M}_1$ and $\mathscr{M}$ is of degree at most one.

Remark 5. This result generalizes (**7**, Theorem 2) for a hereditary class containing all zero rings is a P_2-class inductive over $\mathscr{Z}$.

LEMMA 9. *Let $\mathscr{X}$ be a class of rings such that every accessible subring of an $\mathscr{X}$-ring X is an ideal of X. Then, given a class $\mathscr{M}$, the class $\mathscr{M}_1$ is inductive over $\mathscr{X}$.*

Proof. Let I be an $\mathscr{X}$-ideal of ring R. Put $K = \sum_{\lambda \in \Lambda} K_\lambda$, where $\{K_\lambda; \lambda \in \Lambda\}$ is the set of all $\mathscr{M}_1$-ideals of R contained in I.

Suppose that U is an ideal of K and $U \neq K$. Then there is a K_μ, $\mu \in \Lambda$, such that $K_\mu \nsubseteq U$. Hence, $[U + K_\mu]/U \cong K_\mu/U \cap K_\mu$ is a non-zero $\mathscr{M}_1$-ring. Therefore, $[U + K_\mu]/U$ contains a non-zero $\mathscr{M}$-ideal, V/U say.

Now $V \subseteq U + K_\mu \subseteq K \subseteq I$ with each ring an ideal in its successor. However, $I \in \mathscr{X}$, therefore by the given property of $\mathscr{X}$ we have that V is an ideal in K. Hence, V/U is a non-zero $\mathscr{M}$-ideal of K/U.

Therefore, every non-zero homomorphic image of K contains a non-zero $\mathscr{M}$-ideal. This proves that $K \in \mathscr{M}_1$ and that $\mathscr{M}_1$ is inductive over $\mathscr{X}$.

As examples of classes of rings satisfying the above condition on $\mathscr{X}$ we have any class of zero rings and any class of hereditarily idempotent rings, that is, rings in which every ideal is idempotent.

THEOREM 6. *If $\mathscr{M}$ is a P_1-class of rings, then the lower radical class determined by $\mathscr{M}$ is $\mathscr{M}_2$ and this is hereditary.*

Proof. By Theorem 4, both $\mathscr{M}_1$ and $\mathscr{M}_2$ are hereditary. By Lemma 9 we have that $\mathscr{M}_1$ (which is hereditary, and therefore, in particular, a P_2-class) is inductive over $\mathscr{Z}$. Finally, by applying Theorem 5 to the class $\mathscr{M}_1$ we have that $\mathscr{M}_2 = \mathscr{M}_3$. Hence, $\mathscr{M}_2$ is the lower radical class determined by the class $\mathscr{M}$.

As a corollary we have the following result obtained independently by Armendáriz and Leavitt (**3**).

COROLLARY 4. *The lower radical class determined by a hereditary class $\mathscr{M}$ is $\mathscr{M}_2$ and this also is hereditary.*

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