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Abstract

The moduli space of holomorphic maps from Riemann surfaces to the Grassmannian is known to have two kinds of compactifications: Kontsevich's stable map compactification and Marian–Oprea–Pandharipande's stable quotient compactification. Over a non-singular curve, the latter moduli space is Grothendieck's Quot scheme. In this paper, we give the notion of 'ϵ-stable quotients' for a positive real number ϵ, and show that stable maps and stable quotients are related by wall-crossing phenomena. We will also discuss Gromov–Witten type invariants associated to ϵ-stable quotients, and investigate them under wall crossing.

1. Introduction

The purpose of this paper is to investigate wall-crossing phenomena of several compactifications of the moduli spaces of holomorphic maps from Riemann surfaces to the Grassmannian. So far, two kinds of compactifications are known: Kontsevich's stable map compactification [Kon95] and Marian–Oprea–Pandharipande's stable quotient compactification [MOP09]. The latter moduli space was introduced rather recently, and it is Grothendieck's Quot scheme over a non-singular curve. In this paper, we will introduce the notion of *ϵ-stable quotients* for a positive real number $\epsilon \in \mathbb{R}_{>0}$, and show that the moduli space of ϵ-stable quotients is a proper Deligne–Mumford stack over $\mathbb{C}$ with a perfect obstruction theory. It will turn out that there is a wall and chamber structure on the space of stability conditions $\epsilon \in \mathbb{R}_{>0}$, and the moduli spaces are constant at chambers but jump at walls, i.e. wall-crossing phenomena occur. We will see that stable maps and stable quotients are related by the above wall-crossing phenomena. We will also consider the virtual fundamental classes on the moduli spaces of ϵ-stable quotients, the associated enumerative invariants, and investigate them under the change of $\epsilon \in \mathbb{R}_{>0}$. This is interpreted as a wall-crossing formula of Gromov–Witten (GW) type invariants.

1.1 Stable maps and stable quotients

Let C be a smooth projective curve over $\mathbb{C}$ of genus g, and $\mathbb{G}(r, n)$ the Grassmannian which parameterizes r-dimensional $\mathbb{C}$-vector subspaces in $\mathbb{C}^n$. Let us consider a holomorphic map

$$f : C \to \mathbb{G}(r, n) \tag{1}$$

satisfying the following:

$$f_*[C] = d \in H_2(\mathbb{G}(r, n), \mathbb{Z}) \cong \mathbb{Z}.$$

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By the universal property of $\mathbb{G}(r,n)$, giving a map (1) is equivalent to giving a quotient

$$\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q, \tag{2}$$

where Q is a locally free sheaf of rank $n-r$ and degree d. The moduli space of maps (1) is not compact, and two kinds of compactifications are known: compactification as maps (1) or compactification as quotients (2).

- **Stable map compactification.** We attach trees of rational curves to C, and consider the moduli space of maps from the attached nodal curves to $\mathbb{G}(r,n)$ with finite automorphisms.
- **Quot scheme compactification.** We consider the moduli space of quotients (2), allowing torsion subsheaves in Q. The resulting moduli space is Grothendieck's Quot scheme on C.

In the above compactifications, the (stabilization of the) source curve C is fixed in the moduli. If we vary the curve C as a nodal curve and give m marked points on it, we obtain two kinds of compact moduli spaces

$$\overline{M}_{g,m}(\mathbb{G}(r,n),d), \tag{3}$$

$$\overline{Q}_{g,m}(\mathbb{G}(r,n),d). \tag{4}$$

The space (3) is a moduli space of Kontsevich's *stable maps* [Kon95]. Namely, this is the moduli space of data

$$(C, p_1, \dots, p_m, f : C \to \mathbb{G}(r,n)),$$

where C is a genus g, m-pointed nodal curve and f is a morphism with finite automorphisms.

The space (4) is a moduli space of Marian–Oprea–Pandharipande's stable quotients [MOP09], which we call *MOP-stable quotients*. By definition, an MOP-stable quotient consists of data

$$(C, p_1, \dots, p_m, \mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q) \tag{5}$$

for an m-pointed nodal curve C and a quotient sheaf Q on it, satisfying the following stability conditions.

- The coherent sheaf Q is locally free near nodes and markings. In particular, the determinant line bundle $\det(Q)$ is well defined.
- The $\mathbb{R}$-line bundle

$$\omega_C(p_1 + \cdots + p_m) \otimes \det(Q)^{\otimes \epsilon} \tag{6}$$

  is ample for *every* $\epsilon > 0$.

The space (4) is the moduli space of MOP-stable quotients (5) with C genus g, $\mathrm{rank}(Q) = n-r$ and $\deg(Q) = d$. Both moduli spaces (3) and (4) have the following properties.

- The moduli spaces (3) and (4) are proper Deligne–Mumford stacks over $\mathbb{C}$ with perfect obstruction theories [Beh97, MOP09].
- The moduli spaces (3) and (4) carry proper morphisms as follows.

$$\begin{array}{ccc} \overline{M}_{g,m}(\mathbb{G}(r,n),d) & & \overline{Q}_{g,m}(\mathbb{G}(r,n),d) \\ & \searrow \quad \swarrow & \\ & \overline{M}_{g,m} & \end{array} \tag{7}$$

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Here $\overline{M}_{g,m}$ is the moduli space of genus g, m-pointed stable curves. Taking the fibers of the diagram (7) over a non-singular curve $[C] \in \overline{M}_{g,0}$, we obtain the compactifications as maps (1) and quotients (2), respectively. Also, the associated virtual fundamental classes on the moduli spaces (3) and (4) are compared in [MOP09, § 7].

1.2 ϵ-stable quotients

The purpose of this paper is to introduce a variant of stable quotient theory, depending on a positive real number

$$\epsilon \in \mathbb{R}_{>0}. \tag{8}$$

We define an *ϵ-stable quotient* to be data (5), which has the same property as MOP-stable quotients except for the following.

- The $\mathbb{R}$-line bundle (6) is only ample with respect to the *fixed* stability parameter $\epsilon \in \mathbb{R}_{>0}$.
- For any $p \in C$, the torsion subsheaf $\tau(Q) \subset Q$ satisfies

$$\epsilon \cdot \operatorname{length} \tau(Q)_p \leqslant 1.$$

The idea of ϵ-stable quotients originates from Hassett's weighted pointed stable curves. In [Has03], Hassett introduced the notion of weighted pointed stable curves $(C, p_1, \ldots, p_m)$, where C is a nodal curve and $p_i \in C$ are marked points. The stability condition depends on a choice of a weight

$$(a_1, a_2, \ldots, a_m) \in (0, 1]^m, \tag{9}$$

which puts a similar constraint for the pointed curve $(C, p_1, \ldots, p_m)$ to our ϵ-stability. (See Definition 3.1.) A choice of ϵ in our situation corresponds to a choice of a weight (9) for weighted pointed stable curves.

The moduli space of ϵ-stable quotients (5) with C genus g, $\operatorname{rank}(Q) = n - r$ and $\deg(Q) = d$ is denoted by

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d). \tag{10}$$

We show the following result. (See Theorem 2.12, § 2.3, Propositions 2.16 and 2.18 and Theorem 2.19.)

Theorem 1.1. (i) *The moduli space $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d)$ is a proper Deligne–Mumford stack over $\mathbb{C}$ with a perfect obstruction theory. Also, there is a proper morphism*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) \to \overline{M}_{g,m}. \tag{11}$$

(ii) *There is a finite number of values*

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_k < \epsilon_{k+1} = \infty$$

such that we have

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) = \overline{Q}^{\epsilon_i}_{g,m}(\mathbb{G}(r, n), d)$$

for $\epsilon \in (\epsilon_{i-1}, \epsilon_i]$.

(iii) *We have the following:*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) \cong \overline{M}_{g,m}(\mathbb{G}(r, n), d), \quad \epsilon > 2,$$
$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) \cong \overline{Q}_{g,m}(\mathbb{G}(r, n), d), \quad 0 < \epsilon \leqslant 1/d.$$

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By Theorem 1.1(i), there is the associated virtual fundamental class

$$[\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} \in A_*(\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d),\mathbb{Q}).$$

A comparison of the above virtual fundamental classes under change of ϵ is obtained as follows. (See Theorem 2.25.)

Theorem 1.2. *For $\epsilon \geqslant \epsilon' > 0$ satisfying $2g-2+\epsilon'\cdot d>0$, there is a diagram*

$$\begin{array}{ccc} \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) & \xrightarrow{\iota^{\epsilon}} & \overline{Q}^{\epsilon}_{g,m}\left(\mathbb{G}\left(1,\binom{n}{r}\right)d\right) \\ & & \Big\downarrow c_{\epsilon,\epsilon'} \\ \overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d) & \xrightarrow{\iota^{\epsilon'}} & \overline{Q}^{\epsilon'}_{g,m}\left(\mathbb{G}\left(1,\binom{n}{r}\right),d\right) \end{array}$$

such that we have

$$c_{\epsilon,\epsilon'*}\iota^{\epsilon}_{*}[\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} = \iota^{\epsilon'}_{*}[\overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}.$$

The above theorem, which is a refinement of the result in [MOP09, § 7], is interpreted as a wall-crossing formula relevant to the GW theory.

1.3 Invariants on Calabi–Yau 3-folds

The idea of ϵ-stable quotients is also applied to define new quantum invariants on some compact or non-compact Calabi–Yau 3-folds. One of the interesting examples is a system of invariants on a quintic Calabi–Yau 3-fold $X \subset \mathbb{P}^4$. In § 6, we associate the substack

$$\overline{Q}^{\epsilon}_{0,m}(X,d) \subset \overline{Q}^{\epsilon}_{0,m}(\mathbb{P}^4,d)$$

such that when $\epsilon>2$, it coincides with the moduli space of genus zero, degree d stable maps to X. There is a perfect obstruction theory on the space $\overline{Q}^{\epsilon}_{0,m}(X,d)$ and hence the virtual class

$$[\overline{Q}^{\epsilon}_{0,m}(X,d)]^{\mathrm{vir}} \in A_*(\overline{Q}^{\epsilon}_{0,m}(X,d),\mathbb{Q})$$

with virtual dimension m. In particular, the zero-pointed moduli space yields the invariant

$$N^{\epsilon}_{0,d}(X) = \int_{[\overline{Q}^{\epsilon}_{0,0}(X,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

For $\epsilon>2$, the invariant $N^{\epsilon}_{0,d}(X)$ coincides with the GW invariant counting genus zero, degree d stable maps to X. However, for a smaller ϵ, the above invariant may be different from the GW invariant of X. The understanding of wall-crossing phenomena of such invariants seems relevant to the study of the GW theory. In § 6, we will also discuss such invariants in several other cases.

1.4 Relation to other works

As pointed out in [MOP09, § 1], only a few proper moduli spaces carrying virtual classes are known, e.g. stable maps [Beh97], stable sheaves on surfaces or 3-folds [LT98, Tho00], Grothendieck's Quot scheme on non-singular curves [MO07] and MOP-stable quotients [MOP09]. By the result of Theorem 1.1, we have constructed a new family of moduli spaces which have virtual classes.

Before the appearance of stable maps [Kon95], the Quot scheme was used for an enumeration problem of curves on the Grassmannian [Ber94, Ber97, BDW96]. A relationship between

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compactifications as maps (1) and quotients (2) is discussed in [PR03]. The fiber of the morphism (11) over a non-singular curve is an intermediate moduli space between the above two compactifications. This fact seems to give a new insight to the work [PR03].

Wall-crossing phenomena for stable maps or GW type invariants are discussed in [AG08, BM09, Has03]. In these works, a stability condition is a weight on the marked points, not on maps. In particular, there is no wall-crossing phenomenon if there is no point insertion.

After the work of this paper was completed, a closely related work of Mustaţă–Mustaţă [MM07] was brought to the author's attention. They constructed some compactifications of the moduli space of maps from Riemann surfaces to the projective space, which are interpreted as moduli spaces of ϵ-stable quotients of rank one. However, they do not address higher rank quotients, virtual classes nor the wall-crossing formula. In this sense, the present work is interpreted as a combination of the works [MOP09, MM07].

Recently, wall-crossing formulae of Donaldson–Thomas (DT) type invariants have been developed by Kontsevich–Soibelman [KS08] and Joyce–Song [JS08]. The DT invariant is a counting invariant of stable sheaves on a Calabi–Yau 3-fold, while the GW invariant is a counting invariant of stable maps. The relationship between GW invariants and DT invariants was proposed by Maulik–Nekrasov–Okounkov–Pandharipande (MNOP) [MNOP06], called the *GW/DT correspondence*. On the DT side, a number of applications of the wall-crossing formula to the MNOP conjecture have been found recently, such as the *DT/PT correspondence* and the *rationality conjecture*. (See [Bri10, ST09, Tod10a, Tod10b].) It seems worth trying to find a similar wall-crossing phenomenon on the GW side and give an application to the MNOP conjecture. The work of this paper grew from such an attempt.

2. Stable quotients

In this section, we introduce the notion of ϵ-stable quotients for a positive real number $\epsilon \in \mathbb{R}_{>0}$, study their properties and give some examples. The ϵ-stable quotients are an extended notion of stable quotients introduced by Marian–Oprea–Pandharipande [MOP09].

2.1 Definition of ϵ-stable quotients

Let C be a connected projective curve over $\mathbb{C}$ with at worst nodal singularities. Suppose that the arithmetic genus of C is g,

$$g = \dim H^1(C, \mathcal{O}_C).$$

Let $C^{\mathrm{ns}} \subset C$ be the non-singular locus of C. We say that the data

$$(C, p_1, \ldots, p_m)$$

with distinct markings $p_i \in C^{\mathrm{ns}} \subset C$ is a genus g, m-pointed, *quasi-stable curve*. The notion of quasi-stable quotients was introduced in [MOP09, § 2].

Definition 2.1. Let C be a pointed quasi-stable curve and q a quotient,

$$\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q.$$

We say that q is a *quasi-stable quotient* if Q is locally free near nodes and markings. In particular, the torsion subsheaf $\tau(Q) \subset Q$ satisfies

$$\mathrm{Supp}\, \tau(Q) \subset C^{\mathrm{ns}} \backslash \{p_1, \ldots, p_m\}.$$

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Let $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ be a quasi-stable quotient. The quasi-stability implies that the sheaf Q is perfect, i.e. there is a finite locally free resolution $P^\bullet$ of Q. In particular, the determinant line bundle

$$\det(Q) = \bigotimes_i \left(\bigwedge^{\mathrm{rk}\, P^i} P^i \right)^{\otimes(-1)^i} \in \mathrm{Pic}(C)$$

makes sense. The degree of Q is defined by the degree of $\det(Q)$. We say that a quasi-stable quotient $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ is of *type* (r, n, d) if the following holds:

$$\mathrm{rank}\, Q = n - r, \quad \deg Q = d.$$

For a quasi-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ and $\epsilon \in \mathbb{R}_{>0}$, the $\mathbb{R}$-line bundle $\mathcal{L}(q, \epsilon)$ is defined by

$$\mathcal{L}(q, \epsilon) := \omega_C(p_1 + \cdots + p_m) \otimes (\det Q)^{\otimes \epsilon}. \tag{12}$$

The notion of stable quotients introduced in [MOP09], which we call *MOP-stable quotients*, is defined as follows.

DEFINITION 2.2 [MOP09]. A quasi-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ is a *MOP-stable quotient* if the $\mathbb{R}$-line bundle $\mathcal{L}(q, \epsilon)$ is ample for every $\epsilon > 0$.

The idea of ϵ-stable quotient is that we only require the ampleness of $\mathcal{L}(q, \epsilon)$ for a fixed ϵ (not every $\epsilon > 0$), and put an additional condition on the length of the torsion subsheaf of the quotient sheaf.

DEFINITION 2.3. Let $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ be a quasi-stable quotient and ϵ a positive real number. We say that q is an *ϵ-stable quotient* if the following conditions are satisfied.

- The $\mathbb{R}$-line bundle $\mathcal{L}(q, \epsilon)$ is ample.
- For any point $p \in C$, the torsion subsheaf $\tau(Q) \subset Q$ satisfies the following inequality:

$$\epsilon \cdot \mathrm{length}\, \tau(Q)_p \leqslant 1. \tag{13}$$

Here we give some remarks.

Remark 2.4. As we mentioned in the introduction, the definition of ϵ-stable quotients is motivated by Hassett's weighted pointed stable curves [Has03]. We will discuss the relationship between ϵ-stable quotients and weighted pointed stable curves in § 3.1.

Remark 2.5. The ampleness of $\mathcal{L}(q, \epsilon)$ for every $\epsilon > 0$ is equivalent to the ampleness of $\mathcal{L}(q, \epsilon)$ for $0 < \epsilon \ll 1$. If $\epsilon > 0$ is sufficiently small, then the condition (13) does not say anything, so MOP-stable quotients coincide with ϵ-stable quotients for $0 < \epsilon \ll 1$.

Remark 2.6. For a quasi-stable quotient $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$, take the exact sequence

$$0 \to S \to \mathcal{O}_C^{\oplus n} \to Q \to 0.$$

The quasi-stability implies that S is locally free. By taking the dual of the above exact sequence, giving a quasi-stable quotient is equivalent to giving a locally free sheaf $S^\vee$ and a morphism

$$\mathcal{O}_C^{\oplus n} \stackrel{s}{\to} S^\vee,$$

which is surjective on nodes and marked points. The ϵ-stability is also defined in terms of data $(S^\vee, s)$.

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Remark 2.7. By definition, a quasi-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ of type (r, n, d) induces a rational map

$$f : C \dashrightarrow \mathbb{G}(r, n)$$

such that we have

$$\deg f_*[C] + \operatorname{length} \tau(Q) = d. \tag{14}$$

If $\epsilon > 1$, then the condition (13) is equivalent to that Q is a locally free sheaf. Hence, f is an actual map, and the quotient q is isomorphic to the pull-back of the universal quotient on $\mathbb{G}(r, n)$.

Let C be a marked quasi-stable curve. A point $p \in C$ is called *special* if p is a singular point of C or a marked point. For an irreducible component $P \subset C$, we denote by $s(P)$ the number of special points in P. The following lemma is obvious.

Lemma 2.8. *Let $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ be a quasi-stable quotient and take $\epsilon \in \mathbb{R}_{>0}$. Then the $\mathbb{R}$-line bundle $\mathcal{L}(q, \epsilon)$ is ample if and only if for any irreducible component $P \subset C$ with genus $g(P)$, the following conditions hold:*

$$\deg(Q|_P) > 0, \quad (s(P), g(P)) = (2, 0), (0, 1), \tag{15}$$

$$\deg(Q|_P) > 1/\epsilon, \quad (s(P), g(P)) = (1, 0), \tag{16}$$

$$\deg(Q|_P) > 2/\epsilon, \quad (s(P), g(P)) = (0, 0). \tag{17}$$

Proof. For an irreducible component $P \subset C$, we have

$$\deg(\mathcal{L}(q, \epsilon)|_P) = 2g(P) - 2 + s(P) + \epsilon \cdot \deg(Q|_P).$$

Also, since q is surjective, we have $\deg(Q|_P) \geqslant 0$. Therefore, the lemma follows. □

Here we give some examples. We will discuss some more examples in § 3.

Example 2.9. (i) Let C be a smooth projective curve of genus g and $f : C \to \mathbb{G}(r, n)$ a map. Suppose that f is non-constant if $g \leqslant 1$. By pulling back the universal quotient

$$\mathcal{O}_{\mathbb{G}(r,n)}^{\oplus n} \twoheadrightarrow \mathcal{Q}_{\mathbb{G}(r,n)}$$

on $\mathbb{G}(r, n)$, we obtain the quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$. It is easy to see that the quotient q is an ϵ-stable quotient for $\epsilon > 2$.

(ii) Let C be as in (i) and take distinct points $p_1, \ldots, p_m \in C$. For an effective divisor $D = a_1p_1 + \cdots + a_mp_m$ with $a_i > 0$, the quotient

$$\mathcal{O}_C \stackrel{q}{\twoheadrightarrow} \mathcal{O}_D$$

is an ϵ-stable quotient if and only if

$$2g - 2 + \epsilon \cdot \sum_{i=1}^{m} a_i > 0, \quad 0 < \epsilon \leqslant 1/a_i$$

for all $1 \leqslant i \leqslant m$. In this case, the quotient q is MOP-stable if $g \geqslant 1$, but this is not the case in genus zero.

(iii) Let $\mathbb{P}^1 \cong C \subset \mathbb{P}^n$ be a line and take distinct points $p_1, p_2 \in C$. By restricting the Euler sequence to C, we obtain the exact sequence

$$0 \to \mathcal{O}_C(-1) \stackrel{s}{\to} \mathcal{O}_C^{\oplus n+1} \to T_{\mathbb{P}^n}(-1)|_C \to 0.$$

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Composing the natural inclusion $\mathcal{O}_C(-p_1-p_2-1)\subset\mathcal{O}_C(-1)$ with s, we obtain the exact sequence

$$0\to\mathcal{O}_C(-p_1-p_2-1)\to\mathcal{O}_C^{\oplus n+1}\xrightarrow{q}Q\to 0.$$

It is easy to see that the quotient q is ϵ-stable for $\epsilon=1$. Note that q is not an MOP-stable quotient nor a quotient corresponding to a stable map as in (i).

2.2 Moduli spaces of ϵ-stable quotients

Here we define the moduli functor of the family of ϵ-stable quotients. We use the language of stacks, and the reader can refer to [LM00] for their introduction. First we recall the moduli stack of quasi-stable curves. For a $\mathbb{C}$-scheme B, a *family of genus g, m-pointed quasi-stable curves over B* is defined to be the data

$$(\pi:\mathcal{C}\to B, p_1,\dots,p_m),$$

which satisfies the following.

- The morphism $\pi:\mathcal{C}\to B$ is flat, proper and locally of finite presentation. Its relative dimension is one and $p_1,\dots,p_m$ are sections of π.
- For each closed point $b\in B$, the data

$$(\mathcal{C}_b:=\pi^{-1}(b), p_1(b),\dots,p_m(b))$$

  is an m-pointed quasi-stable curve.

The families of genus g, m-pointed quasi-stable curves form a groupoid $\mathcal{M}_{g,m}(B)$ with the set of isomorphisms

$$\mathrm{Isom}_{\mathcal{M}_{g,m}(B)}((\mathcal{C},p_1,\dots,p_m)(\mathcal{C}',p_1',\dots,p_m'))$$

given by the isomorphisms of schemes over B,

$$\phi:\mathcal{C}\xrightarrow{\cong}\mathcal{C}'$$

satisfying $\phi(p_i)=p_i'$ for each $1\leqslant i\leqslant m$. The assignment $B\mapsto\mathcal{M}_{g,m}(B)$ forms a 2-functor

$$\mathcal{M}_{g,m}:\mathrm{Sch}/\mathbb{C}\to(\mathrm{groupoid}),$$

which is known to be an algebraic stack locally of finite type over $\mathbb{C}$.

DEFINITION 2.10. For a given data

$$\epsilon\in\mathbb{R}_{>0},\quad (r,n,d)\in\mathbb{Z}^{\oplus 3},$$

we define the *stack of genus g, m-pointed ϵ-stable quotients of type* (r,n,d) to be the 2-functor

$$\overline{\mathcal{Q}}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d):\mathrm{Sch}/\mathbb{C}\to(\mathrm{groupoid}),\tag{18}$$

which sends a $\mathbb{C}$-scheme B to the groupoid whose objects consist of data

$$(\pi:\mathcal{C}\to B,p_1,\dots,p_m,\mathcal{O}_{\mathcal{C}}^{\oplus n}\overset{q}{\twoheadrightarrow}\mathcal{Q})\tag{19}$$

satisfying the following.

- $(\pi:\mathcal{C}\to B,p_1,\dots,p_m)$ is a family of genus g, m-pointed quasi-stable curves over B.
- $\mathcal{Q}$ is flat over B such that for any $b\in B$, the data

$$(\mathcal{C}_b,p_1(b),\dots,p_m(b),\mathcal{O}_{\mathcal{C}_b}^{\oplus n}\overset{q_b}{\twoheadrightarrow}\mathcal{Q}_b)$$

  is an ϵ-stable quotient of type (r,n,d).

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For another object over B,

$$(\pi' : \mathcal{C}' \to B, p_1', \ldots, p_m', \mathcal{O}_{\mathcal{C}'}^{\oplus n} \stackrel{q'}{\twoheadrightarrow} \mathcal{Q}'), \tag{20}$$

the set of isomorphisms between (19) and (20) is given by

$$\{\phi \in \mathrm{Isom}_{\mathcal{M}_{g,m}(B)}((\mathcal{C}, p_1, \ldots, p_m), (\mathcal{C}', p_1', \ldots, p_m')) : \ker(q) = \ker(\phi^*(q'))\}.$$

By the construction, there is an obvious forgetting 1-morphism

$$\overline{\mathcal{Q}}_{g,m}^{\epsilon}(\mathbb{G}(r,n), d) \to \mathcal{M}_{g,m}. \tag{21}$$

The following lemma shows that the automorphism groups in $\overline{\mathcal{Q}}_{g,m}^{\epsilon}(\mathbb{G}(r,n), d)$ are finite.

Lemma 2.11. *For a genus g, m-pointed ϵ-stable quotient $(\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q)$ of type (r, n, d), we have*

$$\sharp \mathrm{Aut}(\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q) < \infty \tag{22}$$

in the groupoid $\overline{\mathcal{Q}}_{g,m}^{\epsilon}(\mathbb{G}(r,n), d)(\mathrm{Spec}\, \mathbb{C})$.

Proof. It is enough to show that for each irreducible component $P \subset C$, we have

$$\sharp \mathrm{Aut}(\mathcal{O}_P^{\oplus n} \stackrel{q|_P}{\twoheadrightarrow} Q|_P) < \infty.$$

Hence, we may assume that C is irreducible. The cases we need to consider are the following:

$$(s(C), g(C)) = (0,0), (0,1), (1,0), (2,0).$$

Here we have used the notation in Lemma 2.8. For simplicity, we treat the case of $(s(C), g(C)) = (1, 0)$. The other cases are similarly discussed.

Let f be a rational map

$$f : C \dashrightarrow \mathbb{G}(r,n)$$

determined by the quotient q. (See Remark 2.7.) If f is non-constant, then (22) is obviously satisfied. Hence, we may assume that f is a constant rational map. By the equality (14), this implies that the torsion subsheaf $\tau(Q) \subset Q$ satisfies

$$\mathrm{length}\, \tau(Q) = \deg Q.$$

Also, if $\sharp \mathrm{Supp}\, \tau(Q) \geqslant 2$, then (22) is satisfied, since any automorphism preserves torsion points and special points. Hence, we may assume that there is a unique $p \in C$ such that

$$\mathrm{length}\, \tau(Q)_p = \mathrm{length}\, \tau(Q) = \deg Q.$$

However, this contradicts the condition (13) and Lemma 2.8. □

We will show the following theorem.

Theorem 2.12. *The 2-functor $\overline{\mathcal{Q}}_{g,m}^{\epsilon}(\mathbb{G}(r,n), d)$ is a proper Deligne–Mumford stack of finite type over $\mathbb{C}$ with a perfect obstruction theory.*

Proof. The construction of the moduli space and the properness will be postponed to § 4. The existence of the perfect obstruction theory will be discussed in Theorem 2.14. □

By Theorem 2.12, the 2-functor (18) is interpreted as a geometric object, rather than an abstract 2-functor. In order to emphasize this, we slightly change the notation as follows.

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Definition 2.13. We denote the Deligne–Mumford moduli stack of genus g, m-pointed ϵ-stable quotients of type (r,n,d) by

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d).$$

When $r=1$, we occasionally write

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{P}^{n-1},d) := \overline{Q}^{\epsilon}_{g,m}(\mathbb{P}^{n-1},d).$$

The universal curve is denoted by

$$\pi^{\epsilon} : U^{\epsilon} \to \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d), \tag{23}$$

and we have the universal quotient

$$0 \to S_{U^{\epsilon}} \to \mathcal{O}_{U^{\epsilon}}^{\oplus n} \xrightarrow{q_{U^{\epsilon}}} Q_{U^{\epsilon}} \to 0. \tag{24}$$

2.3 Structures of the moduli spaces of ϵ-stable quotients

Below we discuss some structures on the moduli spaces of ϵ-stable quotients. Similar structures for MOP-stable quotients are discussed in [MOP09, § 3].

Let $\overline{M}_{g,m}$ be the moduli stack of genus g, m-pointed stable curves. By composing (21) with the stabilization morphism, we obtain the proper morphism between Deligne–Mumford stacks

$$\nu^{\epsilon} : \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \to \overline{M}_{g,m}.$$

For an ϵ-stable quotient $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ with markings $p_1,\dots,p_m$, the sheaf Q is locally free at p_i. Hence, it determines an evaluation map

$$\mathrm{ev}_i : \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \to \mathbb{G}(r,n). \tag{25}$$

Taking the fiber product

$$\begin{array}{ccc} \overline{Q}^{\epsilon}_{g_1,m_1+1}(\mathbb{G}(r,n),d_1) \times_{\mathrm{ev}} \overline{Q}^{\epsilon}_{g_2,m_2+1}(\mathbb{G}(r,n),d_2) & \longrightarrow & \overline{Q}^{\epsilon}_{g_1,m_1+1}(\mathbb{G}(r,n),d_1)\ , \\ \downarrow & & \downarrow \mathrm{ev}_{m_1+1} \\ \overline{Q}^{\epsilon}_{g_2,m_2+1}(\mathbb{G}(r,n),d_2) & \xrightarrow{\mathrm{ev}_1} & \mathbb{G}(r,n) \end{array} \tag{26}$$

we have the natural morphism

$$\overline{Q}^{\epsilon}_{g_1,m_1+1}(\mathbb{G}(r,n),d_1) \times_{\mathrm{ev}} \overline{Q}^{\epsilon}_{g_2,m_2+1}(\mathbb{G}(r,n),d_2) \to \overline{Q}^{\epsilon}_{g_1+g_2,m_1+m_2}(\mathbb{G}(r,n)d_1+d_2) \tag{27}$$

defined by gluing ϵ-stable quotients at the marked points. The standard $\mathrm{GL}_n(\mathbb{C})$-action on $\mathcal{O}_C^{\oplus n}$ induces a $\mathrm{GL}_n(\mathbb{C})$-action on $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)$, i.e.

$$g \cdot (\mathcal{O}_C^{\oplus n} \overset{q}{\twoheadrightarrow} Q) = (\mathcal{O}_C^{\oplus n} \overset{q\circ g}{\twoheadrightarrow} Q)$$

for $g \in \mathrm{GL}_n(\mathbb{C})$. The morphisms (25) and (27) are $\mathrm{GL}_n(\mathbb{C})$-equivariant.

2.4 Virtual fundamental classes

The moduli space of ϵ-stable quotients has the associated virtual fundamental class. The following is an analogue of [MOP09, Theorem 2 and Lemma 4] in our situation.

Theorem 2.14. *There is a $\mathrm{GL}_n(\mathbb{C})$-equivariant two-term perfect obstruction theory on $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)$. In particular, there is a virtual fundamental class*

$$[\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} \in A_*^{\mathrm{GL}_n(\mathbb{C})}(\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d),\mathbb{Q})$$

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in the $\mathrm{GL}_n(\mathbb{C})$*-equivariant Chow group. The virtual dimension is given by*

$$nd + r(n-r)(1-g) + 3g - 3 + m,$$

which does not depend on a choice of ϵ*.*

Proof. The same argument of [MOP09, Theorem 2 and Lemma 4] works. For the reader's convenience, we provide the argument. For a fixed marked quasi-stable curve

$$(C, p_1, \ldots, p_m) \in \mathcal{M}_{g,m},$$

the moduli space of ϵ-stable quotients is an open set of the Quot scheme. On the other hand, the deformation theory of the Quot scheme on a non-singular curve is obtained in [CK09, MO07]. Noting that any quasi-stable quotient is locally free near nodes, the analogous construction yields the two-term obstruction theory relative to the forgetting 1-morphism ν,

$$\nu : \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n), d) \to \mathcal{M}_{g,m},$$

given by $\mathbf{R}\pi^{\epsilon}_{*}\mathcal{H}om(S_{U^{\epsilon}}, Q_{U^{\epsilon}})^{*}$. (See (23) and (24).) The absolute obstruction theory is given by the cone $E^{\bullet}$ of the morphism [BF97, GP99],

$$\mathbf{R}\pi^{\epsilon}_{*}\mathcal{H}om(S_{U^{\epsilon}}, Q_{U^{\epsilon}})^{*} \to \nu^{*}\mathbb{L}_{\mathcal{M}_{g,m}}[1],$$

where $\mathbb{L}_{\mathcal{M}_{g,m}}$ is the cotangent complex of the algebraic stack $\mathcal{M}_{g,m}$. By Lemma 2.11, the complex $E^{\bullet}$ is concentrated on $[-1, 0]$. Let $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ be an ϵ-stable quotient with kernel S and marked points $p_1, \ldots, p_m$. By the above description of the obstruction theory and the Riemann–Roch theorem, the virtual dimension is given by

$$\chi(S, Q) - \chi\bigg(T_C\bigg(-\sum_{i=1}^{m} p_i\bigg)\bigg) = nd + r(n-r)(1-g) + 3g - 3 + m. \qquad \square$$

By the proof of the above theorem, the tangent space Tan_q and the obstruction space Obs_q at the ϵ-stable quotient $q : \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ with kernel S and marked points $p_1, \ldots, p_m$ fit into the exact sequence

$$\begin{aligned} 0 &\to H^0\bigg(C, T_C\bigg(-\sum_{i=1}^{m} p_i\bigg)\bigg) \to \mathrm{Hom}(S, Q) \to \mathrm{Tan}_q \\ &\to H^1\bigg(C, T_C\bigg(-\sum_{i=1}^{m} p_i\bigg)\bigg) \to \mathrm{Ext}^1(S, Q) \to \mathrm{Obs}_q \to 0. \end{aligned} \tag{28}$$

In the genus zero case, the obstruction space vanishes and hence the moduli space is non-singular.

Lemma 2.15. *The Deligne–Mumford stack* $\overline{Q}^{\epsilon}_{0,m}(\mathbb{G}(r,n), d)$ *is non-singular of expected dimension* $nd + r(n-r) + m - 3$*.*

Proof. In the notation of the exact sequence (28), it is enough to see that

$$\mathrm{Ext}^1(S, Q) = H^0(C, S \otimes \widetilde{Q}^{\vee} \otimes \omega_C)^* = 0 \tag{29}$$

when the genus of C is zero. Here $\widetilde{Q}$ is the free part of Q, i.e. $Q/\tau(Q)$ for the torsion subsheaf $\tau(Q) \subset Q$. For any irreducible component $P \subset C$ with $s(P) = 1$, it is easy to see that

$$\deg(S \otimes \widetilde{Q}^{*} \otimes \omega_C)|_P < 0,$$

by Lemma 2.8. Then it is easy to deduce the vanishing (29). $\square$

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2.5 Wall-crossing phenomena of ϵ-stable quotients

Here we see that there is a finite number of values in $\mathbb{R}_{>0}$ so that the moduli spaces of ϵ-stable quotients are constant on each interval. First we treat the case of

$$(g, m) \neq (0, 0). \tag{30}$$

We set

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_d < \epsilon_{d+1} = \infty,$$

as follows:

$$\epsilon_i = \frac{1}{d - i + 1}, \quad 1 \leqslant i \leqslant d. \tag{31}$$

Proposition 2.16. *Under the condition (30), take $\epsilon \in (\epsilon_{i-1}, \epsilon_i]$, where ϵ_i is given by (31). Then we have*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) = \overline{Q}^{\epsilon_i}_{g,m}(\mathbb{G}(r, n), d).$$

Proof. Let us take a quasi-stable quotient of type (r, n, d),

$$(\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q). \tag{32}$$

First we show that if (32) is ϵ-stable, then it is also ϵ_i-stable. Since $\epsilon \leqslant \epsilon_i$, the ampleness of $\mathcal{L}(q, \epsilon)$ also implies the ampleness of $\mathcal{L}(q, \epsilon_i)$. For $p \in C$, let us denote by l_p the length of $\tau(Q)$ at p. If $l_p \neq 0$, the condition (13) implies that

$$0 < \epsilon \leqslant \frac{1}{l_p}. \tag{33}$$

Since $l_p \leqslant d$, the inequality (33) also implies that $\epsilon_i \leqslant 1/l_p$, which in turn implies the condition (13) for ϵ_i.

Conversely, suppose that the quasi-stable quotient (32) is ϵ_i-stable. The inequality (13) for ϵ_i also implies (13) for ϵ since $\epsilon \leqslant \epsilon_i$. In order to see that $\mathcal{L}(q, \epsilon)$ is ample, take an irreducible component $P \subset C$ and check (15), (16) and (17). The condition (15) does not depend on ϵ, so (15) is satisfied. Also, the assumption (30) implies that the case (17) does not occur; hence, we only have to check (16). We denote by d_P the degree of $Q|_P$. If $s(P) = 1$ and $g(P) = 0$, we have

$$\epsilon_i > \frac{1}{d_P}, \tag{34}$$

by the condition (16) for ϵ_i. Since $d_P \leqslant d$, (34) implies that

$$\epsilon > \epsilon_{i-1} \geqslant \frac{1}{d_P},$$

which in turn implies the condition (16) for ϵ. Hence, (32) is ϵ-stable. □

Next we treat the case of $(g, m) = (0, 0)$. In this case, the moduli space is empty for small ϵ.

Lemma 2.17. *For $0 < \epsilon \leqslant 2/d$, we have*

$$\overline{Q}^{\epsilon}_{0,0}(\mathbb{G}(r, n), d) = \emptyset.$$

Proof. If $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d)$ is non-empty, the ampleness of $\mathcal{L}(q, \epsilon)$ yields

$$2g - 2 + m + \epsilon \cdot d > 0.$$

Hence, if $g = m = 0$, ϵ should satisfy $\epsilon > 2/d$. □

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Let $d' \in \mathbb{Z}$ be the integer part of $d/2$. We set $0 = \epsilon_0 < \epsilon_1 < \cdots$ in the following way.

$$\epsilon_1 = 2, \quad \epsilon_2 = \infty \quad (d=1),$$
$$\epsilon_1 = \frac{2}{d}, \quad \epsilon_i = \frac{1}{d'-i+2} \quad (2 \leqslant i \leqslant d'+1), \quad \epsilon_{d'+2} = \infty \quad (d \geqslant 3 \text{ is odd}), \tag{35}$$
$$\epsilon_i = \frac{1}{d'-i+1} \quad (1 \leqslant i \leqslant d'), \quad \epsilon_{d'+1} = \infty \quad (d \text{ is even}). \tag{36}$$

We have the following:

Proposition 2.18. *For $\epsilon_\bullet$ as above, we have*

$$\overline{Q}^{\epsilon}_{0,0}(\mathbb{G}(r,n),d) = \overline{Q}^{\epsilon_i}_{0,0}(\mathbb{G}(r,n),d)$$

for $\epsilon \in (\epsilon_{i-1}, \epsilon_i]$.

Proof. By Lemma 2.17, we may assume that $\epsilon_{i-1} \geqslant 2/d$. Then we can follow essentially the same argument of Proposition 2.16. The argument is more subtle since we have to take the condition (17) into consideration, but we leave the detail to the reader. □

Let $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ be the moduli space of genus g, m-pointed stable maps $f: C \to \mathbb{G}(r,n)$ satisfying

$$f_*[C] = d \in H_2(\mathbb{G}(r,n),\mathbb{Z}) \cong \mathbb{Z}.$$

(See [Kon95].) Also, we denote by $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ the moduli space of MOP-stable quotients of type (r,n,d), constructed in [MOP09]. By the following result, we see that both moduli spaces are related by wall-crossing phenomena of ϵ-stable quotients.

Theorem 2.19. (i) *For $\epsilon > 2$, we have*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \cong \overline{M}_{g,m}(\mathbb{G}(r,n),d). \tag{37}$$

(ii) *For $0 < \epsilon \leqslant 1/d$, we have*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \cong \overline{Q}_{g,m}(\mathbb{G}(r,n),d). \tag{38}$$

Proof. (i) First take an ϵ-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ for some $\epsilon > 2$ with marked points $p_1, \ldots, p_m$. By Propositions 2.16 and 2.18, we may take $\epsilon = 3$. The condition (13) implies that Q is locally free; hence, q determines a map

$$f: C \to \mathbb{G}(r,n). \tag{39}$$

Also, the ampleness of $\mathcal{L}(q,3)$ is equivalent to the ampleness of the line bundle

$$\omega_C(p_1 + \cdots + p_m) \otimes f^* O_G(3), \tag{40}$$

where $\mathcal{O}_G(1)$ is the restriction of $\mathcal{O}(1)$ to $\mathbb{G}(r,n)$ via the Plücker embedding. The ampleness of (40) implies that the map f is a stable map.

Conversely, take an m-pointed stable map

$$f: C \to \mathbb{G}(r,n), \quad p_1, \ldots, p_m \in C$$

and a quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ by pulling back the universal quotient on $\mathbb{G}(r,n)$ via f. Then the stability of the map f implies the ampleness of the line bundle (40) and hence the ampleness of $\mathcal{L}(q,3)$. Also, the condition (13) is automatically satisfied for $\epsilon = 3$ since Q is locally free. Hence, we obtain the isomorphism (37).

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(ii) If $(g, m) = (0, 0)$, then both sides of (38) are empty, so we may assume that $(g, m) \neq (0, 0)$. Let us take an ϵ-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ for $0 < \epsilon \leqslant 1/d$. For any irreducible component $P \subset C$, we have $\deg(Q|_P) \leqslant d$. By Lemma 2.8, this implies that there is no irreducible component $P \subset C$ with

$$(s(P), g(P)) = (0, 0) \text{ or } (0, 1).$$

Hence, applying Lemma 2.8 again, we see that q is MOP-stable.

Conversely, take an MOP-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ and $0 < \epsilon \leqslant 1/d$. By the definition of MOP-stable quotient, the line bundle $\mathcal{L}(q, \epsilon)$ is ample. Also, for any point $p \in C$, the length of the torsion part of Q is less than or equal to d (cf. Remark 2.7). Hence, the condition (13) is satisfied and q is ϵ-stable. Therefore, the desired isomorphism (38) holds. □

2.6 Morphisms between moduli spaces of ϵ-stable quotients

In this subsection, we construct some natural morphisms between moduli spaces of ϵ-stable quotients. The first one is an analogue of the Plücker embedding. (See [MOP09, § 5] for the corresponding morphism between MOP-stable quotients.)

LEMMA 2.20. *There is a natural morphism*

$$\iota^{\epsilon} : \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d) \to \overline{Q}^{\epsilon}_{g,m}\left(\mathbb{G}\left(1, \binom{n}{r}\right), d\right). \tag{41}$$

Proof. For a quasi-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ of type (r, n, d) with kernel S, we associate the exact sequence

$$0 \to \wedge^r S \to \wedge^r \mathcal{O}_C^{\oplus n} \stackrel{q'}{\twoheadrightarrow} Q' \to 0.$$

It is easy to see that q is ϵ-stable if and only if q' is ϵ-stable. The map $q \mapsto q'$ gives the desired morphism. □

Next we treat the case of $r = 1$.

PROPOSITION 2.21. *For $\epsilon \geqslant \epsilon'$, there is a natural morphism*

$$c_{\epsilon,\epsilon'} : \overline{Q}^{\epsilon}_{g,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}^{\epsilon'}_{g,m}(\mathbb{P}^{n-1}, d). \tag{42}$$

Proof. For simplicity, we deal with the case of $(g, m) \neq (0, 0)$. By Proposition 2.16, it is enough to construct a morphism

$$c_{i+1,i} : \overline{Q}^{\epsilon_{i+1}}_{g,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}^{\epsilon_i}_{g,m}(\mathbb{P}^{n-1}, d), \tag{43}$$

where ϵ_i is given by (31). Let us take an ϵ_{i+1}-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$, and the set of irreducible components $T_1, \ldots, T_k$ of C satisfying

$$(s(T_j), g(T_j)) = (1, 0), \quad \deg(Q|_{T_j}) = d - i + 1. \tag{44}$$

Note that T_j and $T_{j'}$ are disjoint for $j \neq j'$, by the assumption that $(g, m) \neq (0, 0)$. We set T and C' to be

$$T = \coprod_{j=1}^{k} T_j, \quad C' = \overline{C \setminus T}. \tag{45}$$

The intersection $T_j \cap C'$ consists of one point x_j, unless $(g, m) = (0, 1)$, $k = 1$ and $i = 1$. In the latter case, the space $\overline{Q}^{\epsilon_1}_{0,1}(\mathbb{P}^{n-1}, d)$ is empty, so there is nothing to prove. Let S be the kernel

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of q. We have the sequence of inclusions

$$S' := S|_{C'}\left(-\sum_{j=1}^{k}(d-i+1)x_j\right) \hookrightarrow S|_{C'} \hookrightarrow \mathcal{O}_{C'}^{\oplus n}$$

and the exact sequence

$$0 \to S' \to \mathcal{O}_{C'}^{\oplus n} \xrightarrow{q'} Q' \to 0.$$

It is easy to see that q' is an ϵ_i-stable quotient. Then the map $q \mapsto q'$ gives the desired morphism (43). □

Remark 2.22. Suppose that $(g, m) \neq (0, 0)$. By Propositions 2.21 and 2.16 and Theorem 2.19, we have the sequence of morphisms

$$\begin{aligned}\overline{M}_{g,m}(\mathbb{P}^{n-1}, d) = \overline{Q}^{\epsilon_{d+1}}_{g,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}^{\epsilon_d}_{g,m}(\mathbb{P}^{n-1}, d) \to \cdots \\ \to \overline{Q}^{\epsilon_2}_{g,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}^{\epsilon_1}_{g,m}(\mathbb{P}^{n-1}, d) = \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d).\end{aligned} \tag{46}$$

The composition of the above morphism

$$c : \overline{M}_{g,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d) \tag{47}$$

coincides with the morphism constructed in [MOP09, § 5]. The morphism c also appears for the Quot scheme of a fixed non-singular curve in [PR03].

Let us investigate the morphism (43) more precisely. For $k \in \mathbb{Z}_{\geqslant 0}$, we consider a subspace

$$\overline{Q}^{\epsilon_{i+1},k+}_{g,m}(\mathbb{P}^{n-1}, d) \subset \overline{Q}^{\epsilon_{i+1}}_{g,m}(\mathbb{P}^{n-1}, d) \tag{48}$$

consisting of ϵ_{i+1}-stable quotients with exactly k irreducible components $T_1, \dots, T_k$ satisfying (44). Setting $d_i = d - i + 1$, the subspace (48) fits into the following Cartesian diagram.

$$\begin{array}{ccc} \overline{Q}^{\epsilon_{i+1},k+}_{g,m}(\mathbb{P}^{n-1}, d) & \longrightarrow & \overline{Q}^{\epsilon_{i+1}}_{0,1}(\mathbb{P}^{n-1}, d_i)^{\times k} \\ \downarrow & & \downarrow (\mathrm{ev}_1)^{\times k} \\ \overline{Q}^{\epsilon_i,\epsilon_{i+1}}_{g,k+m}(\mathbb{P}^{n-1}, d - kd_i) & \longrightarrow & (\mathbb{P}^{n-1})^{\times k} \end{array} \tag{49}$$

Here the bottom arrow is the evaluation map with respect to the first k marked points, and the space

$$\overline{Q}^{\epsilon_i,\epsilon_{i+1}}_{g,m}(\mathbb{P}^{n-1}, d) \tag{50}$$

is the moduli space of genus g, m-marked quasi-stable quotients of type $(1, n, d)$, which is both ϵ_i- and ϵ_{i+1}-stable. The space (50) is an open Deligne–Mumford substack of $\overline{Q}^{\epsilon}_{g,m}(\mathbb{P}^{n-1}, d)$ for both $\epsilon = \epsilon_i$ and ϵ_{i+1}. Note that the left-hand arrow of (49) is surjective since the right-hand arrow is surjective.

We also consider a subspace

$$\overline{Q}^{\epsilon_i,k-}_{g,m}(\mathbb{P}^{n-1}, d) \subset \overline{Q}^{\epsilon_i}_{g,m}(\mathbb{P}^{n-1}, d)$$

consisting of ϵ_i-stable quotients $\mathcal{O}_C^{\oplus n} \xrightarrow{q} Q$ with exactly k distinct points $x_1, \dots, x_k \in C$ satisfying

$$\text{length}\, \tau(Q)_{x_j} = d_i, \quad 1 \leqslant j \leqslant k.$$

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Obviously, we have the isomorphism

$$\overline{Q}^{\epsilon_i,k-}_{g,m}(\mathbb{P}^{n-1},d) \cong \overline{Q}^{\epsilon_i,\epsilon_{i+1}}_{g,k+m}(\mathbb{P}^{n-1},d-kd_i), \tag{51}$$

and the construction of (43) yields the following Cartesian diagram.

$$\begin{array}{ccc} \overline{Q}^{\epsilon_{i+1},k+}_{g,m}(\mathbb{P}^{n-1},d) & \longrightarrow & \overline{Q}^{\epsilon_{i+1}}_{g,m}(\mathbb{P}^{n-1},d) \\ \downarrow & & \downarrow{\scriptstyle c_{i+1,i}} \\ \overline{Q}^{\epsilon_i,k-}_{g,m}(\mathbb{P}^{n-1},d) & \longrightarrow & \overline{Q}^{\epsilon_i}_{g,m}(\mathbb{P}^{n-1},d) \end{array} \tag{52}$$

The left-hand arrow of the diagram (52) coincides with the left-hand arrow of (49) under the isomorphism (51), and in particular it is surjective. The above argument implies the following.

LEMMA 2.23. *The morphism $c_{\epsilon,\epsilon'}$ constructed in Proposition 2.21 is surjective.*

For $r>1$, it seems that there is no natural morphism between $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ and $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$, as pointed out in [MOP09, PR03]. However, for $\epsilon=1$, there is a natural morphism between moduli spaces of stable maps and those of ϵ-stable quotients. The following lemma will be used in Lemma 5.1 below.

LEMMA 2.24. *There is a natural surjective morphism*

$$c' : \overline{M}_{g,m}(\mathbb{G}(r,n),d) \to \overline{Q}^{\epsilon=1}_{g,m}(\mathbb{G}(r,n),d).$$

Proof. For simplicity, we assume that $(g,m)\neq(0,0)$. For a stable map $f:C\to\mathbb{G}(r,n)$ of degree d, pulling back the universal quotient yields the exact sequence

$$0\to S\to\mathcal{O}_C^{\oplus n}\xrightarrow{q} Q\to 0. \tag{53}$$

Here Q is a locally free sheaf on C and the quotient q is of type (r,n,d). Let $T_1,\dots,T_k$ be the set of irreducible components of C satisfying the following:

$$(s(T_j),g(T_j))=(1,0), \quad \deg(Q|_{T_j})=1.$$

By the exact sequence (53) and the degree reason, the following isomorphisms exist:

$$Q|_{T_j}\cong\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^{\oplus n-r-1}, \quad S|_{T_j}\cong\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}^{\oplus r-1}. \tag{54}$$

We set T and C' as in (45), and set $x_j=T_j\cap C'$. Let π be the morphism

$$\pi : C\to C',$$

which is the identity outside T and contracts T_j to x_j. The exact sequences

$$0\to Q|_T\bigg(-\sum_{j=1}^k x_j\bigg)\to Q\to Q|_{C'}\to 0, \tag{55}$$

$$0\to S|_{C'}\bigg(-\sum_{j=1}^k x_j\bigg)\to S\to S|_T\to 0 \tag{56}$$

and the isomorphisms (54) show that π_*Q has torsion at x_j with length one and $R^1\pi_*S=0$. Therefore, applying π_* to (53) yields the exact sequence

$$0\to\pi_*S\to\mathcal{O}_{C'}^{\oplus n}\xrightarrow{q'}\pi_*Q\to 0.$$

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It is easy to see that q' is an ϵ-stable quotient with $\epsilon = 1$, and the map $f \mapsto q'$ gives the desired morphism c'. An argument similar to Lemma 2.23 shows that the morphism c' is surjective. $\square$

2.7 Wall-crossing formula of virtual fundamental classes

In [MOP09, Theorems 3 and 4], the virtual fundamental classes on moduli spaces of stable maps and those of MOP-stable quotients are compared. Such a comparison result also holds for ϵ-stable quotients. Note that the arguments in §§ 2.3 and 2.6 yield the following diagram.

$$
\begin{array}{ccc}
 & \overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \xrightarrow{\iota^{\epsilon}} & \overline{Q}^{\epsilon}_{g,m}\left(\mathbb{G}\left(1,\binom{n}{r}\right),d\right) \\
 \swarrow^{\mathrm{ev}_i} & & \\
\mathbb{G}(r,n) & & \downarrow c_{\epsilon,\epsilon'} \\
 \nwarrow_{\mathrm{ev}_i} & & \\
 & \overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d) \xrightarrow{\iota^{\epsilon'}} & \overline{Q}^{\epsilon'}_{g,m}\left(\mathbb{G}\left(1,\binom{n}{r}\right),d\right)
\end{array}
$$

The following theorem, which is a refinement of [MOP09, Theorem 4], is interpreted as a wall-crossing formula of GW type invariants. The proof will be given in § 5.

Theorem 2.25. *Take $\epsilon \geqslant \epsilon' > 0$ satisfying $2g - 2 + \epsilon' \cdot d > 0$. We have the formula*

$$c_{\epsilon,\epsilon' *}\iota^{\epsilon}_{*}[\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} = \iota^{\epsilon'}_{*}[\overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}. \tag{57}$$

In particular, for classes $\gamma_i \in A^{*}_{\mathrm{GL}_n(\mathbb{C})}(\mathbb{G}(r,n),\mathbb{Q})$, the following holds:

$$\begin{aligned} c_{\epsilon,\epsilon' *}\iota^{\epsilon}_{*}&\left(\prod_{i=1}^{m}\mathrm{ev}_i^{*}(\gamma_i) \cap [\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}\right) \\ &= \iota^{\epsilon'}_{*}\left(\prod_{i=1}^{m}\mathrm{ev}_i^{*}(\gamma_i) \cap [\overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}\right). \end{aligned} \tag{58}$$

Remark 2.26. The formula (57) in particular implies the formula

$$c_{*}[\overline{Q}^{\epsilon}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}} = [\overline{Q}^{\epsilon'}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}}. \tag{59}$$

Here the morphism c is given by (47). Applying the formula (59) to the diagram (46) repeatedly, we obtain the following formula:

$$c_{*}[\overline{M}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}} = [\overline{Q}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}},$$

which reconstructs the result of [MOP09, Theorem 3].

3. Type $(1,1,d)$-quotients

In this section, we investigate the moduli spaces of ϵ-stable quotients of type $(1,1,d)$ and relevant wall-crossing phenomena.

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3.1 Relation to Hassett's weighted pointed stable curves

Here we see that ϵ-stable quotients of type $(1, 1, d)$ are closely related to Hassett's weighted pointed stable curves [Has03], which we recall here. Let us take a sequence

$$a = (a_1, a_2, \dots, a_m) \in (0, 1]^m.$$

Definition 3.1. A data $(C, p_1, \dots, p_m)$ of a nodal curve C and (possibly not distinct) marked points $p_i \in C^{\mathrm{ns}}$ is called *a-stable* if the following conditions hold.

- The $\mathbb{R}$-divisor $K_C + \sum_{i=1}^m a_i p_i$ is ample.
- For any $p \in C$, we have $\sum_{p_i = p} a_i \leqslant 1$.

Note that setting $a_i = 1$ for all i yields the usual m-pointed stable curves. The moduli space of genus g, m-pointed a-stable curves is constructed in [Has03] as a proper smooth Deligne–Mumford stack over $\mathbb{C}$. Among weights, we only use the following weight for $\epsilon \in (0, 1]$:

$$a(m, d, \epsilon) := \Big(\overbrace{1, \dots, 1}^{m}, \overbrace{\epsilon, \dots, \epsilon}^{d}\Big). \tag{60}$$

The moduli space of genus g, $m + d$-pointed $a(m, d, \epsilon)$-stable curves is denoted by

$$\overline{M}^{\epsilon}_{g,m|d}. \tag{61}$$

If $m = 0$, we simply write (61) as $\overline{M}^{\epsilon}_{g,d}$. For $\epsilon \geqslant \epsilon'$, there is a natural birational contraction [Has03, Theorem 4.3]

$$c_{\epsilon,\epsilon'} : \overline{M}^{\epsilon}_{g,m|d} \to \overline{M}^{\epsilon'}_{g,m|d}. \tag{62}$$

Now we describe the moduli spaces of ϵ-stable quotients of type $(1, 1, d)$ and relevant wall-crossing phenomena. In what follows, we denote

$$\mathrm{pt} := \mathbb{P}^0 = \mathbb{G}(1, 1) \cong \operatorname{Spec} \mathbb{C}.$$

We have the following proposition. (See [MOP09, Proposition 3] for the corresponding result of MOP-stable quotients.)

Proposition 3.2. *We have the isomorphism*

$$\phi : \overline{M}^{\epsilon}_{g,m|d}/S_d \xrightarrow{\sim} \overline{Q}^{\epsilon}_{g,m}(\mathrm{pt}, d), \tag{63}$$

where the symmetric group S_d acts by permuting the last d marked points.

Proof. Take a genus g, $m + d$-pointed $a(m, d, \epsilon)$-stable curve

$$(C, p_1, \dots, p_m, \widehat{p}_1, \dots, \widehat{p}_d).$$

We associate the genus g, m-pointed quasi-stable quotient of type $(1, 1, d)$ by the exact sequence

$$0 \to \mathcal{O}_C\left(-\sum_{j=1}^{d} \widehat{p}_j\right) \to \mathcal{O}_C \xrightarrow{q} Q \to 0$$

with m marked points $p_1, \dots, p_m$. The $a(m, d, \epsilon)$-stability immediately implies the ϵ-stability for the quotient q. The map $(C, p_\bullet, \widehat{p}_\bullet) \mapsto q$ is S_d-equivariant; hence, we obtain the map ϕ. It is straightforward to check that ϕ is an isomorphism. □

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Remark 3.3. The morphism (62) is S_d-equivariant and hence it determines a morphism

$$c_{\epsilon,\epsilon'}:\overline{M}_{g,m|d}^{\epsilon}/S_d\to\overline{M}_{g,m|d}^{\epsilon'}/S_d.$$

It is easy to see that the above morphism coincides with (42) under the isomorphism (63).

3.2 The case of $(g,m)=(0,0)$

Here we investigate ϵ-stable quotients of type $(1,1,d)$ with $(g,m)=(0,0)$. First we take d to be an odd integer with $d=2d'+1$, $d'\geqslant 1$. We take $\epsilon_\bullet$ as in (35). Applying the morphism (62) repeatedly, we obtain the sequence of birational morphisms

$$\overline{M}_{0,d}=\overline{M}_{0,d}^{\epsilon_{d'+1}=1}\to\overline{M}_{0,d}^{\epsilon_{d'}}\to\cdots\to\overline{M}_{0,d}^{\epsilon_3}\to\overline{M}_{0,d}^{\epsilon_2=1/d'}. \tag{64}$$

It is easy to see that $\overline{M}_{0,d}^{1/d'}$ is the moduli space of configurations of d points in $\mathbb{P}^1$ in which at most d' points coincide. This space is well known to be isomorphic to the GIT quotient [MFK94]

$$\overline{M}_{0,d}^{1/d'}\cong(\mathbb{P}^1)^d/\!\!/\mathrm{SL}_2(\mathbb{C}). \tag{65}$$

Here $\mathrm{SL}_2(\mathbb{C})$ acts on $(\mathbb{P}^1)^d$ diagonally, and we take the linearization on $\mathcal{O}(\overbrace{1,\dots,1}^{d})$ induced by the standard linearization on $\mathcal{O}_{\mathbb{P}^1}(1)$. Since the sequence (64) is S_d-equivariant, taking the quotients of (64) and combining with the isomorphism (63) yield the sequence of birational morphisms

$$\overline{Q}_{0,0}^{\epsilon_{d'+1}=1}(\mathrm{pt},d)\to\overline{Q}_{0,0}^{\epsilon_{d'}}(\mathrm{pt},d)\to\cdots\to\overline{Q}_{0,0}^{\epsilon_3}(\mathrm{pt},d)\to\overline{Q}_{0,0}^{\epsilon_2=1/d'}(\mathrm{pt},d)\cong\mathbb{P}^d/\!\!/\mathrm{SL}_2(\mathbb{C}). \tag{66}$$

Here the last isomorphism is obtained by taking the quotient of (65) by the S_d-action. By Remark 3.3, each morphism in (66) coincides with the morphism (42). Recently, Kiem–Moon [KM10] showed that each birational morphism in the sequence (64) is a blow-up at a union of transversal smooth subvarieties of the same dimension. As pointed out in [KM98, Remark 4.5], the sequence (66) is a sequence of weighted blow-ups from $\mathbb{P}^d/\!\!/\mathrm{SL}_2(\mathbb{C})$.

When d is even with $d=2d'$, let us take $\epsilon_\bullet$ as in (36). We also have a similar sequence to (64),

$$\overline{M}_{0,d}=\overline{M}_{0,d}^{\epsilon_{d'}=1}\to\overline{M}_{0,d}^{\epsilon_{d'-1}}\to\cdots\to\overline{M}_{0,d}^{\epsilon_3}\to\overline{M}_{0,d}^{\epsilon_2=1/(d'-1)},$$

which is a sequence of blow-ups [KM10]. In this case, instead of the isomorphism (65), there is a birational morphism (cf. [KM10, Theorem 1.1])

$$\overline{M}_{0,d}^{1/(d'-1)}\to(\mathbb{P}^1)^d/\!\!/\mathrm{SL}_2(\mathbb{C})$$

obtained by the blow-up along the singular locus which consists of $\frac{1}{2}\binom{d}{d'}$ points in the right-hand side. As mentioned in [KM10], $\overline{M}_{0,d}^{1/(d'-1)}$ is Kirwan's partial desingularization [Kir85] of the GIT quotient $(\mathbb{P}^1)^d/\!\!/\mathrm{SL}_2(\mathbb{C})$. By taking the quotients with respect to the S_d-actions, we obtain a sequence similar to (66),

$$\overline{Q}_{0,0}^{\epsilon_{d'}=1}(\mathrm{pt},d)\to\overline{Q}_{0,0}^{\epsilon_{d'-1}}(\mathrm{pt},d)\to\cdots\to\overline{Q}_{0,0}^{\epsilon_2=1/(d'-1)}(\mathrm{pt},d)\to(\mathbb{P}^d)/\!\!/\mathrm{SL}_2(\mathbb{C}), \tag{67}$$

a sequence of weighted blow-ups. Finally, Theorem 2.19 yields that

$$\overline{Q}_{0,0}^{\epsilon}(\mathrm{pt},d)=\emptyset,\quad \epsilon>1 \text{ or } d=1.$$

As a summary, we obtain the following.

Theorem 3.4. *The moduli space $\overline{Q}_{0,0}^{\epsilon}(\mathrm{pt},d)$ is either empty or obtained by a sequence of weighted blow-ups starting from the GIT quotient $\mathbb{P}^d/\!\!/\mathrm{SL}_2(\mathbb{C})$.*

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3.3 The case of $(g, m) = (0, 1), (0, 2)$

In this subsection, we study moduli spaces of genus zero, 1- or 2-pointed ϵ-stable quotients of type $(1, 1, d)$. Note that for small ϵ, we have

$$\overline{Q}^{\epsilon}_{0,1}(\mathrm{pt}, d) = \emptyset, \quad 0 < \epsilon \leqslant 1/d.$$

The first interesting situation happens at $\epsilon = 1/(d-1)$ and $d \geqslant 2$. For an object

$$(C, p, \widehat{p}_1, \ldots, \widehat{p}_d) \in \overline{M}^{1/(d-1)}_{0,1|d},$$

applying Lemma 2.8 immediately implies that $C \cong \mathbb{P}^1$. We may assume that $p = \infty \in \mathbb{P}^1$ and hence $\widehat{p}_i \in \mathbb{A}^1$. The stability condition is equivalent to that at least two points among $\widehat{p}_1, \ldots, \widehat{p}_d$ are distinct. Let Δ be the small diagonal,

$$\Delta = \{(x_1, \ldots, x_d) \in \mathbb{A}^d : x_1 = x_2 = \cdots = x_d\}.$$

Noting that the subgroup of automorphisms of $\mathbb{P}^1$ preserving $p \in \mathbb{P}^1$ is $\mathbb{A}^1 \rtimes \mathbb{G}_m$, we have

$$\begin{aligned} \overline{M}^{1/(d-1)}_{0,1|d} &\cong (\mathbb{A}^d \backslash \Delta)/\mathbb{A} \rtimes \mathbb{G}_m \\ &\cong \mathbb{P}^{d-2}. \end{aligned}$$

By Proposition 3.2, we obtain

$$\overline{Q}^{1/(d-1)}_{0,1}(\mathrm{pt}, d) \cong \mathbb{P}^{d-2}/S_d. \tag{68}$$

In particular, for each $\epsilon \in \mathbb{R}_{>0}$, the moduli space $\overline{Q}^{\epsilon}_{0,1}(\mathrm{pt}, d)$ is either empty or admits a birational morphism to $\mathbb{P}^{d-2}/S_d$.

Next we look at the case of $(g, m) = (0, 2)$. An ϵ-stable quotient is an MOP-stable quotient for $0 < \epsilon \leqslant 1/d$, and in this case the moduli space is described in [MOP09, § 4]. In fact, for any MOP-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$, the curve C is a chain of rational curves and two marked points lie at distinct rational tails if C is not irreducible. If k is the number of irreducible components of C, then giving an MOP-stable quotient is equivalent to giving a partition $d_1 + \cdots + d_k = d$ and length d_i divisors on each irreducible component up to rotations. Therefore, we have (set theoretically)

$$\overline{Q}_{0,2}(\mathrm{pt}, d) = \coprod_{\substack{k \geqslant 1 \\ d_1 + \cdots + d_k = d}} \prod_{j=1}^{k} \mathrm{Sym}^{d_i}(\mathbb{C}^*)/\mathbb{C}^*. \tag{69}$$

For $1/d < \epsilon \leqslant 1/(d-1)$, an MOP-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ is not ϵ-stable if and only if $C \cong \mathbb{P}^1$ and the support of $\tau(Q)$ consists of one point. Such stable quotients consist of one point in the right-hand side of (69). Noting the isomorphism (68), the Cartesian diagram (52) is described as follows.

$$\begin{array}{ccc} \mathbb{P}^{d-2}/S_d & \longrightarrow & \overline{Q}^{\epsilon}_{0,2}(\mathrm{pt}, d) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbb{C} & \longrightarrow & \overline{Q}_{0,2}(\mathrm{pt}, d) \end{array}$$

4. Proof of Theorem 2.12

In this section, we give a proof of Theorem 2.12. We first show that $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d)$ is a Deligne–Mumford stack of finite type over $\mathbb{C}$, following the argument of [Has03, MOP09]. Next we show

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the properness of $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)$ using the valuative criterion. The argument to show the properness of MOP-stable quotients [MOP09, § 6] is not applied for ϵ-stable quotients. Instead, we give an alternative argument, which also gives another proof of [MOP09, Theorem 1].

4.1 Construction of the moduli space

The same arguments of Propositions 2.16 and 2.18 show the similar result for the 2-functors (18). For $\epsilon > 1$, the moduli space of ϵ-stable quotients is either empty or isomorphic to the moduli space of stable maps to the Grassmannian. Therefore, we assume that

$$\epsilon = \frac{1}{l}, \quad l = 1, 2, \dots, d,$$

and construct the moduli space $\overline{Q}^{1/l}_{g,m}(\mathbb{G}(r,n),d)$ as a global quotient stack. If $\epsilon = 1/d$, then the moduli space coincides with that of MOP-stable quotients (cf. Theorem 2.19), and the construction is given in [MOP09, § 6]. We need to slightly modify the argument to construct the moduli spaces for a general ϵ, but the essential idea is the same. First we show the following lemma.

Lemma 4.1. *Take an $\epsilon = 1/l$-stable quotient $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ and an integer $k \geqslant 5$. Then the line bundle $\mathcal{L}(q,1/l)^{\otimes lk}$ is very ample. Here $\mathcal{L}(q,1/l)$ is defined in (12).*

Proof. It is enough to show that for $x_1, x_2 \in C$, we have

$$H^1(C, \mathcal{L}(q,1/l)^{\otimes lk} \otimes I_{x_1} I_{x_2}) = 0. \tag{70}$$

Here I_{x_i} is the ideal sheaf of x_i. By the Serre duality, (70) is equivalent to

$$\mathrm{Hom}(I_{x_1} I_{x_2}, \omega_C \otimes \mathcal{L}(q,1/l)^{\otimes(-lk)}) = 0. \tag{71}$$

Suppose that $x_1, x_2 \in C^{\mathrm{ns}}$. For an irreducible component $P \subset C$, we set $d_P = \deg(Q|_P)$. In the notation of Lemma 2.8, we have

$$\begin{aligned} &\deg(\omega_C(x_1+x_2) \otimes \mathcal{L}(q,1/l)^{\otimes(-lk)}|_P) \\ &\quad \leqslant 2g(P) - 2 + s(P) + 2 - lk(2g(P) - 2 + s(P) + d_P/l) \\ &\quad = (2g(P) - 2 + s(P))(1 - lk) + 2 - d_P k. \end{aligned} \tag{72}$$

In the case of

$$2g(P) - 2 + s(P) > 0,$$

(72) is obviously negative. Otherwise, $(g(P), s(P))$ is one of the following:

$$(g(P), s(P)) = (1,0), (0,2), (0,1), (0,0).$$

In these cases, (72) is negative by Lemma 2.8. Therefore, (71) holds.

When x_1 or x_2 or both of them are nodes, for instance x_1 is a node and $x_2 \in C^{\mathrm{ns}}$, then we take the normalization at x_1,

$$\pi : \widetilde{C} \to C,$$

with $\pi^{-1}(x_1) = \{x_1', x_1''\}$. Then (71) is equivalent to

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}(x_1' + x_1'' + x_2) \otimes \mathcal{L}(q,1/l)^{\otimes(-lk)}) = 0, \tag{73}$$

and the same calculation as above shows (73). The other cases are also similarly discussed. □

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By Lemma 4.1, we have

$$h^0(C, \mathcal{L}(q, 1/l)^{\otimes kl}) = 1 - g + kl(2g-2) + kd + m, \tag{74}$$

which does not depend on a choice of a $1/l$-stable quotient of type (r, n, d). Let V be a $\mathbb{C}$-vector space of dimension (74). The very ample line bundle $\mathcal{L}(q, 1/l)^{\otimes kl}$ on C determines an embedding

$$C \hookrightarrow \mathbb{P}(V),$$

and marked points determine points in $\mathbb{P}(V)$. Therefore, the $1/l$-stable quotient associates a point

$$(C, p_1, \dots, p_m) \in \mathrm{Hilb}(\mathbb{P}(V)) \times \mathbb{P}(V)^{\times m}. \tag{75}$$

Let

$$\mathcal{H} \subset \mathrm{Hilb}(\mathbb{P}(V)) \times \mathbb{P}(V)^{\times m}$$

be the locally closed subscheme which parameterizes $(C, p_1, \dots, p_m)$ satisfying the following.

- The subscheme $C \subset \mathbb{P}(V)$ is a connected nodal curve of genus g.
- We have $p_i \in C^{\mathrm{ns}}$ and $p_i \neq p_j$ for $i \neq j$.

Let $\pi : \mathcal{C} \to \mathcal{H}$ be the universal curve and

$$\mathrm{Quot}(n-r, d) \to \mathcal{H}$$

the relative Quot scheme which parameterizes rank $n-r$, degree d quotients $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ on the fibers of π. We define

$$\mathcal{Q} \subset \mathrm{Quot}(n-r, d)$$

to be the locally closed subscheme corresponding to quotients $\mathcal{O}_C^{\oplus n} \stackrel{q}{\twoheadrightarrow} Q$ satisfying the following.

- The coherent sheaf Q is locally free near nodes and p_i.
- For any $p \in C$, we have length $\tau(Q)_p \leqslant l$.
- The line bundle $\mathcal{L}(q, 1/l)^{\otimes lk}$ coincides with $\mathcal{O}_{\mathbb{P}(V)}(1)|_C$.

The natural $\mathrm{PGL}_2(\mathbb{C})$-action on $\mathcal{H}$ lifts to the action on $\mathcal{Q}$, and the desired moduli space is the following quotient stack:

$$\overline{Q}^{1/l}_{g,m}(\mathbb{G}(r, n), d) = [\mathcal{Q}/\mathrm{PGL}_2(\mathbb{C})].$$

By Lemma 2.11, the stabilizer groups of closed points in $\overline{Q}^{1/l}_{g,m}(\mathbb{G}(r, n), d)$ are finite. Hence, this is a Deligne–Mumford stack of finite type over $\mathbb{C}$.

4.2 Valuative criterion

In this subsection, we prove the properness of the moduli stack $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d)$. Before this, we introduce some notation. Let X be a variety and F a locally free sheaf of rank r on X. For $n \geqslant r$ and a morphism

$$s : \mathcal{O}_X^{\oplus n} \to F,$$

we associate the degenerate locus

$$Z(s) \subset X.$$

Namely, $Z(s)$ is defined by the ideal, locally generated by $r \times r$-minors of the matrix given by s. For a point $g \in \mathbb{G}(r, n)$, let us choose a lift of g to an embedding

$$g : \mathbb{C}^r \hookrightarrow \mathbb{C}^n. \tag{76}$$

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Here, by abuse of notation, we have also denoted the above embedding by g. We have the sequence

$$s_g : \mathcal{O}_X^{\oplus r} \xrightarrow{g} \mathcal{O}_X^{\oplus n} \xrightarrow{s} F.$$

The morphism s_g is determined by $g \in \mathbb{G}(r, n)$ up to the $\mathrm{GL}_r(\mathbb{C})$-action on $\mathcal{O}_X^{\oplus r}$. Note that if s_g is injective, then $Z(s_g)$ is a divisor on X which does not depend on a choice of a lift (76). The divisor $Z(s_g)$ fits into the exact sequence

$$0 \to \bigwedge^r F^\vee \to \mathcal{O}_X \to \mathcal{O}_{Z(s_g)} \to 0.$$

When $X = \mathbb{G}(n-r, n)$ and s is a universal rank r quotient, s_g is injective and $H_g := Z(s_g)$ is a divisor in $\mathbb{G}(n-r, n)$.

Lemma 4.2. *Let $\mathcal{O}_C^{\oplus n} \xrightarrow{q} Q$ be an ϵ-stable quotient with kernel S and marked points $p_1, \ldots, p_m$. Let $s : \mathcal{O}_C^{\oplus n} \to S^\vee$ be the dual of the inclusion $S \hookrightarrow \mathcal{O}_C^{\oplus n}$. Then for a general choice of $g \in \mathbb{G}(r, n)$, the degenerate locus $Z(s_g) \subset C$ is a divisor written as*

$$Z(s_g) = Z(s) + D_g.$$

Here D_g is a reduced divisor on C satisfying

$$D_g \cap \{Z(s) \cup \{p_1, \ldots, p_m\}\} = \emptyset.$$

Proof. Let $F \subset S^\vee$ be the image of s. Note that F is a locally free sheaf of rank r; hence, it determines a map

$$\pi_F : C \to \mathbb{G}(n-r, n).$$

It is easy to see that a general $g \in \mathbb{G}(r, n)$ satisfies the following.

- The divisor $H_g \subset \mathbb{G}(n-r, n)$ intersects the image of π_F transversally. (Or the intersection is empty if $\pi_F(C)$ is a point.)
- For $p \in \mathrm{Supp}\,\tau(Q) \cup \{p_1, \ldots, p_m\}$, we have $\pi_F(p) \notin H_g$.

Then we have

$$Z(s_g) = Z(s) + \pi_F^* H_g,$$

and $D_g := \pi_F^* H_g$ satisfies the desired property. □

In the next proposition, we show that the moduli space of ϵ-stable quotients is separated. Let Δ be a non-singular curve with a closed point $0 \in \Delta$. We set

$$\Delta^* = \Delta \setminus \{0\}.$$

Proposition 4.3. *For $i = 1, 2$, let $\pi_i : \mathcal{X}_i \to \Delta$ be flat families of quasi-stable curves with disjoint sections $p_1^{(i)}, \ldots, p_m^{(i)} : \Delta \to \mathcal{X}_i$. Let $q_i : \mathcal{O}_{\mathcal{X}_i}^{\oplus n} \twoheadrightarrow \mathcal{Q}_i$ be flat families of ϵ-stable quotients of type (r, n, d) which are isomorphic over Δ^*. Then, possibly after base change ramified over 0, there are an isomorphism $\phi : \mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ over Δ and an isomorphism $\psi : \phi^* \mathcal{Q}_2 \xrightarrow{\sim} \mathcal{Q}_1$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}_1}^{\oplus n} & \xrightarrow{\phi^* q_2} & \phi^* \mathcal{Q}_2 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \psi} \\ \mathcal{O}_{\mathcal{X}_1}^{\oplus n} & \xrightarrow{q_1} & \mathcal{Q}_1 \end{array}$$

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Proof. Since the relative Quot scheme is separated, it is enough to show that the isomorphism over Δ^* extends to the families of marked curves $\pi_i : \mathcal{X}_i \to \Delta$. By taking the base change and the normalization, we may assume that the general fibers of π_i are non-singular irreducible curves, by adding the preimage of the nodes to the marking points. Let us take exact sequences

$$0 \to \mathcal{S}_i \to \mathcal{O}_{\mathcal{X}_i}^{\oplus n} \to \mathcal{Q}_i \to 0.$$

Since $\mathcal{S}_i|_{\mathcal{X}_{i,t}}$ is locally free for any $t \in \Delta$, where $\mathcal{X}_{i,t} := \pi_i^{-1}(t)$, the sheaf $\mathcal{S}_i$ is a locally free sheaf on $\mathcal{X}_i$. Taking the dual, we obtain the morphism

$$s_i : \mathcal{O}_{\mathcal{X}_i}^{\oplus n} \to \mathcal{S}_i^{\vee}.$$

Let us take a general point $g \in \mathbb{G}(r, n)$ and the degenerate locus

$$D_i := Z(s_{i,g}) \subset \mathcal{X}_i.$$

By Lemma 4.2, the divisor $D_{i,t} := D_i|_{\mathcal{X}_{i,t}}$ is written as

$$D_{i,t} = Z(s_{i,t}) + D_{i,t}^{\circ},$$

where $D_{i,t}^{\circ}$ is a reduced divisor on $\mathcal{X}_{i,t}$ satisfying

$$D_{i,t}^{\circ} \cap \{Z(s_{i,t}) \cup \{p_1(t), \ldots, p_m(t)\}\} = \emptyset.$$

Then the ϵ-stability of $\mathcal{O}_{\mathcal{X}_{i,t}}^{\oplus n} \stackrel{q_{i,t}}{\twoheadrightarrow} \mathcal{Q}_i|_{\mathcal{X}_{i,t}}$ implies the following.

- The coefficients of the $\mathbb{R}$-divisor $\sum_{j=1}^{m} p_j^{(i)}(t) + \epsilon \cdot D_{i,t}$ are less than or equal to 1.
- The $\mathbb{R}$-divisor $K_{\mathcal{X}_{i,t}} + \sum_{j=1}^{m} p_j^{(i)}(t) + \epsilon \cdot D_{i,t}$ is ample on $\mathcal{X}_{i,t}$.

The first condition implies that the pairs

$$\left(\mathcal{X}_i, \sum_{j=1}^{m} p_j^{(i)} + \epsilon \cdot D_i\right), \quad i = 1, 2, \tag{77}$$

have only log canonical singularities. (See [KMM87, KM98].) Also, since the divisors $\sum_{j=1}^{m} p_j^{(i)} + \epsilon \cdot D_i$ do not contain curves supported on the central fibers, we have

$$\phi_*\left(\sum_{j=1}^{m} p_j^{(1)} + \epsilon \cdot D_1\right) = \sum_{j=1}^{m} p_j^{(2)} + \epsilon \cdot D_2,$$

where ϕ is the birational map $\phi : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2$. Therefore, the pairs (77) are birational log canonical models over Δ. Since two birational log canonical models are isomorphic, the birational map ϕ extends to an isomorphism $\phi : \mathcal{X}_1 \xrightarrow{\cong} \mathcal{X}_2$. □

Finally, we show that the moduli space $\overline{Q}_{g,m}^{\epsilon}(\mathbb{G}(r, n), d)$ is complete.

Proposition 4.4. *Suppose that the following data is a flat family of m-pointed ϵ-stable quotients of type (r, n, d) over Δ^*:*

$$\pi^* : \mathcal{X}^* \to \Delta^*, \quad p_1^*, \ldots, p_m^* : \Delta^* \to \mathcal{X}^*, \quad q^* : \mathcal{O}_{\mathcal{X}^*}^{\oplus n} \twoheadrightarrow \mathcal{Q}^*. \tag{78}$$

Then, possibly after base change ramified over $0 \in \Delta$, there is a flat family of m-pointed ϵ-stable quotients over Δ,

$$\pi : \mathcal{X} \to \Delta, \quad p_1, \ldots, p_m : \Delta \to \mathcal{X}, \quad q : \mathcal{O}_{\mathcal{X}}^{\oplus n} \twoheadrightarrow \mathcal{Q}, \tag{79}$$

which is isomorphic to (78) over Δ^*.

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Proof. As in the proof of Proposition 4.3, we may assume that the general fibers of π^* are non-singular irreducible curves. Let $\mathcal{S}^*$ be the kernel of q^*. Taking the dual of the inclusion $\mathcal{S}^* \subset \mathcal{O}_{\mathcal{X}^*}^{\oplus n}$, we obtain the morphism

$$s^* : \mathcal{O}_{\mathcal{X}^*}^{\oplus n} \to \mathcal{S}^{*\vee}.$$

We choose a general point

$$g \in \mathbb{G}(r, n), \tag{80}$$

and set $D^* := Z(s_g^*) \subset \mathcal{X}^*$. As in the proof of Proposition 4.3, the ϵ-stability implies that the pair

$$\left(\mathcal{X}^*, \sum_{j=1}^{m} p_j^* + \epsilon \cdot D^*\right) \tag{81}$$

is a log canonical model over Δ^*.

Indeed, the family (81) can be interpreted as a family of Hassett's weighted pointed stable curves [Has03]. Let us write

$$D^* = \sum_{j=1}^{k} m_j D_j^*$$

for distinct irreducible divisors D_j^* and $m_j \geqslant 1$. Since the family (78) is of type (r, n, d), we have

$$m_1 + m_2 + \cdots + m_k = d.$$

By shrinking Δ if necessary, we may assume that each D_j^* is a section of π^*. Then the data

$$\left(\pi^* : \mathcal{X}^* \to \Delta^*, p_1^*, \ldots, p_m^*, \overbrace{D_1^*, \ldots, D_1^*}^{m_1}, \ldots, \overbrace{D_k^*, \ldots, D_k^*}^{m_k}\right) \tag{82}$$

is a family of $a(m, d, \epsilon)$-stable $m + d$-pointed curves [Has03] over Δ^*. (See Definition 3.1 and (60).) By the properness of $\overline{M}_{g,m|d}^{\epsilon}$ (cf. [Has03, (61)]), there is a family of $a(m, d, \epsilon)$-stable $m + d$-pointed curves over Δ,

$$\left(\pi : \mathcal{X} \to \Delta, p_1, \ldots, p_m, \overbrace{D_1, \ldots, D_1}^{m_1}, \ldots, \overbrace{D_k, \ldots, D_k}^{m_k}\right), \tag{83}$$

which is isomorphic to the family (82) over Δ^*. In particular, we have an extension of D^* to $\mathcal{X}$,

$$D = \sum_{j=1}^{k} m_j D_j, \quad D|_{\mathcal{X}^*} = D^*.$$

By the properness of the relative Quot scheme, there is an exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathcal{X}}^{\oplus n} \xrightarrow{q} \mathcal{Q} \to 0 \tag{84}$$

such that q is isomorphic to q^* over Δ^*. Restricting to $\mathcal{X}_0$, we obtain the exact sequence

$$0 \to \mathcal{S}_0 \to \mathcal{O}_{\mathcal{X}_0}^{\oplus n} \xrightarrow{q_0} \mathcal{Q}_0 \to 0. \tag{85}$$

We claim that the quotient q_0 is an ϵ-stable quotient and hence the family $(\mathcal{X}, p_1, \ldots, p_m)$ and q gives a desired extension (79). We prove the following lemma.

Lemma 4.5. *The sheaf $\mathcal{S}$ is a locally free sheaf on $\mathcal{X}$.*

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Proof. First we see that the sheaf $\mathcal{S}$ is reflexive, i.e. $\mathcal{S}^{\vee\vee} \cong \mathcal{S}$. We have the morphism of exact sequence of sheaves on $\mathcal{X}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{O}_{\mathcal{X}}^{\oplus n} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S}^{\vee\vee} & \longrightarrow & \mathcal{O}_{\mathcal{X}}^{\oplus n} & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \end{array}$$

where the left-hand arrow is an injection. By the snake lemma, there is an inclusion

$$\mathcal{S}^{\vee\vee}/\mathcal{S} \hookrightarrow \mathcal{Q},$$

and $\mathcal{S}^{\vee\vee}/\mathcal{S}$ is supported on $\mathcal{X}_0$, which contradicts that $\mathcal{Q}$ is flat over Δ. In particular, setting

$$U = \mathcal{X} \backslash (\text{nodes of } \mathcal{X}_0),$$

the sheaf $\mathcal{S}$ is a push-forward of some locally free sheaf on U to $\mathcal{X}$. We only need to check that $\mathcal{S}$ is free at nodes on $\mathcal{X}_0$.

Taking the dual of the inclusion $\mathcal{S} \hookrightarrow \mathcal{O}_{\mathcal{X}}^{\oplus n}$ and composing with $g : \mathcal{O}_{\mathcal{X}}^{\oplus r} \hookrightarrow \mathcal{O}_{\mathcal{X}}^{\oplus n}$, where g is taken in (80), we obtain a morphism

$$s_g : \mathcal{O}_{\mathcal{X}}^{\oplus r} \stackrel{g}{\hookrightarrow} \mathcal{O}_{\mathcal{X}}^{\oplus n} \to \mathcal{S}^{\vee}.$$

Restricting to U, we obtain the divisor in U,

$$D_U^{\dagger} := Z(s_g|_U) \subset U,$$

and the closure of $D_U^{\dagger}$ in $\mathcal{X}$ is denoted by $D^{\dagger}$. We have the following.

- By the construction, we have $D|_{\mathcal{X}^*} = D^{\dagger}|_{\mathcal{X}^*}$.
- Replacing g by another general point in $\mathbb{G}(r, n)$ if necessary, the divisors $D^{\dagger}$ and D do not contain any irreducible component of $\mathcal{X}_0$.

These properties imply that $D^{\dagger} = D$. Noting that the divisor D has support away from nodes of $\mathcal{X}_0$, the support of the cokernel of s_g is written as

$$\operatorname{Supp} \operatorname{Cok}(s_g) = \operatorname{Supp}(D) \amalg V, \tag{86}$$

where V is a finite set of points contained in the nodes of $\mathcal{X}_0$. However, if V is non-empty, then there are a nodal point $x \in \mathcal{X}_0$ and an injection $\mathcal{O}_x \hookrightarrow \mathcal{S}^{\vee}$, which contradict that $\mathcal{S}$ is torsion free. Therefore, V is empty, and the morphism s_g is isomorphic on nodes of $\mathcal{X}_0$. Hence, $\mathcal{S}^{\vee}$ is a locally free sheaf on $\mathcal{X}_0$, and the sheaf $\mathcal{S}$ is also locally free since $\mathcal{S} \cong \mathcal{S}^{\vee\vee}$. $\square$

Note that the local freeness of $\mathcal{S}$ implies that the divisor $Z(s_g)$ is well defined, and the proof of the above lemma immediately implies that

$$Z(s_g) = D^{\dagger} = D. \tag{87}$$

Next let us see that q_0 is a quasi-stable quotient. Taking $\mathcal{H}om(*, \mathcal{O}_{\mathcal{X}_0})$ to the exact sequence (85), we obtain the exact sequence

$$0 \to \mathcal{Q}_0^{\vee} \to \mathcal{O}_{\mathcal{X}_0}^{\oplus n} \xrightarrow{s_0} \mathcal{S}_0^{\vee} \to \mathcal{E}xt^1_{\mathcal{X}_0}(\mathcal{Q}_0, \mathcal{O}_{\mathcal{X}_0}) \to 0,$$

and the vanishing $\mathcal{E}xt^i_{\mathcal{X}_0}(\mathcal{Q}_0, \mathcal{O}_{\mathcal{X}_0}) = 0$ for $i \geqslant 2$. We have the surjection

$$\operatorname{Cok}(s_{0,g}) \twoheadrightarrow \mathcal{E}xt^1_{\mathcal{X}_0}(\mathcal{Q}_0, \mathcal{O}_{\mathcal{X}_0}), \tag{88}$$

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and the left-hand side of (88) has support away from nodes and markings by (87). Therefore, for a nodal point or marked point $p \in \mathcal{X}_0$, we have

$$\mathcal{E}xt^i_{\mathcal{X}_0}(\mathcal{Q}_0, \mathcal{O}_{\mathcal{X}_0})_p = 0, \quad i \geqslant 1,$$

which implies that $\mathcal{Q}_0$ is locally free at p, i.e. $q_0 : \mathcal{O}_{\mathcal{X}_0}^{\oplus n} \twoheadrightarrow \mathcal{Q}_0$ is a quasi-stable quotient.

Finally, we check the ϵ-stability of q_0. The ampleness of $\mathcal{L}(q_0, \epsilon)$ is equivalent to the ampleness of the divisor

$$K_{\mathcal{X}_0} + p_1(0) + \cdots + p_m(0) + \epsilon \cdot Z(s_{g,0}). \tag{89}$$

Noting the equality (87), we have $Z(s_{g,0}) = D|_{\mathcal{X}_0}$. Since the data (83) is a family of $a(m, d, \epsilon)$-stable curves, the divisor (89) on $\mathcal{X}_0$ is ample. Also, the surjection (88) and the fact $Z(s_{g,0}) = D|_{\mathcal{X}_0}$ imply that

$$\begin{aligned} \epsilon \cdot \text{length}\, \tau(\mathcal{Q}_0)_p &\leqslant \epsilon \cdot \text{length}\, \text{Cok}(s_{g,0})_p \\ &= \epsilon \cdot \text{length}\, \mathcal{O}_{Z_{s_{g,0}},p}, \\ &= \epsilon \cdot \text{length}\, \mathcal{O}_{D|_{\mathcal{X}_0},p} \end{aligned} \tag{90}$$

for any $p \in \mathcal{X}_0$. Again noting that (83) is a family of $a(m, d, \epsilon)$-stable curves, we conclude that (90) $\leqslant 1$. Therefore, q_0 is an ϵ-stable quotient. □

5. Wall-crossing formula

The purpose of this section is to give an argument to prove Theorem 2.25. Our strategy is to modify [MOP09, § 7] so that ϵ is involved in the argument. Therefore, we only focus on the arguments to be modified, and we leave several details to the reader.

5.1 Localization

Let T be a torus $T = \mathbb{G}_m^n$ acting on $\mathbb{C}^n$ via

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n).$$

The above T-action induces a T-action on $\mathbb{G}(r, n)$ and $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r, n), d)$. Over the moduli space of MOP-stable quotients, the T-fixed loci are obtained in [MOP09, § 7] via certain combinatorial data. The T-fixed loci of ϵ-stable quotients are similarly obtained, but we need to take the ϵ-stability into consideration. They are indexed by the following data:

$$\theta = (\Gamma, \iota, \gamma, s, \beta, \delta, \mu). \tag{91}$$

- $\Gamma = (V, E)$ is a connected graph, where V is the vertex set and E is the edge set with no self edges.
- ι is an assignment of an inclusion

  $$\iota_v : \{1, \ldots, r\} \to \{1, \ldots, n\}$$

  to each $v \in V$. In particular, the induced subspace $\mathbb{C}^r \hookrightarrow \mathbb{C}^n$ by ι_v determines a map

  $$\nu : V \to \mathbb{G}(r, n)^T.$$

- γ is a genus assignment $\gamma : V \to \mathbb{Z}_{\geqslant 1}$ satisfying

  $$\sum_{v \in V} \gamma(v) + h^1(\Gamma) = g.$$

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- For each $v \in V$, $s(v) = (s_1(v), \dots, s_r(v))$ with $s_i(v) \in \mathbb{Z}_{\geqslant 0}$. We set
$$\mathbf{s}(v) = \sum_{i=1}^{r} s_i(v).$$
- β is an assignment to each $e \in E$ of a T-invariant curve $\beta(e)$ of $\mathbb{G}(r,n)$. The two vertices incident to $e \in E$ are mapped via ν to the two T-fixed points incident to $\beta(e)$.
- $\delta : E \to \mathbb{Z}_{\geqslant 1}$ is an assignment of a covering number satisfying
$$\sum_{v \in V} \mathbf{s}(v) + \sum_{e \in E} \delta(e) = d.$$
- μ is a distribution of the m-markings to the vertices of V.
- For each $v \in V$, we set
$$w(v) = \min\{0, 2\gamma(v) - 2 + \epsilon \cdot \mathbf{s}(v) + \mathrm{val}(v)\}.$$
Then for each edge $e \in E$ with incident vertex $v_1, v_2 \in V$, we have
$$\epsilon \cdot \delta(e) + w(v_1) + w(v_2) > 0. \tag{92}$$

The condition (92) corresponds to the ampleness of (12) at the irreducible component determined by e. Given a data θ as in (91), the isomorphism classes of T-fixed ϵ-stable quotients indexed by θ form a product of the quotients of the moduli spaces of weighted pointed stable curves,
$$Q^T(\theta) = \prod_{v \in V} \left(\overline{M}^{\epsilon}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)} \Big/ \prod_{i=1}^{r} S_{s_i(v)} \right). \tag{93}$$
Here, if $v \in V$ does not satisfy the condition
$$2\gamma(v) - 2 + \epsilon \cdot s(v) + \mathrm{val}(v) > 0, \tag{94}$$
we set
$$\overline{M}^{\epsilon}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)} = \begin{cases} \mathrm{Spec}\,\mathbb{C}, & V \neq \{v\}, \\ \emptyset, & V = \{v\}. \end{cases}$$
The corresponding T-fixed ϵ-stable quotients are described in the following way.

- For $v \in V$, suppose that the condition (94) holds. A point in the v-factor of (93) determines a curve C_v and an r-tuple of divisors on it $D_1, \dots, D_r$ with $\deg(D_i) = s_i(v)$. Then an ϵ-stable quotient is obtained by the exact sequence
$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{C_v}(-D_i) \to \mathcal{O}_{C_v}^{\oplus n} \to Q \to 0. \tag{95}$$
Here the first inclusion is the composition of the natural inclusion
$$\bigoplus_{i=1}^{r} \mathcal{O}_{C_v}(-D_i) \hookrightarrow \mathcal{O}_{C_v}^{\oplus r}$$
and the inclusion $\mathcal{O}_{C_v}^{\oplus r} \hookrightarrow \mathcal{O}_{C_v}^{\oplus n}$ induced by ι_v.
- For $e \in E$, consider the degree $\delta(e)$-covering ramified over the two torus fixed points,
$$f_e : C_e \to \beta(e) \subset \mathbb{G}(r,n). \tag{96}$$

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Note that f_e is a finite map between projective lines. We obtain the exact sequence

$$0 \to S \to \mathcal{O}_{C_e}^{\oplus n} \xrightarrow{q} Q \to 0, \tag{97}$$

and hence a quotient q, by pulling back the tautological sequence on $\mathbb{G}(r,n)$ to C_e. Let v and v' be the two vertices incident to e, and $x, x' \in C_e$ the corresponding ramification points, respectively. We have the following cases.

(i) Suppose that both of v and v' satisfy (94). Then we take the quotient q.

(ii) Suppose that exactly one of v or v', say v, does not satisfy (94). For simplicity, we assume that $\iota_v(j) = j$ for $1 \leqslant j \leqslant n$, and

$$\iota_{v'}(j) = j, \quad 1 \leqslant j \leqslant r-1, \quad \iota_{v'}(r) = r+1.$$

Then the exact sequence (97) is identified with the sequence

$$0 \to \mathcal{O}_{C_e}^{\oplus r-1} \oplus \mathcal{O}_{C_e}(-\delta(e)) \to \mathcal{O}_{C_e}^{\oplus n} \to \mathcal{O}_{C_e}(\delta(e)) \oplus \mathcal{O}_{C_e}^{\oplus n-r-1} \to 0. \tag{98}$$

Here the embedding

$$\mathcal{O}_{C_e}^{\oplus r-1} \oplus \mathcal{O}_{C_e}(-\delta(e)) \subset \mathcal{O}_{C_e}^{\oplus n}$$

is the composition

$$\mathcal{O}_{C_e}^{\oplus r-1} \oplus \mathcal{O}_{C_e}(-\delta(e)) \subset \mathcal{O}_{C_e}^{\oplus r-1} \oplus \mathcal{O}_{C_e}^{\oplus 2} \subset \mathcal{O}_{C_e}^{\oplus n},$$

where the first embedding is the direct sum of the identity and the pull-back of the tautological embedding via f_e, and the second one is the embedding into the first $r+1$ factors. Composing the embedding

$$0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}_{C_e}(-s_i(v)x) \oplus \mathcal{O}_{C_e}(-s_r(v)x - \delta(e)) \to \mathcal{O}_{C_e}^{\oplus r-1} \oplus \mathcal{O}_{C_e}(-\delta(e))$$

with the sequence (98), we obtain the exact sequence

$$0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}_{C_e}(-s_i(v)x) \oplus \mathcal{O}_{C_e}(-s_r(v)x - \delta(e)) \to \mathcal{O}_{C_e}^{\oplus n} \xrightarrow{q'} Q' \to 0.$$

Then we take the quotient q'.

(iii) Suppose that both v and v' do not satisfy (94). Then, as above, we take the exact sequence

$$0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}_{C_e}(-s_i(v)x - s_i(v')x') \oplus \mathcal{O}_{C_e}(-s_r(v)x - s_r(v')x' - \delta(e))$$
$$\to \mathcal{O}_{C_e}^{\oplus n} \xrightarrow{q''} Q'' \to 0,$$

and we take the quotient q''.

By gluing the above quotients, we obtain a curve C and a quotient from $\mathcal{O}_C^{\oplus n}$. The condition (92) ensures that the resulting quotient is ϵ-stable.

5.2 Virtual localization formula

Let $Q^T(\theta)$ be the T-fixed locus (93), and i_θ the inclusion

$$i_\theta : Q^T(\theta) \hookrightarrow \overline{Q}_{g,m}^{\epsilon}(\mathbb{G}(r,n), d).$$

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We denote by $N^{\mathrm{vir}}(\theta)$ the virtual normal bundle of $Q^T(\theta)$ in $\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)$. The virtual localization formula [GP99] in this case is written as

$$[\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} = \sum_{\theta} i_{\theta !}\bigg(\frac{[Q^T(\theta)]}{\mathbf{e}(N^{\mathrm{vir}}(\theta))}\bigg) \in A_*^T(\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d),\mathbb{Q}) \otimes_R \mathbb{Q}(\lambda_1,\ldots,\lambda_n). \tag{99}$$

Here R is the equivariant Chow ring of a point with respect to the trivial T-action,

$$R = \mathbb{Q}[\lambda_1,\ldots,\lambda_n],$$

with λ_i equivariant parameters.

Let v be a vertex in the data (91) which satisfies the condition (94). We see the contribution of v to the right-hand side of (99). For simplicity, we assume that $\iota_v(j)=j$ for $1\leqslant j\leqslant r$. The vertex v corresponds to the space

$$\overline{M}^{\epsilon}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)} \Big/ \prod_{i=1}^{r} S_{s_i(v)}.$$

Similarly to the sequence (95), each point on the above space corresponds to an exact sequence

$$0 \to S = \bigoplus_{i=1}^{r} S_i \to \mathcal{O}_C^{\oplus n} \to Q \to 0$$

for $S_i = \mathcal{O}_{C_v}(-D_i)$ with $\deg(D_i)=s_i(v)$, and $\mathrm{val}(v)$-marked points. The exact sequence (28) and the argument of [MOP09, § 7] show that the contribution of the vertex v is

$$\mathrm{Cont}(v) = \frac{\mathbf{e}(\mathbb{E}^*\otimes T_{\nu(v)})}{\mathbf{e}(T_{\nu(v)})} \frac{1}{\prod_e (\lambda(e)/\delta(e)) - \psi_e} \tag{100}$$

$$\times \frac{1}{\prod_{i\neq j} \mathbf{e}(H^0(O_C(S_i)|_{S_j}) \otimes [\lambda_j - \lambda_i])} \tag{101}$$

$$\times \frac{1}{\prod_{i\neq j^*} \mathbf{e}(H^0(O_C(S_i)|_{S_i}) \otimes [\lambda_{j^*} - \lambda_i])}. \tag{102}$$

Here each factor is as follows.

- The symbol $\mathbf{e}$ denotes the Euler class, $T_{\nu(v)}$ is the T-representation on the tangent space of $\mathbb{G}(r,n)$ at $\nu(v)$ and $\mathbb{E}$ is the Hodge bundle
$$\mathbb{E} \to \overline{M}^{\epsilon}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)}. \tag{103}$$
- The product in the denominator of (100) is over all half edges e incident to v. The factor $\lambda(e)$ denotes the T-weight of the tangent representation along the corresponding T-fixed edge, and ψ_e is the first Chern class of the cotangent line at the corresponding marking of $\overline{M}^{\epsilon}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)}$. (See (104) below.)
- The products in (101) and (102) satisfy the following conditions:
$$1\leqslant i\leqslant r, \quad 1\leqslant j\leqslant r, \quad r+1\leqslant j^*\leqslant n.$$
The brackets $[\lambda_j-\lambda_i]$ denote the trivial bundle with specified weights.

The same argument describing $\mathrm{Cont}(v)$ as above is also applied to see the contribution term of the edge e to the formula (99). However, we do not need to know its precise formula, and it

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is enough to notice that

$$\mathrm{Cont}(e) \in \mathbb{Q}(\lambda_1, \ldots, \lambda_n).$$

The above fact is easily seen by the description of the T-fixed ϵ-stable quotients in the last subsection. Then the right-hand side of (99) is the sum of the products

$$\sum_{\theta} i_{\theta!}\bigg(\prod_{e} \mathrm{Cont}(e) \prod_{v} \mathrm{Cont}(v)[Q^T(\theta)]\bigg).$$

5.3 Classes on $\overline{M}^{\epsilon}_{g,m|d}$

As we have seen, each term of the virtual localization formula is a class on the moduli space of weighted pointed stable curves $\overline{M}^{\epsilon}_{g,m|d}$. The relevant classes on $\overline{M}^{\epsilon}_{g,m|d}$ for a sufficiently small ϵ are discussed in [MOP09, § 4]. For arbitrary $0 < \epsilon \leqslant 1$, the similar classes are also available, which we recall here.

For every subset $J \subset \{1, \ldots, d\}$ of size at least 2, there is a diagonal class

$$D_J \in A^{|J|-1}(\overline{M}^{\epsilon}_{g,m|d}, \mathbb{Q})$$

corresponding to the weighted pointed stable curves

$$(C, p_1, \ldots, p_m, \widehat{p}_1, \ldots, \widehat{p}_d)$$

satisfying

$$\widehat{p}_j = \widehat{p}_{j'}, \quad j, j' \in J.$$

Note that $D_J = 0$ if $\epsilon \cdot |J| > 1$.

Next we have the cotangent line bundles

$$\mathbb{L}_i \to \overline{M}^{\epsilon}_{g,m|d}, \quad \widehat{\mathbb{L}}_j \to \overline{M}^{\epsilon}_{g,m|d}$$

for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant d$, corresponding to the respective markings. We have the associated first Chern classes

$$\psi_i = c_1(\mathbb{L}_i), \quad \widehat{\psi}_j = c_1(\widehat{\mathbb{L}}_j) \in A^1(\overline{M}^{\epsilon}_{g,m|d}, \mathbb{Q}). \tag{104}$$

The above classes are related as follows. For a subset $J \subset \{1, \ldots, d\}$, the class

$$\widehat{\psi}_J := \widehat{\psi}_j|_{D_J} \tag{105}$$

does not depend on $j \in J$. If J and J' have non-trivial intersections, it is easy to see that

$$D_J \cdot D_{J'} = (-\widehat{\psi}_{J\cup J'})^{|J\cap J'|-1} D_{J\cup J'}. \tag{106}$$

By the above properties, we obtain the notion of *canonical forms* (cf. [MOP09, § 4]), for any monomial $M(\widehat{\psi}_j, D_J)$ of $\widehat{\psi}_j$ and D_J. It is obtained as follows.

- We multiply the classes D_J using the formula (106) until we obtain the product of classes $\widehat{\psi}_j$ and $D_{J_1} D_{J_2} \cdots D_{J_l}$ with all J_i disjoint.
- Using (105), we collect the equal cotangent classes.

By extending the above operation linearly, we obtain the canonical form for any polynomial $P(\widehat{\psi}_j, D_J)$.

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5.4 Standard classes under change of ϵ

For $\epsilon \geqslant \epsilon'$, recall that there is a birational morphism (cf. (62) and Remark 3.3)

$$c_{\epsilon,\epsilon'} : \overline{M}^{\epsilon}_{g,m|d} \to \overline{M}^{\epsilon'}_{g,m|d}.$$

For simplicity, we write $c_{\epsilon,\epsilon'}$ as c. Then we have the following:

$$c^*\psi_i = \psi_i, \quad 1 \leqslant i \leqslant m, \tag{107}$$
$$c^*\widehat{\psi}_j = \widehat{\psi}_j - \Delta_j, \quad 1 \leqslant j \leqslant d. \tag{108}$$

Here Δ_j is given by

$$\Delta_j = \sum_{j \in J \subset \{1,\dots,d\}} \Delta_J, \tag{109}$$

where $\Delta_J \subset \overline{M}^{\epsilon}_{g,m|d}$ correspond to curves

$$C = C_1 \cup C_2, \quad g(C_1) = 0, \quad g(C_2) = g$$

with a single node which separates C_1 and C_2, and the markings of J are distributed to C_1. The subsets J in the sum (109) should satisfy

$$\epsilon \cdot |J| - 1 > 0,$$
$$\epsilon' \cdot |J| - 1 \leqslant 0.$$

Applying (107) and (108) and the projection formula, we obtain the universal formula

$$c_*\left(\prod_{i=1}^{m} \psi_i^{m_i} \prod_{j=1}^{d} \widehat{\psi}_j^{n_j}\right) = \prod_{i=1}^{m} \psi_i^{m_i}\left(\prod_{j=1}^{d} \widehat{\psi}_j^{n_j} + \cdots\right). \tag{110}$$

If $\epsilon = 1$ and $0 < \epsilon' \ll 1$, the above formula coincides with the formula obtained in [MOP09, Lemma 3].

Also, the Hodge bundle (103) satisfies

$$c^*\mathbb{E} \cong \mathbb{E}, \tag{111}$$

since c contracts only rational tails.

5.5 The case of genus zero

In genus zero, note that the moduli space

$$\overline{Q}^{\epsilon}_{0,m}(\mathbb{G}(r,n),d)$$

is non-singular by Lemma 2.15. If it is also connected, then it is irreducible and there is a birational map

$$\overline{Q}^{\epsilon_1}_{0,m}(\mathbb{G}(r,n),d) \dashrightarrow \overline{Q}^{\epsilon_2}_{0,m}(\mathbb{G}(r,n),d)$$

as long as $\epsilon_i > (2-m)/d$. In fact, we have the following.

LEMMA 5.1. *The moduli stack*

$$\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d) \tag{112}$$

is connected.

Proof. The connectedness of the stable map moduli spaces is proved in [KP01], and we reduce the connectedness of (112) to that of the stable map moduli spaces. To do this, it is enough to

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see that any ϵ-stable quotient $q : \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$ is deformed to a quotient obtained by a stable map. By applying the T-action, we may assume that q is a T-fixed quotient. Then q fits into an exact sequence

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_C(-D_i) \to \mathcal{O}_C^{\oplus n} \xrightarrow{q} Q \to 0$$

for r-tuple divisors D_i on C. (See § 5.1.) By deforming D_i to reduced divisors D_i', we can deform the quotient q to $q' : \mathcal{O}_C \twoheadrightarrow Q'$, which is ϵ-stable for $\epsilon = 1$. Then, by Lemma 2.24, we can deform q' to a quotient corresponding to a stable map. □

The smoothness of the genus zero moduli spaces and the above lemma show the formula

$$c_{\epsilon,\epsilon' *}\iota_*^{\epsilon}([\overline{Q}_{0,m}^{\epsilon}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}) = \iota_*^{\epsilon'}([\overline{Q}_{0,m}^{\epsilon'}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}). \tag{113}$$

Hence, Theorem 2.25 in the genus zero case is proved.

5.6 Sketch of the proof of Theorem 2.25

Under the map to $\overline{Q}_{g,m}^{\epsilon'}(\mathbb{G}(1,\binom{n}{r}),d)$, several rational tails on $\overline{Q}_{g,m}^{\epsilon}(\mathbb{G}(r,n),d)$ with small degree collapse. Also, the T-fixed *loci* of $\overline{Q}_{g,m}^{\epsilon}(\mathbb{G}(r,n),d)$ have many splitting types of the subbundle S which are collapsed. For a non-collapsed edge, its contribution exactly coincides, and we just need to show the matching on each vertex.

The equality (113) implies that both sides are equal after T-equivariant localization. For each vertex v on $\overline{Q}_{0,m}^{\epsilon}(\mathbb{G}(r,n),d)$, the contribution

$$\mathrm{Cont}(v) \in A_*^T(\overline{M}_{0,\mathrm{val}(v)|\mathbf{s}(v)}^{\epsilon}, \mathbb{Q}) \otimes_R \mathbb{Q}(\lambda_1,\ldots,\lambda_n)$$

is given in § 5.2. In genus zero the Hodge bundle is trivial, and the class $\mathrm{Cont}(v)$ is easily seen to be written as an element

$$\mathrm{Cont}(v) \in \mathbb{Q}(\lambda_1,\ldots,\lambda_n)[\psi_i, \widehat{\psi}_j, D_J]$$

symmetric with respect to the variables $\widehat{\psi}_j$. Let us take the push-forward to $\overline{Q}_{0,m}^{\epsilon'}(\mathbb{G}(1,\binom{n}{r}),d)$ using (110), and take the canonical form (cf. § 5.3). At each vertex on $\overline{Q}_{0,m}^{\epsilon'}(\mathbb{G}(1,\binom{n}{r}),d)$, the vertices and the collapsed edges on $\overline{Q}_{0,m}^{\epsilon}(\mathbb{G}(r,n),d)$ contribute to the left-hand side of (113) by the polynomial

$$L^C(\psi_i, \widehat{\psi}_j, D_J).$$

Also, the vertices on $\overline{Q}_{0,m}^{\epsilon'}(\mathbb{G}(r,n),d)$ with collapsed splitting types contribute to the right-hand side of (113) by the polynomial

$$R^C(\psi_i, \widehat{\psi}_j, D_J).$$

The equality (113) implies the equality

$$L^C(\psi_i, \widehat{\psi}_j, D_J) = R^C(\psi_i, \widehat{\psi}_j, D_J) \tag{114}$$

as *classes* in the equivariant Chow ring.

Although (114) is an equality after taking classes, exactly the same argument of [MOP09, Lemma 5] shows that the equality (114) holds as *abstract polynomials*. Also note that the genus-dependent part involving Hodge bundles (100) in the virtual localization formula (99) does not

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depend on ϵ by (111). Therefore, the above argument immediately implies that

$$c_{\epsilon,\epsilon'*}\iota^{\epsilon}_{*}([\overline{Q}^{\epsilon}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}) = \iota^{\epsilon'}_{*}([\overline{Q}^{\epsilon'}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}) \tag{115}$$

for any $g \geqslant 0$. Hence, we obtain the formula (57).

6. Invariants on (local) Calabi–Yau 3-folds

In this section, we introduce some enumerative invariants of curves on (local) Calabi–Yau 3-folds and propose related problems. Similar invariants for MOP-stable quotients are discussed in [MOP09, §§ 9 and 10]. In what follows, we use the notation (23) and (24) for universal curves and quotients.

6.1 Invariants on a local (−1, −1)-curve

Let us consider a crepant small resolution of a conifold singularity, that is, the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$,

$$X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1.$$

In a similar way to [MOP09, § 9], we define the $\mathbb{Q}$-valued invariant by

$$N^{\epsilon}_{g,d}(X) := \int_{[\overline{Q}^{\epsilon}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} \mathbf{e}(R^1\pi^{\epsilon}_{*}(S_{U^{\epsilon}}) \oplus R^1\pi^{\epsilon}_{*}(S_{U^{\epsilon}})). \tag{116}$$

It is easy to see that

$$\pi^{\epsilon}_{*}(S_{U^{\epsilon}}) = 0;$$

hence, $R^1\pi^{\epsilon}_{*}(S_{U^{\epsilon}})$ is a vector bundle and (116) is well defined. By Theorem 2.19(i) and Lemma 2.17, we have

$$\begin{aligned} N^{\epsilon}_{g,d}(X) &= N^{\mathrm{GW}}_{g,d}(X), \quad \epsilon > 2,\\ N^{\epsilon}_{0,d}(X) &= 0, \quad 0 < \epsilon \leqslant 2/d. \end{aligned}$$

Here $N^{\mathrm{GW}}_{g,d}(X)$ is the genus g, degree d local GW invariant of X. The following result is obtained by the same method of [MOP09, Propositions 6 and 7], using the localization with respect to the twisted $\mathbb{C}^*$-action on X, and the vanishing result similar to [FP00]. We leave the reader to check the detail.

Theorem 6.1. *We have the following:*

$$N^{\epsilon}_{g,d}(X) = \begin{cases} N^{\mathrm{GW}}_{g,d}(X), & 2g-2+\epsilon\cdot d > 0,\\ 0, & 2g-2+\epsilon\cdot d \leqslant 0. \end{cases}$$

Let $F^{\mathrm{GW}}(X)$ be the generating series

$$F^{\mathrm{GW}}(X) = \sum_{g\geqslant 0, d>0} N^{\mathrm{GW}}_{g,d}(X)\lambda^{2g-2}t^d.$$

Recall that we have the following Gopakumar–Vafa formula:

$$F^{\mathrm{GW}}(X) = \sum_{d\geqslant 1} \frac{t^d}{4d\sin^2(d\lambda/2)}. \tag{117}$$

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By Theorem 6.1 and the formula (117), the generating series of $N^{\epsilon}_{g,d}(X)$ satisfies the formula

$$\begin{aligned} F^{\epsilon}(X) &:= \sum_{g\geqslant 0, d>0} N^{\epsilon}_{g,d}(X)\lambda^{2g-2}t^d \\ &= \sum_{d\geqslant 1} \frac{t^d}{4d\sin^2(d\lambda/2)} - \sum_{0<d\leqslant 2/\epsilon} \frac{1}{d^3}\lambda^{-2}t^d. \end{aligned}$$

6.2 Invariants on a local projective plane

Let us consider the total space of the canonical line bundle of $\mathbb{P}^2$,

$$X = \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2.$$

As in the case of a $(-1,-1)$-curve, we can define the invariant by

$$N^{\epsilon}_{g,d}(X) := \int_{[\overline{Q}^{\epsilon}_{g,0}(\mathbb{P}^2,d)]^{\mathrm{vir}}} \mathbf{e}(R^1\pi^{\epsilon}_*(S^{\otimes 3}_{U^{\epsilon}})) \in \mathbb{Q}$$

since we have the vanishing

$$\pi^{\epsilon}_*(S^{\otimes 3}_{U^{\epsilon}}) = 0.$$

Note that $N^{\epsilon}_{g,d}(X)$ is a local GW invariant of X when $\epsilon > 2$. However, for a small ϵ, the following example shows that $N^{\epsilon}_{g,d}(X)$ is different from the local GW invariant of X.

Example 6.2. For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, an explicit computation shows that

$$N^{\epsilon}_{1,1}(X) = \begin{cases} \frac{1}{4}, & \epsilon > 1, \\ \frac{3}{4}, & 0 < \epsilon \leqslant 1. \end{cases}$$

In fact, if $\epsilon > 1$, then $N^{\epsilon}_{1,1}(X)$ coincides with the local GW invariant of X, and it is already computed. A list is available in [AMV04, Table 1] in a Gopakumar–Vafa form.

Let us compute $N^{\epsilon}_{1,1}(X)$ for $0 < \epsilon \leqslant 1$. In this case, any ϵ-stable quotient of type $(1,3,1)$ is MOP-stable, and the moduli space is described as

$$\overline{Q}^{\epsilon}_{1,0}(\mathbb{P}^2, 1) \cong \overline{M}_{1,1} \times \mathbb{P}^2.$$

(See [MOP09, Example 5.4].) Also, there is no obstruction in this case,

$$[\overline{Q}^{\epsilon}_{1,0}(\mathbb{P}^2, 1)]^{\mathrm{vir}} = [\overline{Q}^{\epsilon}_{1,0}(\mathbb{P}^2, 1)].$$

Let

$$\pi : U \to \overline{M}_{1,1}$$

be the universal curve with a section $D \subset U$. Then

$$U^{\epsilon} = U \times \mathbb{P}^2 \to \overline{Q}^{\epsilon}_{1,0}(\mathbb{P}^2, 1)$$

is the universal curve, and the universal subsheaf $S_{U^{\epsilon}} \subset \mathcal{O}^{\oplus 3}_{U^{\epsilon}}$ is given by

$$S_{U^{\epsilon}} \cong \mathcal{O}_U(-D) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1).$$

Therefore, we have

$$R^1\pi^{\epsilon}_*(S^{\otimes 3}_{U^{\epsilon}}) \cong R^1\pi_*\mathcal{O}_U(-3D) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-3).$$

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The vector bundle $R^1\pi_*\mathcal{O}_U(-3D)$ on $\overline{M}_{1,1}$ admits a filtration whose subquotients are line bundles $\mathbb{E}^\vee$, $\mathbb{L}_1$ and $\mathbb{L}_1^{\otimes 2}$. Therefore, the integration of the Euler class is given by

$$\int_{\overline{Q}^\epsilon_{1,0}(\mathbb{P}^2,1)} \mathbf{e}(R^1\pi^\epsilon_*(S^{\otimes 3}_{U^\epsilon})) = 9\cdot\int_{\overline{M}_{1,1}}(3\psi_1 - c_1(\mathbb{E})),$$
$$= \frac{3}{4}.$$

Here the last equality follows from the computation in [FP00],

$$\int_{\overline{M}_{1,1}} c_1(\mathbb{E}) = \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

By the above example, the following problem seems to be interesting.

Problem 6.3. How do the invariants $N^\epsilon_{g,d}(X)$ depend on ϵ when $X = \mathcal{O}_{\mathbb{P}^2}(-3)$?

6.3 Generalized tree level GW systems on hypersurfaces

Let X be a smooth projective variety, defined by the degree N homogeneous polynomial f of $n+1$ variables,

$$X = \{f=0\} \subset \mathbb{P}^n.$$

Recall that in Lemma 2.15, the moduli stack $\overline{Q}^\epsilon_{0,m}(\mathbb{P}^n, d)$ is shown to be smooth of the expected dimension. We construct the closed substack

$$\overline{Q}^\epsilon_{0,m}(X,d) \subset \overline{Q}^\epsilon_{0,m}(\mathbb{P}^n, d) \tag{118}$$

as follows. For an ϵ-stable quotient of type $(1, n+1, d)$,

$$0 \to S \to \mathcal{O}_C^{\oplus n+1} \to Q \to 0,$$

we take the dual of the first inclusion,

$$(s_0, s_1, \dots, s_n) : \mathcal{O}_C^{\oplus n+1} \to S^\vee.$$

Applying f, we obtain the section

$$f(s_0, s_1, \dots, s_n) \in H^0(C, S^{\otimes -N}). \tag{119}$$

In genus zero, we have the vanishing

$$R^1\pi^*_\epsilon(S^{\otimes -N}_{U^\epsilon}) = 0;$$

hence, (119) determines a section of the vector bundle $\pi^\epsilon_*(S^{\otimes -N}_{U^\epsilon})$, which we denote as

$$s_f \in H^0(\overline{Q}_{0,m}(\mathbb{P}^n, d), \pi^\epsilon_*(S^{\otimes -N}_{U^\epsilon})).$$

Then we define (scheme theoretically)

$$\overline{Q}^\epsilon_{0,m}(X,d) = \{s_f = 0\}. \tag{120}$$

Note that if $\epsilon > 2$, then the above space coincides with the moduli stack of genus zero, degree d stable maps to X. Since (120) is a zero locus of a section of a vector bundle on a smooth stack, there is a perfect obstruction theory on it, determined by the two-term complex

$$(\pi^\epsilon_*(S^{\otimes -N}))^\vee \to \Omega_{\overline{Q}^\epsilon_{0,m}(\mathbb{P}^n,d)}|_{\overline{Q}^\epsilon_{0,m}(X,d)}.$$

The associated virtual class is denoted by

$$[\overline{Q}^\epsilon_{0,m}(X,d)]^{\mathrm{vir}} \in A_*(\overline{Q}^\epsilon_{0,m}(X,d), \mathbb{Q}).$$

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The evaluation map factors through X,

$$\mathrm{ev}_i : \overline{Q}^{\epsilon}_{0,m}(X, d) \to X,$$

for $1 \leqslant i \leqslant m$. Hence we obtain the diagram

$$\begin{array}{ccc} \overline{Q}^{\epsilon}_{0,m}(X,d) & \xrightarrow{\alpha} & \overline{M}_{0,m} \\ {\scriptstyle (\mathrm{ev}_1,\dots,\mathrm{ev}_m)}\downarrow & & \\ X \times \cdots \times X & & \end{array}$$

and a system of maps

$$I^{\epsilon}_{0,m,d} = \alpha_*(\mathrm{ev}_1, \dots, \mathrm{ev}_m)^* : H^*(X, \mathbb{Q})^{\otimes m} \to H^*(\overline{M}_{0,m}, \mathbb{Q}). \tag{121}$$

It is straightforward to check that the above system of maps (121) satisfies the axiom of the tree level GW system [KM94]. In particular, we have the genus zero GW type invariants

$$\langle I^{\epsilon}_{0,m,d}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_m) = \int_{\overline{M}_{0,m}} I^{\epsilon}_{0,m,d}(\gamma_1 \otimes \cdots \otimes \gamma_m)$$

for $\gamma_i \in H^*(X, \mathbb{Q})$. The formal function

$$\Phi^{\epsilon}(\gamma) = \sum_{m \geqslant 3, d \geqslant 0} \frac{1}{n!} \langle I^{\epsilon}_{0,m,d}\rangle(\gamma^{\otimes m}) q^d$$

satisfies the WDVV equation [KM94], and induces the generalized big (small) quantum cohomology ring

$$(H^*(X, \mathbb{Q})[\![q]\!], \circ^{\epsilon})$$

depending on $\epsilon \in \mathbb{R}_{>0}$. For $\epsilon > 2$, the above ring coincides with the big (small) quantum cohomology ring defined by the GW theory on X.

Remark 6.4. The above construction of the generalized tree level GW system can be easily generalized to any complete intersection of the Grassmannian $X \subset \mathbb{G}(r, n)$.

Remark 6.5. As discussed in [MOP09, § 10] for MOP-stable quotients, it might be possible to define the substack (118) and the virtual class on it for every genera.

6.4 Enumerative invariants on projective Calabi–Yau 3-folds

The construction in the previous subsection enables us to construct genus zero GW type invariants without point insertions on several projective Calabi–Yau 3-folds. One of the interesting examples is a quintic 3-fold

$$X \subset \mathbb{P}^4.$$

We can define the invariant

$$\begin{aligned} N^{\epsilon}_{0,d}(X) &= \int_{[\overline{Q}^{\epsilon}_{0,d}(X,d)]^{\mathrm{vir}}} 1 \\ &= \int_{\overline{Q}^{\epsilon}_{0,d}(\mathbb{P}^4,d)} \mathbf{e}(\pi^{\epsilon}_*(S^{\vee \otimes 5}_{U^{\epsilon}})) \in \mathbb{Q}. \end{aligned} \tag{122}$$

Another interesting example is a Calabi–Yau 3-fold obtained as a complete intersection of the Grassmannian $\mathbb{G}(2, 7)$. Let us consider the Plücker embedding

$$\mathbb{G}(2, 7) \hookrightarrow \mathbb{P}^{20},$$

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and take general hyperplanes

$$H_1, \dots, H_7 \subset \mathbb{P}^{20}. \tag{123}$$

Then the intersection

$$X = \mathbb{G}(2,7) \cap H_1 \cap \cdots \cap H_7$$

is a projective Calabi–Yau 3-fold. The hyperplanes (123) define the section

$$s_H \in H^0(\overline{Q}^{\epsilon}_{0,m}(\mathbb{G}(2,7), d), \pi^{\epsilon}_*(\wedge^2 S^{\vee}_{U^{\epsilon}})^{\oplus 7}),$$

and we define

$$\overline{Q}^{\epsilon}_{0,m}(X, d) = \{s_H = 0\}. \tag{124}$$

As in the previous subsection, there is a perfect obstruction theory and the virtual class on (124). In particular, we can define

$$\begin{aligned} N^{\epsilon}_{0,d}(X) &= \int_{[\overline{Q}^{\epsilon}_{0,d}(X,d)]^{\mathrm{vir}}} 1 \\ &= \int_{\overline{Q}^{\epsilon}_{0,d}(\mathbb{G}(2,7),d)} \mathbf{e}(\pi^{\epsilon}_*(\wedge^2 S^{\vee}_{U^{\epsilon}})^{\oplus 7}) \in \mathbb{Q}. \end{aligned} \tag{125}$$

For $\epsilon > 2$, both invariants (122) and (125) coincide with the GW invariants of X. As in Problem 6.3, we can address the following problem.

Problem 6.6. How do the invariants $N^{\epsilon}_{0,d}(X)$ depend on ϵ when X is a quintic 3-fold in $\mathbb{P}^4$ or a complete intersection of $\mathbb{G}(2,7)$ of codimension seven?

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