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THE LINEARISATIONS OF CYCLIC PERMUTATIONS HAVE RATIONAL ZETA FUNCTIONS

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Let $n \geqslant 2$ be an integer. Let P be the set of all integers in $[1, n+1]$ and let σ be a cyclic permutation on P. Assume that f is the linearisation of σ on P. Then we show that f has rational Artin-Mazur zeta function which is closely related to the characteristic polynomial of some $n \times n$ matrix with entries either zero or one. Some examples of non-conjugate maps with the same Artin-Mazur zeta function are also given.

Let $[a, b]$ be a nondegenerate compact interval on the real line and let f be a continuous map from $[a, b]$ into itself. For every positive integer k, let f^k, the k^{th} iterate of f, be defined by: $f^1 = f$ and $f^k = f \circ f^{k-1}$ if $k > 1$. For $x_0 \in [a, b]$, we call x_0 a periodic point of f if $f^m(x_0) = x_0$ for some positive integer m and call the smallest such positive integer m the least period of x_0 (under f). We call the set $\{f^k(x_0) \mid k \text{ is any nonnegative integer}\}$ the periodic orbit of x_0 (under f). It is easy to see that, if $f^m(x_0) = x_0$ for some positive integer m, then the least period of x_0 must divide m. We shall need this fact later. A periodic point of least period 1 is also called a fixed point. In discrete dynamical systems theory, one problem related to the numbers of periodic points is: For every positive integer k, let $P_k = \{x \in [a, b] \mid f^k(x) = x\}$. Let $N(f^k)$ be the number of points in $\{x \in P_k \mid x \text{ is isolated in } P_m \text{ for some positive integer } m \text{ dividing } k\}$. Assume that, for every positive integer k, the number $N(f^k)$ is finite. (Note that this definition of $N(f^k)$ is a slight generalisation of that of Artin and Mazur [**1**].) Find the reduced Artin-Mazur zeta function [**5**] $\overline{\zeta}_f(z) = \sum_{k=1}^{\infty} N(f^k)z^k$ of f or find the Artin-Mazur zeta function [**1, 5**] $\zeta_f(z) = \exp\left(\sum_{k=1}^{\infty} \left(N(f^k)/k\right)z^k\right)$ of f, where z is the complex variable. When we actually compute [**2, 3**] the reduced Artin-Mazur zeta functions of some special types of continuous piecewise linear maps f on $[a, b]$, we find some $n \times n$ matrices A_f (depending on f) such that $N(f^k) = \operatorname{tr}\left(A_f^k\right)$, the trace of A_f^k, for every positive integer k (see also [**6**]) and the reduced Artin-Mazur zeta functions of f are closely

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related to the characteristic polynomials of the matrices A_f. In this note, we extend this result to a class called the linearisations of cyclic permutations. To this end, we shall need the following result from matrix theory.

THEOREM 1. *Let A be an $n \times n$ complex matrix and let $\det(xE_n - A) = x^n + \sum_{j=0}^{n-1} \beta_j x^j$ be the characteristic polynomial of A, where E_n is the $n \times n$ identity matrix. Then*

$$\sum_{k=1}^{\infty} \operatorname{tr}(A^k) z^k = -z \frac{\frac{d}{dz}\left(1 + \sum_{j=1}^{n} \beta_{n-j} z^j\right)}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j} \quad \text{or} \quad \exp\left(\sum_{k=1}^{\infty} \frac{\operatorname{tr}(A^k)}{k} z^k\right) = \frac{1}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j}.$$

PROOF: Write $z^n + \sum_{j=0}^{n-1} \beta_j z^j = \prod_{j=1}^{n} (z - \lambda_j)$. Then, by replacing z by $1/z$ and simplifying, we obtain that $1 + \sum_{j=1}^{n} \beta_{n-j} z^j = \prod_{j=1}^{n} (1 - \lambda_j z)$. Since A is similar to an upper triangular matrix with main diagonal entries the eigenvalues of A, we easily obtain that $\operatorname{tr}(A^k) = \sum_{j=1}^{n} \lambda_j^k$ for all positive integers k. This fact will also be used later. So,

$$\begin{aligned} \sum_{k=1}^{\infty} \operatorname{tr}(A^k) z^k &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n} \lambda_j^k\right) z^k = \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} (\lambda_j z)^k\right) = \sum_{j=1}^{n} \frac{\lambda_j z}{1 - \lambda_j z} \\ &= -z \frac{\frac{d}{dz}\left(\prod_{j=1}^{n} (1 - \lambda_j z)\right)}{\prod_{j=1}^{n} (1 - \lambda_j z)} = -z \frac{\frac{d}{dz}\left(1 + \sum_{j=1}^{n} \beta_{n-j} z^j\right)}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j}. \end{aligned}$$

Or, by formal integration,

$$\exp\left(\sum_{k=1}^{\infty} \frac{\operatorname{tr}(A^k)}{k} z^k\right) = \frac{1}{1 + \sum_{j=1}^{n} \beta_{n-j} z^j}.$$

∎

We now return to the discrete dynamical systems theory on the interval. Let $n \geqslant 2$ be an integer. Let P be the set of all integers in $[1, n+1]$ and let σ be a map from P into itself. For every integer $1 \leqslant k \leqslant n$, let $J_k = [k, k+1]$. Assume that f is the continuous map from $[1, n+1]$ into itself such that $f(i) = \sigma(i)$ for every integer $1 \leqslant i \leqslant n+1$ and f is linear on J_k (and so, the absolute value of the slope of f on J_k is $\geqslant 1$) for every integer $1 \leqslant k \leqslant n$. This continuous map f is called the linearisation of σ on P. Let $A_f = (\alpha_{ij})$ be the $n \times n$ matrix defined by $\alpha_{ij} = 1$ if $f(J_i) \supset J_j$ and $\alpha_{ij} = 0$ otherwise. This matrix A_f is called the Markov matrix of f. Note that, for every positive integer k, since f is piecewise linear, so is f^k and the slope of every linear piece of f^k is the product of the slopes of k linear pieces of f. Now, for every positive integer k, if $\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_k i_{k+1}}$ is nonzero (in this case, this product is 1) for some integers $i_1, i_2, \cdots, i_{k+1}$ in $[1, n]$, then, by definition of A_f, $f(J_{i_s}) \supset J_{i_{s+1}}$ for every integer $1 \leqslant s \leqslant k$. If I and J are closed subintervals of $[1, n+1]$ such that $f(I) \supset J$, then it is well-known that there is a closed interval $L \subset I$ such that $f(L) = J$. So, since $f(J_{i_k}) \supset J_{i_{k+1}}$, there is a closed interval $L_1 \subset J_{i_k}$ such that $f(L_1) = J_{i_{k+1}}$. Since $L_1 \subset J_{i_k} \subset f(J_{i_{k-1}})$, there is a closed interval $L_2 \subset J_{i_{k-1}}$ such that $f(L_2) = L_1$. Inductively, there are closed intervals $L_1, L_2, \cdots, L_k$ such that $L_s \subset J_{i_{k+1-s}}$ and $f(L_s) = L_{s-1}$ for every integer $1 \leqslant s \leqslant k$, where we define $L_0 = J_{i_{k+1}}$. Consequently, f^k is *linear* on $L_k \subset J_{i_1}$ *and* $f^k(L_k) = J_{i_{k+1}}$. Note that, since f^k is linear on L_k and $f^k(L_k) = J_{i_{k+1}}$, the slope of f^k on L_k is nonzero.

Conversely, let k be a fixed positive integer and let T_k be a maximum closed interval on which f^k is linear *with nonzero slope.* Since the y-coordinates of the turning points and the boundary points, $(1, f(1))$ and $(n+1, f(n+1))$, of f are contained in P, $f^k(T_k)$ is a compact interval whose endpoints are distinct and contained in P. So, for some integers $1 \leqslant u_k \leqslant v_k \leqslant n$, $f^k(T_k) = \bigcup_{i=u_k}^{v_k} J_i$. Let i_{k+1} be any integer such that $u_k \leqslant i_{k+1} \leqslant v_k$. Then $f^k(T_k) \supset J_{i_{k+1}}$ and there is a closed subinterval L_k of T_k such that $f^k(L_k) = J_{i_{k+1}}$. Since f^k is linear on L_k, f is linear on the interval $f^s(L_k)$ for every integer $0 \leqslant s \leqslant k-1$. In particular, f is linear on the interval $f^{k-1}(L_k)$ and since the interior of $f(f^{k-1}(L_k)) = J_{i_{k+1}}$ contains no point of P, neither does the interior of $f^{k-1}(L_k)$. This implies that $f^{k-1}(L_k) \subset J_{i_k}$ for some integer $1 \leqslant i_k \leqslant n$. Similarly, since f is linear on the interval $f^{k-2}(L_k)$ and the interior of $f(f^{k-2}(L_k)) = f^{k-1}(L_k) \subset J_{i_k}$ contains no point of P, neither does the interior of $f^{k-2}(L_k)$. Therefore, $f^{k-2}(L_k) \subset J_{i_{k-1}}$ for some integer $1 \leqslant i_{k-1} \leqslant n$. Similar arguments imply that there are integers $i_k, i_{k-1}, \cdots, i_3, i_2, i_1$ in $[1, n]$ such that $f^t(L_k) \subset J_{i_{t+1}}$ for every integer $0 \leqslant t \leqslant k-1$ and $f^k(L_k) = J_{i_{k+1}}$. Note that, when $t = 0$, $L_k \subset J_{i_1}$. Thus, $f^s(L_k) \subset f(J_{i_s})$ for every integer $1 \leqslant s \leqslant k$. Consequently, $f^s(L_k) \subset J_{i_{s+1}} \cap f(J_{i_s})$ for every integer $1 \leqslant s \leqslant k-1$. So, $f(J_{i_s})$ contains some interior points of $J_{i_{s+1}}$ for every integer $1 \leqslant s \leqslant k-1$. Since both endpoints of $f(J_{i_s}) (\supset f^s(L_k))$ are distinct points in P, this implies that $f(J_{i_s}) \supset J_{i_{s+1}}$, and so, $\alpha_{i_s i_{s+1}} = 1$ for every integer $1 \leqslant s \leqslant k-1$. Furthermore, since $J_{i_k} \supset f^{k-1}(L_k)$, we have $f(J_{i_k}) \supset f(f^{k-1}(L_k)) = J_{i_{k+1}}$. Thus, $\alpha_{i_k i_{k+1}} = 1$. Hence $\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_k i_{k+1}} = 1$. Therefore, for every positive integer k, there is a one-to-one correspondence between the collection of nonzero products $\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_k i_{k+1}}$ and the collection of closed intervals $L_k \subset J_{i_1}$ such that f^k is *linear* on L_k *and* $f^k(L_k) = J_{i_{k+1}}$. Consequently, if $A_f^k = \left(\alpha_{ij}^{(k)}\right)$; then every entry $\alpha_{ij}^{(k)}$ represents the number of closed intervals $L_k \subset J_i$ with disjoint interiors such that f^k is *linear* on L_k *and* $f^k(L_k) = J_j$. In the following, we shall show how to relate this number $\sum_{j=1}^{n} \alpha_{jj}^{(k)}$ to the number $N(f^k)$. We have two cases to consider:

CASE 1. Assume that, for some positive integer t, the absolute value of the slope of every linear piece of f^t is > 1. Then, it is easy to see that, for every integer $j > t$, the absolute value of the slope of every linear piece of f^j is > 1. So, for any integers $1 \leqslant i \leqslant n$, $k \geqslant 1$ (we do not require $k \geqslant t$) and any closed interval $L \subset J_i$, if f^k is linear on L and $f^k(L) = J_i \supset L$, then the slope of f^k on L *cannot be equal to* 1 and hence the equation $f^k(x) = x$ has exactly one solution in L. So, we can associate this unique solution to the interval L. This implies that, for every positive integer k, $N(f^k) = \sum_{j=1}^{n} \alpha_{jj}^{(k)} = \operatorname{tr}\left(A_f^k\right) = \sum_{j=1}^{n} \lambda_j^k$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the Markov matrix A_f of f.

CASE 2. Assume that the absolute value of the slope of some linear piece of f^n is 1. Let T_n be a maximum closed interval on which f^n is linear and the absolute value of the slope of f^n on T_n is 1. As was just shown above, there exist integers $i_1, i_2, \cdots, i_{n+1}$ in $[1, n]$ and a closed interval $L_n \subset T_n$ such that $J_{i_{n+1}} \subset f^n(T_n), f^n(L_n) = J_{i_{n+1}}$ and $f^s(L_n) \subset J_{i_{s+1}}$ for every integer $0 \leqslant s \leqslant n-1$. Since the absolute value of the slope of f^n on L_n (which is 1) equals the product of the absolute values of the slopes of f on $f^s(L_n)$ (which are $\geqslant 1$) for all integers $0 \leqslant s \leqslant n-1$, the absolute value of the slope of f on $f^s(L_n)$ must be 1 for every integer $0 \leqslant s \leqslant n-1$. So, the length of $f^s(L_n)$ is equal to that of $f^{s+1}(L_n)$ for every integer $0 \leqslant s \leqslant n-1$. Since the length of $f^n(L_n)\left(= J_{i_{n+1}}\right)$ is 1, we obtain that the length of $f^s(L_n)\left(\subset J_{i_{s+1}}\right)$ is also 1 for every integer $0 \leqslant s \leqslant n-1$. This, together with the fact that $f^n(L_n) = J_{i_{n+1}}$, implies that $f^s(L_n) = J_{i_{s+1}}$ for every integer $0 \leqslant s \leqslant n$. Note that, when $s = 0$, $L_n = J_{i_1}$. Thus, $f(J_{i_s}) = f\left(f^{s-1}(L_n)\right) = f^s(L_n) = J_{i_{s+1}}$ for every integer $1 \leqslant s \leqslant n$. Since there are $n+1$ closed intervals $J_{i_{s+1}}\left(= f^s(L_n)\right)$, $0 \leqslant s \leqslant n$, taken from the n distinct J_i's, some interval J_{i_s} appears at least twice. Without loss of generality, we may assume that $J_{i_1} = J_{i_{m+1}}$ for some integer $1 \leqslant m \leqslant n$ and the J_{i_s}'s are distinct for all integers $1 \leqslant s \leqslant m$. Since $f(J_{i_s}) = J_{i_{s+1}}$ and f is linear on J_{i_s} for all integers $1 \leqslant s \leqslant m$, we obtain that, for every integer $1 \leqslant s \leqslant m$, $f^m(J_{i_s}) = J_{i_s}$ and f^m is linear on J_{i_s}, and hence, we have either $f^m(x) = x$ for all $x \in J_{i_s}$ and all integers $1 \leqslant s \leqslant m$ or $f^m(x) = -x + a_s + b_s$, where $J_{i_s} = [a_s, b_s]$, for all $x \in J_{i_s}$ and all integers $1 \leqslant s \leqslant m$. In the following, we assume, for simplicity, that σ is a *cyclic permutation* on P. If, for some integer $1 \leqslant j \leqslant m$, $f^m(x) = x$ for all $x \in J_{i_j}$, then, in particular,

$f^m(a_j) = a_j$. Since a_j is a periodic point of f with least period $n+1$, we must have $n+1 \leqslant m$ which contradicts the assumption that $1 \leqslant m \leqslant n$. This contradiction implies that $f^m(x) = -x+a_s+b_s$ and hence $f^{2m}(x) = x$ for all $x \in J_{i_s}$ and all integers $1 \leqslant s \leqslant m$. In particular, $f^{2m}(a_s) = a_s$ for all integers $1 \leqslant s \leqslant m$. Since the least period of a_s is $n+1$, $n+1$ divides $2m$. Thus, $2m = r(n+1)$ for some positive integer r. But, since $1 \leqslant m \leqslant n$, we must have $r = 1$, and so, $n+1 = 2m$. Furthermore, since f^m maps every endpoint of J_{i_s} to the other for every integer $1 \leqslant s \leqslant m$, the m closed intervals J_{i_s}'s are pairwise disjoint. Since there are exactly $n = 2m-1 = m+(m-1)$ distinct closed intervals J_i's, we obtain that $\{J_{i_s} \mid 1 \leqslant s \leqslant m\} = \{J_{2j-1} \mid 1 \leqslant j \leqslant m\}$. Consequently, $f^m(x) = -x+4s-1$ and $f^{2m}(x) = x$ for all $x \in J_{2s-1}$ and all integers $1 \leqslant s \leqslant m$. This *also* implies that $T_n = J_{2j-1}$ for some integer $1 \leqslant j \leqslant m$ and the absolute value of the slope of every linear piece of f^n (and hence of f^k for every integer $k \geqslant n$) on any closed interval contained in J_{2i} for any integer $1 \leqslant i \leqslant m-1$ is > 1. Thus, for any positive integer k (we do not require $k \geqslant n$), any integer $1 \leqslant i \leqslant m-1$ and any closed interval $L \subset J_{2i}$, if f^k is linear on L and $f^k(L) = L_{2i} \supset L$, then the slope of f^k on L *cannot be equal to* 1 and hence the equation $f^k(x) = x$ has a unique solution in L. So, we can associate this unique solution to the interval L. Furthermore, since f permutes cyclically the intervals J_{2j-1}, $1 \leqslant j \leqslant m$, and since f^m maps every endpoint of any $J_{2j-1}, 1 \leqslant j \leqslant m$, to the other, we obtain that there exists a cyclic permutation ρ on the set of all integers in $[1,m]$ such that $\sigma^j(1) \in \{2\rho^j(1)-1,\ 2\rho^j(1)\}$ and $\{\sigma^j(1),\ \sigma^{m+j}(1)\} = \{2\rho^j(1)-1,\ 2\rho^j(1)\}$ for all integers $1 \leqslant j \leqslant m$. On the other hand, since f permutes cyclically the intervals J_{2j-1}, $1 \leqslant j \leqslant m$, and since $f^m(x) = -x+4j-1$ and $f^{2m}(x) = x$ for all $x \in J_{2j-1}$ and all integers $1 \leqslant j \leqslant m$, we see that, for any positive integer k, (i) if k is not a multiple of m, then the equation $f^k(x) = x$ has no solution in J_{2j-1} for any integer $1 \leqslant j \leqslant m$; (ii) if k is an odd multiple of m, then each midpoint of J_{2j-1} is the unique (and *isolated*) solution of the equation $f^k(x) = x$ in J_{2j-1} for every integer $1 \leqslant j \leqslant m$; and (iii) if k is an even multiple of m, then, for every integer $1 \leqslant j \leqslant m$, every point of J_{2j-1} is a (*non-isolated*) solution of the equation $f^k(x) = x$, but, the midpoint of J_{2j-1} is an *isolated* solution of the equation $f^m(x) = x$. Therefore, we can associate, for every integer $1 \leqslant j \leqslant m$, the midpoint of J_{2j-1} to the interval J_{2j-1} when k is an (odd or even) multiple of m and nothing otherwise. This implies that, for every positive integer k, $N(f^k) = \sum_{j=1}^{n} \alpha_{jj}^{(k)} = \operatorname{tr}\left(A_f^k\right) = \sum_{j=1}^{n} \lambda_j^k$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the Markov matrix A_f of f. Consequently, by Theorem 1, we have proved the following two results:

THEOREM 2. *Let $n \geqslant 2$ be an integer. Let P be the set of all integers in $[1, n+1]$ and let σ be a map from P into itself. Assume that f is the linearisation of σ on P such that, for some positive integer t, the absolute value of the slope of every linear*

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piece of f^t *is* > 1. *Then the following hold:*

(a) *For every positive integer* k, $N(f^k) = \operatorname{tr}\left(A_f^k\right) = \sum_{j=1}^{n} \lambda_j^k$, *where* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are the eigenvalues of the Markov matrix* A_f *of* f.

(b) *The Artin-Mazur zeta function* $\zeta_f(z)$ *of* f *is* $\zeta_f(z) = 1\Big/\left(1 + \sum_{k=1}^{n} \beta_{n-k} z^k\right)$, *where* $x^n + \sum_{k=0}^{n-1} \beta_k x^k$ *is the characteristic polynomial of the Markov matrix of* f.

THEOREM 3. *Let* $n \geqslant 2$ *be an integer. Let* P *be the set of all integers in* $[1, n+1]$ *and let* σ *be a map from* P *into itself. For every integer* $1 \leqslant k \leqslant n$, *let* $J_k = [k, k+1]$. *Assume that* f *is the linearisation of* σ *on* P *such that the absolute value of the slope of some linear piece of* f^n *is 1. Then there exist an integer* $1 \leqslant m \leqslant n$ *and* m *distinct integers* $i_1, i_2, \cdots, i_m$ *in* $[1, n]$ *such that* f *is linear on* J_{i_s} *and* $f(J_{i_s}) = J_{i_{s+1}}$ *for every integer* $1 \leqslant s \leqslant m$, *where we define* $i_{m+1} = i_1$. *Furthermore, if* σ *is a cyclic permutation on* P, *then the following also hold:*

(a) $n + 1 = 2m$.

(b) $f^m(x) = -x + 4k - 1$ *and* $f^{2m}(x) = x$ *for all* $x \in J_{2k-1}$ *and all integers* $1 \leqslant k \leqslant m$. *In particular,* f *has periodic points of least period* $(n+1)/2$.

(c) *There exists a cyclic permutation* ρ *on the set of all integers in* $[1, m]$ *such that* $\sigma^j(1) \in \{2\rho^j(1) - 1,\ 2\rho^j(1)\}$ *and* $\{\sigma^j(1),\ \sigma^{m+j}(1)\} = \{2\rho^j(1) - 1,\ 2\rho^j(1)\}$ *for all integers* $1 \leqslant j \leqslant m$.

(d) *For every positive integer* k, $N(f^k) = \operatorname{tr}\left(A_f^k\right) = \sum_{j=1}^{n} \lambda_j^k$, *where* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are the eigenvalues of the Markov matrix* A_f *of* f.

(e) *The Artin-Mazur zeta function* $\zeta_f(z)$ *of* f *is* $\zeta_f(z) = 1\Big/\left(1 + \sum_{k=1}^{n} \beta_{n-k} z^k\right)$, *where* $x^n + \sum_{k=0}^{n-1} \beta_k x^k$ *is the characteristic polynomial of the Markov matrix of* f.

REMARK. We require σ to be a cyclic permutation on P in Theorem 3 while not in Theorem 2. This is because the requirement on the slope of f in Theorem 3 is weaker than that in Theorem 2. If we do not make the stronger requirement on σ in Theorem 3, there would be many trivial examples whose linearisations have well-defined Markov matrices while their Artin-Mazur zeta functions are not defined.

The following partial converse of Theorem 3 is easy to prove.

THEOREM 4. *Let* m *and* n *be positive integers such that* $n+1 = 2m$. *Let* ρ *be a cyclic permutation on the set of all integers in* $[1, m]$ *and let* σ *be a cyclic permutation on the set* P *of all integers in* $[1, n+1]$. *For every integer* $1 \leqslant i \leqslant n$, *let* $J_i = [i, i+1]$.

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Assume that f is the linearisation of σ on P and $\sigma^j(1) \in \{2\rho^j(1)-1,\ 2\rho^j(1)\}$ and $\{\sigma^j(1),\ \sigma^{m+j}(1)\} = \{2\rho^j(1)-1,\ 2\rho^j(1)\}$ for all integers $1 \leqslant j \leqslant m$. Then, for every positive integer k, the absolute value of the slope of f^k on J_{2j-1} is 1 for all integers $1 \leqslant j \leqslant m$. Consequently, the Artin-Mazur zeta function $\zeta_f(z)$ of f is $\zeta_f(z) = 1\Big/\left(1 + \sum_{k=1}^{n} \beta_{n-k} z^k\right)$, where $x^n + \sum_{k=0}^{n-1} \beta_k x^k$ is the characteristic polynomial of the Markov matrix of f.

REMARK. If σ is a cyclic permutation on the set P of all integers in $[1, n+1]$ and f is the linearisation of σ on P, then Theorems 2, 3, & 4 give a complete solution of the Artin-Mazur zeta function of f. In particular, the Artin-Mazur zeta function of f is rational with poles at the values $1/\lambda_j$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the Markov matrix of f.

Let f and g be two continuous maps from $[a, b]$ into itself. If they are (topologically) conjugate to each other, then it is clear that they have the same (if defined) Artin-Mazur zeta function. However, if they are not conjugate to each other, they may still have the same Artin-Mazur zeta function. For example, assume that both f and g satisfy the conditions in Theorem 2 above. If their respective Markov matrices are similar to each other, then, since similar matrices have the same characteristic polynomial [**4**], they have, by Theorem 2, the same Artin-Mazur zeta function. In the following, we present some such examples. The following result is taken from [**2**].

THEOREM 5. *For every integer $n \geqslant 3$, let $f_n(x)$ be the continuous map from $[1, n]$ onto itself defined by*

$$f_n(x) = \begin{cases} x+1, & \text{for } 1 \leqslant x \leqslant n-1, \\ -(n-1)x + n^2 - n + 1, & \text{for } n-1 \leqslant x \leqslant n. \end{cases}$$

We also define sequences $\langle b_{k,n} \rangle$ as follows:

$$b_{k,n} = \begin{cases} 2^k - 1, & \text{for } 1 \leqslant k \leqslant n-1, \\ \sum_{i=1}^{n-1} b_{k-i,n}, & \text{for } n \leqslant k. \end{cases}$$

Then, for any integers $k \geqslant 1$ and $n \geqslant 3$, $b_{k,n}$ is the number of distinct fixed points of the map $f_n^k(x)$ in $[1, n]$. Moreover, the Artin-Mazur zeta function $\zeta_{f_n}(z)$ of f_n, for every integer $n \geqslant 3$, is $\zeta_{f_n}(z) = 1\Big/\left(1 - \sum_{k=1}^{n-1} z^k\right)$.

THEOREM 6. *For every odd integer $m \geqslant 3$, let $g_m(x)$ and $h_m(x)$ be the contin-*

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uous maps from $[1, m]$ *onto itself defined by*

$$g_m(x) = \begin{cases} -x + m + 1, & \text{for } 1 \leqslant x \leqslant \frac{1}{2}(m-1), \\ -\frac{1}{2}(m+1)x + \frac{1}{4}(m+1)^2 + 1, & \text{for } \frac{1}{2}(m-1) \leqslant x \leqslant \frac{1}{2}(m+1), \\ \frac{1}{2}(m-1)x - \frac{1}{4}(m^2-1) + 1, & \text{for } \frac{1}{2}(m+1) \leqslant x \leqslant \frac{1}{2}(m+1) + 1, \\ -x + m + 2, & \text{for } \frac{1}{2}(m+1) + 1 \leqslant x \leqslant m. \end{cases}$$

and

$$h_m(x) = \begin{cases} x + \frac{1}{2}(m-1), & \text{for } 1 \leqslant x \leqslant \frac{1}{2}(m+1), \\ -(m-1)x + \frac{1}{2}(m^2 + 2m - 1), & \text{for } \frac{1}{2}(m+1) \leqslant x \leqslant \frac{1}{2}(m+1) + 1, \\ x - \frac{1}{2}(m+1), & \text{for } \frac{1}{2}(m+1) + 1 \leqslant x \leqslant m. \end{cases}$$

Then, for any odd integer $m \geqslant 3$, *both* $g_m(x)$ *and* $h_m(x)$ *have the same Artin-Mazur zeta function as* $f_m(x)$, *where* $f_m(x)$ *is defined as in Theorem 5 above.*

PROOF: Let $m \geqslant 3$ be an odd integer. It suffices to show that the Markov matrices of f_m, g_m, and h_m are similar to one another. Indeed, let P be the set of all integers in $[1, m]$ and let σ be a cyclic permutation on P. Let φ be the linearisation of σ on P and let V_{m-1} be the vector space over the field of real numbers with the set $Q_1 = \{J_1, J_2, \cdots, J_{m-1}\}$ as a basis, where, for every integer $1 \leqslant k \leqslant m-1$, $J_k = [k, k+1]$. Then, φ determines a linear transformation (which we call $\overline{\varphi}$) on V_{m-1} defined by $\overline{\varphi}\left(\sum_{k=1}^{m-1} r_k J_k\right) = \sum_{k=1}^{m-1} r_k \overline{\varphi}(J_k)$, where r_k's are real numbers and $\overline{\varphi}(J_k) = \sum_{s=i_k}^{j_k} J_s$ if $\varphi(J_k) = \bigcup_{s=i_k}^{j_k} J_s$ for some integers $1 \leqslant i_k \leqslant j_k \leqslant m-1$. Furthermore, with respect to the basis Q_1, the linear transformation $\overline{\varphi}$ is also determined [4] by the $(m-1) \times (m-1)$ matrix $B_\varphi = (\beta_{ij})$ in such a way that, for every integer $1 \leqslant k \leqslant m-1$, $\overline{\varphi}(J_k) = \sum_{j=1}^{m-1} \beta_{kj} J_j \left(= \sum_{s=i_k}^{j_k} J_s\right)$ which happens to be the same as the Markov matrix of the map φ on $[1, m]$. Now, if we take $Q_2 = \left\{J_1, J_{m-1}, J_2, J_{m-2}, \cdots, J_i, J_{m-i}, \cdots, J_{(m-3)/2}, J_{(m+3)/2}, J_{(m-1)/2}, \sum_{k=1}^{(m+1)/2} J_k\right\}$ as a new basis for V_{m-1}, then it is easy to see that $\overline{g}_m$ acts on Q_2 like $\overline{f}_m$ on Q_1. Similarly, if we take $Q_3 = \{J_{(m-1)/2}, J_{m-1}, J_{(m-3)/2}, J_{m-2}, J_{(m-5)/2}, J_{m-3}, \ldots, J_3, J_{(m-1)/2+3}, J_2, J_{(m-1)/2+2}, J_1, J_{(m-1)/2+1}\}$ as a new basis for V_{m-1}, then $\overline{h}_m$ acts on Q_3 like $\overline{f}_m$ on Q_1. Therefore, the matrices of the linear transformations $\overline{f}_m, \overline{g}_m$, and $\overline{h}_m$ on the respective bases Q_1, Q_2, and Q_3 are the same. So, the matrices of $\overline{f}_m, \overline{g}_m$, and $\overline{h}_m$ on the basis Q_1 are similar to one another [4]. Consequently, the Markov matrices of the maps f_m, g_m, and h_m on the interval $[1, m]$ are similar to one another and hence, by Theorem 2, f_m, g_m, and h_m have the same Artin-Mazur zeta function. $\square$

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