AN INTEGRAL REPRESENTATION FOR THE PRODUCT OF SPECTRAL MEASURES

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Introduction. Let $\mathscr{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and let $E(\cdot)$ and $E^0(\cdot)$ be spectral measures in $\mathscr{H}$ corresponding to self-adjoint operators $H = \int \lambda E\,(d\lambda)$ and $H_0 = \int \mu E^0(d\mu)$. In this paper we consider the set function $F(I \times J) = E(I)E^0(J)$ defined on the semiring of bounded rectangles, and obtain an integral representation for this set function for disjoint I, J under the hypotheses that $H - H_0$ is a type of Carleman operator.

In case H is a gentle perturbation of H_0 in the sense of Friedrichs (**2**) such a representation follows easily from the conclusion of the theorem on gentle perturbations proved in (**2**). We indicate briefly how this is done. If $B(N)$ denotes the space of bounded linear operators in a Hilbert space N, then gentle operators V, mapping $L_2(-\infty, \infty; N)$ into itself, are of the form

$$Vf(x) = \int K(x, y) f(y)\, dy,$$

where the kernel $K(x, y)$ is a $B(N)$-valued function which satisfies certain Hölder continuity conditions. Friedrichs shows that if V is gentle then there exists U_ϵ, a unitary operator-valued analytic function of ϵ, such that

$$H_\epsilon U_\epsilon = U_\epsilon L,$$

where $(Lf)(x) = xf(x)$, $H_\epsilon = L + \epsilon V$, and

$$U_\epsilon f(x) = f(x) + \epsilon \left\{ \pi i R(\epsilon; x, x) f(x) + \int \frac{R(\epsilon; x, y) f(y)}{y - x}\, dy \right\}. \tag{1}$$

The integral is taken in the principal-value sense and $R(\epsilon; x, y)$ is also a gentle kernel.

Let χ_S denote the projection onto the subspace of functions with support in the set S. Then, if I and J are disjoint compact intervals, it follows from (1) that

$$\begin{aligned}(f, E(I)E^0(J)g) &= (E(I)f, E^0(J)g) = (\chi_I U_\epsilon f, \chi_J g)\\ &= \int_{I\times J} dx\, dy\, \frac{R(\epsilon; x, y) f(y) \overline{g(x)}}{y - x}.\end{aligned} \tag{2}$$

In this paper we obtain a formula analogous to (2) by a different method and under the assumption that V is a type of Carleman operator, an assump-

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tion less restrictive than the gentleness condition. The number ϵ will not play a role in our investigation and we shall simply deal with $H = L + V$. Furthermore, H and L need not be unitarily equivalent; if they are not, however, we restrict our attention to the absolutely continuous part of H which we define presently.

Let $H = \int \lambda E\,(d\lambda)$ be a self-adjoint operator in the Hilbert space $\mathscr{H}$. The absolutely continuous subspace $\mathscr{H}_{ac}$ corresponding to H consists of those $f \in \mathscr{H}$ for which the measure $(E(\cdot)f, f)$ is absolutely continuous with respect to Lebesgue measure. It is known (**5**) that $\mathscr{H}_{ac}$ reduces H, that is, if P is the projection on $\mathscr{H}_{ac}$, then $HP \supseteq PH$. It follows from the spectral representation theorem (**1**) that there exists a Hilbert space N' and a family of subspaces N'_λ such that HP has a representation as a simple multiplication operator in the subspace $\int d\lambda N'_\lambda$ of $\mathscr{H}' = L^2(-\infty, \infty; N')$,

$$\int d\lambda N'_\lambda = \{f\colon f(\lambda) \in N'_\lambda,\ -\infty < \lambda < \infty\}.$$

This means that there exists a partial isometry U having initial set $\mathscr{H}_{ac}$ and final set $\int d\lambda N'_\lambda$ such that $LU = UHP$. If H and L are unitarily equivalent, then $\mathscr{H} = \mathscr{H}_{ac}$ and N'_λ is isomorphic with N' so that we may identify $\mathscr{H}'$ and $\mathscr{H}$ to write $LU = UH$.

Theorem 1 shows that in either case we have a representation of the form (2), while Lemmas 1 and 2 provide the background for Theorem 1.

LEMMA 1. *Let H be a self-adjoint operator on the Hilbert space $\mathscr{H}$. Let $R_\lambda = (\lambda I - H)^{-1}$ be the resolvent of H and E_μ the corresponding resolution of the identity. Then the limits*

$$\lim_{\epsilon \to 0+} (R_{\mu \pm i\epsilon} f, g) \quad \textit{exist } \mu\textit{-a.e.}$$

for each $f, g \in \mathscr{H}$.

Denote these limits by $(R_{\mu \pm i0} f, g)$. If we assume in addition that f (or g) is in the absolutely continuous subspace $\mathscr{H}_{ac}$ of H, then for any interval J we have that

$$(E(J)f, g) = \frac{1}{2\pi i}\int_J d\mu[(R_{\mu - i0}f, g) - (R_{\mu + i0}f, g)].$$

Proof. We begin with the well-known equation (**1**, p. 1196)

$$(R_\lambda f, g) = \int \frac{d_\nu (E_\nu f, g)}{\lambda - \nu}, \qquad \operatorname{Im} \lambda \neq 0.$$

Observe that

$$4(E_\nu f, g) = ||E_\nu(f + g)||^2 - ||E_\nu(f - g)||^2 + i||E_\nu(f + ig)||^2 - i||E_\nu(f - ig)||^2 \tag{3}$$

and that each term on the right is an increasing function of ν. The integral

$$\int \frac{d_\nu\, ||E_\nu h||^2}{\lambda - \nu} = \int \frac{(\mu - \nu) d_\nu\, ||E_\nu h||^2}{(\mu - \nu)^2 + \epsilon^2} - i\epsilon \int \frac{d_\nu\, ||E_\nu h||^2}{(\mu - \nu)^2 + \epsilon^2}, \qquad \lambda = \mu + i\epsilon,$$

defines an analytic function $\phi(\lambda)$ in the upper half-plane having negative imaginary part. Hence $\psi(\lambda) = (i - \phi(\lambda))^{-1}$ defines an analytic function whose modulus is bounded by 1. It is known (**6**) that under these conditions $\lim_{\epsilon\to 0+} \psi(\mu + i\epsilon)$ exists μ-a.e. and equals zero only on a null set. From this we conclude that $\lim_{\epsilon\to 0+} \phi(\mu + i\epsilon)$ exists μ-a.e. and that $\lim_{\epsilon\to 0+} (R_{\mu+i\epsilon}f, g)$ exists μ-a.e. because of (3).

To prove the second assertion of the lemma we argue that if f or $g \in \mathscr{H}_{ac}$, then $(E_\nu f, g)$ is an absolutely continuous function of ν. Let $\rho(\nu)$ be its Radon-Nikodym derivative. Then

$$(R_{\mu-i0}f, g) - (R_{\mu+i0}f, g) = \lim_{\epsilon\to 0+} 2i \int \frac{\epsilon d_\nu(E_\nu f, g)}{(\mu - \nu)^2 + \epsilon^2}$$

$$= \lim_{\epsilon\to 0} 2i \int \frac{\epsilon\rho(\nu)\, d\nu}{(\mu - \nu)^2 + \epsilon^2} = 2\pi i\rho(\mu) \quad \mu\text{-a.e.},$$

where the last equality is obtained by using a standard argument (**3**, Chapter 8) based upon the fact that ρ is integrable and $\pi^{-1}\epsilon[(\mu - \nu)^2 + \epsilon^2]^{-1}$ is an approximate identity. Furthermore, we obtain $(E(J)f, g) = \int_J d\mu\rho(\mu)$, which completes the proof.

LEMMA 2. *Let E_μ be a resolution of the identity in the Hilbert space $\mathscr{H}$. Let $v(x)$ be a strongly measurable function from the real numbers into $\mathscr{H}$ such that $\int_J dx\, ||v(x)|| < \infty$ for every finite interval J. Then for each x:*

(i) $$\frac{d}{d\mu}(E_\mu f, v(x)) = w(\mu, x) \quad \textit{exists } \mu\textit{-a.e.},$$

(ii) $$\left|\frac{d}{d\mu}(E_\mu f, v(x))\right| \leqq \left(\frac{d}{d\mu}||E_\mu f||^2\right)^{1/2}\left(\frac{d}{d\mu}||E_\mu v(x)||^2\right)^{1/2} \quad \mu\text{-a.e.},$$

and

(iii) $$\int_J dx \int d\mu\, |w(\mu, x)| \leqq ||f|| \int_J dx\, ||v(x)||.$$

The proof is straightforward and is therefore omitted.

In Theorem 1, let E_0 and E denote the spectral measures of L and H, respectively, and let $\mathscr{H}_{ac}$ be the absolutely continuous subspace corresponding to H. The inner products for N and $\mathscr{H}$ are denoted by $\langle \cdot , \cdot \rangle$ and $(\cdot , \cdot)$, respectively, i.e., $(f, g) = \int dx \langle f(x), g(x)\rangle$. Equation (4) is analogous to (2). The proof of Theorem 1 is dependent upon the two lemmas which follow it.

THEOREM 1. *Let V be a symmetric linear operator in $\mathscr{H}$ whose domain includes that of L. Suppose that for each f in the domain of L and for each $g \in \mathscr{H}$, $\langle Vf(x), g(x)\rangle = (f, v_g(x))$, where $v_g(\cdot)$ is a locally strongly integrable $\mathscr{H}$-valued function on the real line. Furthermore, assume $H = L + V$ is self-adjoint.*

Then for $f \in \mathscr{H}_{ac}$, $g \in \mathscr{H}$ and disjoint compact intervals I, J,

$$(E(I)f, E_0(J)g) = \int_{I\times J} d\mu\, dx \frac{w_{fg}(\mu, x)}{\mu - x}, \tag{4}$$

where $w_{fg}(\mu, x) = (d/d\mu)(E_\mu f, v_g(x))$.

Proof. Since L and H are self-adjoint, the resolvents $R_\lambda{}^0 = (\lambda I - L)^{-1}$ and $R_\lambda = (\lambda I - H)^{-1}$ exist and are bounded operators on $\mathscr{H}$ for $\operatorname{Im}\lambda \neq 0$. Since L and H have common domain we have the resolvent equation

$$R_\lambda = R_\lambda{}^0 + R_\lambda{}^0 V R_\lambda.$$

It is clear that $R_\lambda{}^0 f(x) = f(x)/(\lambda - x)$, and $E_0(J)f(x) = f(x)$ when $x \in J$ and is zero otherwise. Hence, for $\operatorname{Im}\lambda \neq 0$,

$$\begin{aligned}(R_\lambda f, E_0(J)g) &= \int_J dx \langle R_\lambda f(x), g(x)\rangle \\ &= \int_J dx \left\langle \frac{f(x)}{\lambda - x} + \frac{VR_\lambda f(x)}{\lambda - x}, g(x)\right\rangle \\ &= \int_J dx \left\{\frac{\langle f(x), g(x)\rangle}{\lambda - x} + \frac{\langle VR_\lambda f(x), g(x)\rangle}{\lambda - x}\right\} \\ &= \int_J dx \frac{\langle f(x), g(x)\rangle}{\lambda - x} + \int_J dx \frac{(R_\lambda f, v_g(x))}{\lambda - x}.\end{aligned} \tag{5}$$

If μ is a positive distance from J, then

$$\lim_{\epsilon\to 0+} \int_J dx \frac{\langle f(x), g(x)\rangle}{\mu \pm i\epsilon - x} = \int_J dx \frac{\langle f(x), g(x)\rangle}{\mu - x}. \tag{6}$$

By Lemma 1, $(R_{\mu\pm i0} f, E_0(J)g)$ exists μ-a.e. Thus we can conclude from (5) and (6) that

$$\lim_{\epsilon\to 0+} \int_J dx \frac{(R_{\mu\pm i\epsilon} f, v_g(x))}{\mu \pm i\epsilon - x} \quad \text{exists } \mu\text{-a.e.},$$

and

$$\begin{aligned}(R_{\mu-i0} f, E_0(J)g) - (R_{\mu+i0} f, E_0(J)g) &= \\ \lim_{\epsilon\to 0+} \int_J dx &\left\{\frac{(R_{\mu-i\epsilon} f, v_g(x))}{\mu - i\epsilon - x} - \frac{(R_{\mu+i\epsilon} f, v_g(x))}{\mu + i\epsilon - x}\right\}.\end{aligned} \tag{7}$$

To compute the last limit we begin with the equation

$$\begin{aligned}(R_{\mu\pm i\epsilon} f, v_g(x)) &= \int \frac{d_\nu (E_\nu f, v_g(x))}{\mu \pm i\epsilon - \nu} \\ &= \int \frac{\mu - \nu}{(\mu - \nu)^2 + \epsilon^2} d_\nu (E_\nu f, v_g(x)) \mp i \int \frac{\epsilon}{(\mu - \nu)^2 + \epsilon^2} d_\nu (E_\nu f, v_g(x)).\end{aligned}$$

Then

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$$\int_J dx \left\{ \frac{(R_{\mu-i\epsilon}f, v_g(x))}{\mu - i\epsilon - x} - \frac{(R_{\mu+i\epsilon}f, v_g(x))}{\mu + i\epsilon - x} \right\} =$$

$$\int_J dx \left(\frac{1}{\mu - i\epsilon - x} - \frac{1}{\mu + i\epsilon - x} \right) \int \frac{\mu - \nu}{(\mu-\nu)^2 + \epsilon^2} d_\nu (E_\nu f, v_g(x)) \tag{8}$$

$$+ i \int_J \left(\frac{1}{\mu - i\epsilon - x} + \frac{1}{\mu + i\epsilon - x} \right) \int \frac{\epsilon}{(\mu-\nu)^2 + \epsilon^2} d_\nu (E_\nu f, v_g(x)).$$

If we take the limit as $\epsilon \to 0+$ in (8), the first term on the right tends to zero by Lemma 3, and the second term tends to

$$\int_J dx \frac{2\pi i}{\mu - x} \frac{d}{d\mu} (E_\mu f, v_g(x)) \quad \mu\text{-a.e.}$$

by Lemma 4. Thus (7) becomes

$$(R_{\mu-i0}f, E_0(J)g) - (R_{\mu+i0}f, E_0(J)g) = 2\pi i \int dx \frac{w_{fg}(\mu, x)}{\mu - x} \quad \mu\text{-a.e.},$$

as long as μ remains a positive distance from J. This condition is certainly satisfied if $\mu \in I$. Then Lemma 1 yields

$$(E(I)f, E_0(J)g) = \int_I d\mu \int_J dx \frac{w_{fg}(\mu, x)}{\mu - x}. \tag{9}$$

By Lemma 2, $w_{fg}(\mu, x)$ is locally integrable and therefore $w_{fg}(\mu, x)/(\mu - x)$ is integrable over $I \times J$. Consequently, the right side of (9) can be written as a double integral, yielding (4).

LEMMA 3. *Let E_μ be a resolution of the identity in $\mathscr{H}$ with corresponding absolutely continuous subspace $\mathscr{H}_{ac}$. Let $v(x)$ be a locally strongly integrable function from the real numbers into $\mathscr{H}$. Then, for $f \in \mathscr{H}_{ac}$ and μ a positive distance from the bounded interval J,*

$$\lim_{\epsilon \to 0+} \int_J dx \frac{\epsilon}{(\mu - x)^2 + \epsilon^2} \int \frac{\mu - \nu}{(\mu - \nu)^2 + \epsilon^2} d_\nu (E_\nu f, v(x)) = 0.$$

Proof. Since $f \in \mathscr{H}_{ac}$, $(E_\mu f, v(x))$ is an absolutely continuous function of μ, and

$$w(\mu, x) = \frac{d}{d\mu} (E_\mu f, v(x)) \quad \text{exists } \mu\text{-a.e.}$$

By Lemma 2, $\int_J dx \int d\nu \, |w(\nu, x)| < \infty$. Hence

$$\int_J dx \frac{\epsilon}{(\mu - x)^2 + \epsilon^2} \int \frac{\mu - \nu}{(\mu - \nu)^2 + \epsilon^2} d_\nu (E_\nu f, v(x)) =$$

$$\int_J dx \int d\nu \frac{\epsilon(\mu - \nu)}{[(\mu - x)^2 + \epsilon^2][(\mu - \nu)^2 + \epsilon^2]} w(\nu, x).$$

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Observe that the integrand in the last integral approaches 0 as $\epsilon \to 0$ for each choice of μ, ν, and x. Hence, if we can show that the integrand is dominated by an integrable function of (ν, x) which does not depend on ϵ, our result will follow by the dominated convergence theorem. It is clear that

$$\left|\frac{\epsilon(\mu-\nu)}{(\mu-\nu)^2+\epsilon^2}\right| \leqq \frac{1}{2}.$$

Consequently,

$$\left|\frac{\epsilon(\mu-\nu)w(\nu,x)}{[(\mu-x)^2+\epsilon^2][(\mu-\nu)^2+\epsilon^2]}\right| \leqq \frac{|w(\nu,x)|}{2\inf\{|\mu-x| : x \in J\}},$$

which completes the proof.

Lemma 4. *Under the assumptions of Lemma* 3,

$$\lim_{\epsilon\to 0+} \int_J dx \frac{\mu-x}{(\mu-x)^2+\epsilon^2} \int \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} d_\nu(E_\nu f, v(x)) = \int_J dx \frac{\pi}{\mu-x} \frac{d}{d\mu}(E_\mu f, v(x)) \quad \mu\text{-a.e.}$$

Proof. Since $f \in \mathscr{H}_{ac}$, $(E_\mu f, v(x))$ is an absolutely continuous function of μ. Therefore

$$\frac{d}{d\mu}(E_\mu f, v(x)) = w(\mu, x) \quad \text{exists } \mu\text{-a.e.},$$

and

$$\int \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} d_\nu(E_\nu f, v(x)) = \int d\nu \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} w(\nu, x).$$

Then

$$\left|\int_J dx \frac{\mu-x}{(\mu-x)^2+\epsilon^2} \int d\nu \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} w(\nu,x) - \int_J dx \frac{\pi}{\mu-x} w(\mu,x)\right| \leqq$$

$$\int_J dx \left|\frac{\mu-x}{(\mu-x)^2+\epsilon^2} - \frac{1}{\mu-x}\right| \int d\nu \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} |w(\nu,x)|$$

$$+ \left|\int_J dx \frac{1}{\mu-x} \int d\nu \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} w(\nu,x) - \int_J dx \frac{\pi}{\mu-x} w(\mu,x)\right|.$$

We now use the result of Lemma 2 that $\int_J dx \int d\nu\, |w(\nu,x)| < \infty$, to obtain

$$\int_J dx \left|\frac{\mu-x}{(\mu-x)^2+\epsilon^2} - \frac{1}{\mu-x}\right| \int d\nu \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} |w(\nu,x)|$$

$$= \int_J dx \int d\nu \frac{\epsilon^2}{[(\mu-x)^2+\epsilon^2]\,|\mu-x|} \cdot \frac{\epsilon}{(\mu-\nu)^2+\epsilon^2} |w(\nu,x)|$$

$$\leqq \frac{\epsilon}{\inf\{|\mu-x| : x \in J\}} \int_J dx \int d\nu\, |w(\nu,x)|$$

which tends to zero as $\epsilon \to 0+$.

Next we show that μ-a.e.,

(10) $$\lim_{\epsilon\to 0+}\int_J dx\,\frac{1}{\mu - x}\int d\nu\,\frac{\epsilon}{(\mu - x)^2 + \epsilon^2}\,w(\nu, x) = \int_J \frac{\pi}{\mu - x}\,w(\mu, x).$$

Let I be a bounded interval. Since $\int d\nu \int_I dx\,|w(\nu, x)| < \infty$ and

$$\pi^{-1}\epsilon[(\mu - \nu)^2 + \epsilon^2]^{-1}$$

is an approximate identity, it follows that

$$\lim_{\epsilon\to 0+}\int_I dx\int d\nu\,\frac{\epsilon}{(\mu - \nu)^2 + \epsilon^2}\,w(\nu, x) = \pi\int_I dx\,w(\mu, x) \quad \mu\text{-a.e.}$$

Hence, if $s(x)$ is a step function on J, we have that

(11) $$\lim_{\epsilon\to 0+}\int_J dx\,s(x)\int d\nu\,\frac{\epsilon}{(\mu - \nu)^2 + \epsilon^2}\,w(\nu, x) = \pi\int_J dx\,s(x)w(\mu, x) \quad \mu\text{-a.e.}$$

Denote the expression whose limit we are taking by $F(s, \epsilon, \mu)$. Then

$$|F(s, \epsilon, \mu)| \leqq \sup_x|s(x)|\int d\nu\,\frac{\epsilon}{(\mu - \nu)^2 + \epsilon^2}\int_J dx\,|w(\nu, x)|,$$

so that μ-a.e. there exists $M_\mu > 0$ such that

(12) $$|F(s, \epsilon, \mu)| \leqq M_\mu \sup_x |s(x)|.$$

With the help of (12), (11) can be extended by means of an elementary technique to the case where s is a uniform limit of step functions. Thus (10) is proved.

Application of Theorem 1. Let $\mathscr{H} = L^2(-\infty, \infty)$ and let L be the simple multiplication operator on $\mathscr{H}$. A perturbation V is given by

$$Vf(x) = \int K(x, y)f(y)\,dy,$$

where

(13) $$\int \frac{|K(x, y)|^2}{\sqrt{(1 + y^2)}}\,dx\,dy = \omega_K < \infty.$$

Here and in the following, $\int$ means $\int_{-\infty}^{\infty}$.

Assumption (13) allows us to show that the hypotheses of Theorem 1 are satisfied. Let $h \in \mathfrak{D}(L)$ and define $f \in \mathscr{H}$ by $f(x) = \sqrt{(1 + x^2)}h(x)$. Then

$$|Vh(x)| \leqq \int \frac{|K(x, y)|}{\sqrt{(1 + y^2)}}\,|f(y)|\,dy \leqq \left\{\int \frac{|K(x, y)|^2}{\sqrt{(1 + y^2)}}\,dy\right\}^{1/2} ||f||.$$

Consequently,

(14) $$||Vh|| \leqq \omega_K\,||f|| \leqq \omega_K(||h|| + ||Lh||),$$

implying $h \in \mathfrak{D}(V)$.

We next deduce an inequality which will imply that $H = L + V$ is self-adjoint. Let $r > 0$ and put $K(x, y) = K'(x, y) + K''(x, y)$, where

$$K'(x, y) = \begin{cases} K(x, y) & \text{if } |x|, |y| < r, \\ 0, & \text{otherwise.} \end{cases}$$

Certainly, both K' and K'' are symmetric kernels satisfying (13), and by choosing r sufficiently large we can ensure that

$$\omega_{K''} < 1. \tag{15}$$

Then

$$\iint |K'(x, y)|^2 \, dx\, dy \leqq (1 + r^2) \iint \frac{|K'(x, y)|^2}{1 + y^2} \, dx\, dy \tag{16}$$

so that K' is a Hilbert-Schmidt kernel. If V', V'' are integral operators corresponding to the kernels K', K'', then $Vh = V'h + V''h$, and, combining the results of (14) and (16) we have that

$$||Vh|| \leqq ||V''h|| + ||V'h|| \leqq \omega_{K''}||Lh|| + [(1 + r^2)\omega_{K'} + 1] \, ||h||. \tag{17}$$

Since $\omega_{K''} < 1$, it follows that $L + V$ is self-adjoint by a result of Kato (**4**, Chapter 5, Theorem 4.3).

We proceed to show that V is an operator of the kind envisaged in Theorem 1. Let $f \in \mathfrak{D}(L)$ and $g \in \mathscr{H}$. Then

$$\langle Vf(x), g(x)\rangle = \overline{g(x)}\int K(x, y)f(y)\, dy = (f, v_g(x)),$$

where

$$v_g(x) = g(x)\overline{K(x, \cdot)}.$$

We have that

$$||v_g(x)|| = |g(x)| \, ||K(x, \cdot)||,$$

and

$$\int_J dx \, ||v_g(x)|| \leqq ||g|| \, \{\int_J dx \, ||K(x, \cdot)||^2\}^{1/2} < \infty$$

for every bounded interval J. Thus the hypotheses for Theorem 1 are satisfied.

Let E_0 and E denote the spectral measures corresponding to L and $L + V$ as before. Choose $f \in \mathscr{H}_{ac}$, $g \in \mathscr{H}$ and disjoint compact intervals I, J. Theorem 1 then asserts that

$$(E(I)f, E_0(J)g) = \int_{I\times J} d\mu \, dx \, \frac{w_{fg}(\mu, x)}{\mu - x}, \tag{18}$$

where

$$w_{fg}(\mu, x) = \frac{d}{d\mu} (E_\mu f, v_g(x)).$$

We are now prepared to obtain a representation analogous to (2) of the Introduction. Let $L^2(-\infty, \infty; N)$ be a representation space for the absolutely

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continuous part of H, and let U be the corresponding representation isometry. Denote the inner product in N by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|_N$. Then

$$(E_\mu f, v_g(x)) = \int_{-\infty}^{\mu} \langle [Uf](y), [Uv_g(x)](y) \rangle_N \, dy$$

which implies that

$$w_{fg}(\mu, x) = \langle [Uf](\mu), [Uv_g(x)](\mu) \rangle_N \quad \mu\text{-a.e.}$$

Now,

$$[Uv_g(x)](\mu) = g(x)[\overline{UK(x, \cdot)}](\mu),$$

where

$$\int |[UK(x, \cdot)](\mu)|_N^2 \, d\mu \leqq \int |K(x, y)|^2 \, dy.$$

If we define

$$R(\mu, x) = [\overline{UK(x, \cdot)}](\mu),$$

then

$$\iint \frac{|R(\mu, x)|_N^2}{1 + x^2} \, d\mu \, dx \leqq \omega_K$$

and finally, (18) becomes

$$(E(I)f, E_0(J)g) = \int_{I \times J} d\mu \, dx \, \frac{\langle Uf(\mu), R(\mu, x) \rangle \overline{g(x)}}{\mu - x},$$

which is analogous to (2).

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