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# THE TRANSLATIONAL HULL OF AN INVERSE SEMIGROUP

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**1. Introduction.** Let $S$ be a semigroup. A function $\lambda(\rho)$ on $S$ is a *left* (*right*) *translation* of $S$ if, for all $x, y \in S$, $\lambda(xy) = \lambda(x)y$ $((xy)\rho = x(y\rho))$. A left translation $\lambda$ and a right translation $\rho$ are said to be *linked* if $x(\lambda y) = (x\rho)y$, for all $x, y \in S$, and then the ordered pair $(\lambda, \rho)$ is called a *bitranslation*. Clearly the set $\Lambda(S)$ $(P(S))$ of all left (right) translations is a semigroup with respect to composition of functions. The set of bitranslations forms a subsemigroup of the direct product $\Lambda(S) \times P(S)$ which is called the *translational hull*, $\Omega(S)$, of $S$. A valuable survey of results relating to $\Omega(S)$ and its importance in relation to semigroup extensions will be found in Petrich's review [**6**], to which the reader is referred for basic results on translational hulls.

For each $a \in S$, the *inner left* (*right*) *translation* of $S$ induced by $a$ is the function $\lambda_a(\rho_a)$ defined by $\lambda_a(x) = ax$ $((x)\rho_a = xa)$, for all $x \in S$. Then $\pi_a = (\lambda_a, \rho_a) \in \Omega(S)$ and $\Pi(S) = \{\pi_a : a \in S\}$ is a subsemigroup of $\Omega(S)$. The mapping $\Pi : a \rightarrow \pi_a$ is one-to-one if $S$ is *weakly reductive* (that is, $ax = bx$ and $xa = xb$, for all $x \in S$ implies that $a = b$).

LEMMA 1.1 (Gluskin [**3**]). *If $S$ is weakly reductive then $\Omega(S)$ is the idealizer of $\Pi(S)$ in $\Lambda(S) \times P(S)$.*

Let $\Pi_\Lambda$ be the projection homomorphism of $\Omega(S)$ into $\Lambda(S)$ and $\Gamma(S) = \{\lambda_a : a \in S\}$. Then clearly $\Pi_\Lambda\Pi(S) = \Gamma(S)$. A semigroup $S$ is *reductive* if $ax = bx$, for all $x$, or $xa = xb$, for all $x$, implies that $a = b$.

LEMMA 1.2 (Petrich [**6**]). *If $S$ is reductive then $\Pi_\Lambda$ is an isomorphism of $\Omega(S)$ into $\Lambda(S)$ and $\Pi_\Lambda\Pi$ is an isomorphism of $S$ onto $\Gamma(S)$.*

An *inverse semigroup* $S$ is a semigroup $S$ such that for each $a \in S$ there is a unique element $x \in S$ with $axa = a$ and $xax = x$. We shall denote the idempotents of $S$ by $E_S$, or just $E$, if there is no likelihood of confusion. For basic properties of inverse semigroups the reader is referred to [**2**]. An inverse semigroup is reductive and hence, for an inverse semigroup $S$, $\Pi_\Lambda$ is an isomorphism of $\Omega(S)$ into $\Lambda(S)$. Our objective in this paper is to investigate $\Lambda(S)$, to discuss the relationship between $\Gamma(S)$, $\Pi_\Lambda(\Omega(S))$ and $\Lambda(S)$ and thereby describe $\Omega(S)$ for certain fairly general classes of inverse semigroups. A crucial observation is the following.

LEMMA 1.3 (Ponizovski [**8**]). *If $S$ is an inverse semigroup then so is $\Omega(S)$.*

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The main theorem in Section 2 establishes that $\Pi_\Lambda(\Omega(S))$ is the idealizer of $\Gamma(S)$ in $\Lambda(S)$, the unique maximal inverse subsemigroup of $\Lambda(S)$ containing $\Gamma(S)$ and the unique maximal inverse subsemigroup of $\Lambda(S)$ with $\Lambda(E)$ as its set of idempotents, where $E$ is the semilattice of idempotents of $S$.

A key tool in these discussions is a homomorphism $\theta$ of $\Lambda(S)$ into a semigroup of mappings of the set of idempotents $E$ of $S$ defined by $\theta:\lambda \to \theta_\lambda$ where $\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1}$, for all $e \in E$. The mappings $\phi_\lambda$, where $\lambda$ is such that, for some right translation $\rho$, $(\lambda, \rho)$ is in the unit group $\Sigma(S)$ of $\Omega(S)$, were introduced by Ault [**1**] and used to characterize $\Sigma(S)$ for certain inverse semigroups $S$. In Section 3 we show that the congruence $\theta \circ \theta^{-1}$ induced on $\Pi_\Lambda(\Omega(S))$ by $\theta$ (and therefore the corresponding congruence on $\Omega(S)$) is the maximum idempotent separating congruence on $\Pi_\Lambda(\Omega(S))$ ($\Omega(S)$, respectively).

Then it is shown that the Howie-Munn representation [**5**] of an inverse semigroup $S$ as a semigroup of isomorphisms of principal ideals of the set of idempotents $E$ of $S$ onto principal ideals of $E$ extends to a representation of $\Pi_\Lambda(\Omega(S))$ as a semigroup of isomorphisms of $P$-ideals of $E$ onto $P$-ideals of $E$ (where an ideal $F$ of $E$ is a $P$-ideal if the intersection of $F$ with any principal ideal is a principal ideal). Likewise the Vagner-Preston representation of an inverse semigroup $S$ by one-to-one partial transformations of $S$ is extended to a representation of $\Pi_\Lambda(\Omega(S))$ by one-to-one partial transformations of $S$.

In the final three sections the techniques introduced in earlier sections are used to characterize $\Pi_\Lambda(\Omega(S))$ for $S = T_X$ (the semigroup of isomorphisms of principal ideals of a semilattice $X$ onto principal ideals of $X$) and for Brandt semigroups.

**2. The relationship between $\Gamma(S)$, $\Pi_\Lambda(\Omega(S))$ and $\Lambda(S)$.** If $A$ is a subsemigroup of a semigroup $T$ then the *left idealizer* $L$ of $A$ is $\{t \in T: ta \in A$, for all $a \in A\}$. Then $L$ is the largest subsemigroup of $S$ containing $A$ as a left ideal. The *idealizer* and *right idealizer* of $A$ are defined similarly.

It is straightforward to see that, for any semigroup $S$ such that $S^2 = S$, $\Lambda(S)$ is the *left idealizer* of $\Gamma(S)$ in the full transformation semigroup $\mathscr{T}s$ on $S$. The principal result of this section will show that $\Pi_\Lambda(\Omega(S))$, for $S$ an inverse semigroup, is the idealizer of $\Gamma(S)$ in $\Lambda(S)$. Since $S^2 = S$ for any inverse semigroup, $\Gamma(S)$ is a left ideal in $\Lambda(S)$ and consequently $\Pi_\Lambda\Omega(S)$ can be described as the right idealizer of $\Gamma(S)$ in $\Lambda(S)$. In doing so we shall obtain several other characterizations of $\Pi_\Lambda(\Omega(S))$ as a subsemigroup of $\Lambda(S)$.

An ideal $I$ in a semilattice $X$ will be called a $P$-ideal (*principal intersection ideal*) if the intersection of $I$ with any principal ideal of $X$ is a principal ideal.

LEMMA 2.1 (Petrich [**6**]). *Let $X$ be a semilattice and $\kappa$ be a left translation of $X$. Then $\kappa$ is an idempotent homomorphism of $X$ such that $\kappa(X)$ is a $P$-ideal and $\kappa(x) \leqq x$, for all $x \in X$.*

For the remainder of this note, $S$ will denote an inverse semigroup and $E$ will denote its semilattice of idempotents. If $\kappa \in \Lambda(E)$ then the mapping $\kappa'$

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such that $\kappa'(a) = \kappa(aa^{-1})a$ is an element of $\Lambda(S)$ such that $\kappa'|E = \kappa$ and $\kappa \to \kappa'$ is an isomorphism. Hence we identify $\kappa$ and $\kappa'$ and thereby consider $\Lambda(E)$ as a subsemigroup of $\Lambda(S)$. For any $\lambda \in \Lambda(S)$, $a \in S$, $\lambda(a) = \lambda(aa^{-1}a) = \lambda(aa^{-1})a$ and therefore, for any elements $\lambda$, $\lambda'$ of $\Lambda(S)$, $\lambda = \lambda'$ if and only if $\lambda|E = \lambda'|E$.

LEMMA 2.2. *In* $\Lambda(S)$, *let*

$$E_1 = \{\kappa^2 = \kappa : \kappa(E) \subseteq E\}.$$

*Then*

$$\begin{aligned} E_1 &= \{\kappa : \kappa(E) \subseteq E\} \\ &= \Lambda(E) \\ &= \{\kappa : \kappa|E \in \Lambda(E)\} \\ &= \{\kappa^2 = \kappa : \kappa\lambda_e = \lambda_e\kappa, \textit{ for all } e \in E\} \\ &= \textit{Idealizer of } \Gamma(E) \textit{ in } \Lambda(S) \\ &= \textit{largest commutative subsemigroup of } \Lambda(S) \textit{ consisting of} \\ &\qquad \textit{idempotents and containing } \Gamma(E). \end{aligned}$$

*Proof.* Let the sets on the right side of the various equalities be denoted by $E_2, E_3, \ldots, E_7$, respectively. Clearly $E_1 \subseteq E_2 \subseteq E_3 \subseteq E_4$. Let $\kappa \in E_4$ and $e \in E$. Then

$$\begin{aligned} \kappa^2(e) &= \kappa(\kappa(e)) = \kappa(\kappa(e^2)) = \kappa(\kappa(e)e) \\ &= \kappa(e\kappa(e)) = \kappa(e)\kappa(e) = \kappa(e). \end{aligned}$$

Hence $\kappa^2 = \kappa$. Moreover, for any $e, f \in E$,

$$\kappa\lambda_e(f) = \kappa(ef) = \kappa(fe) = \kappa(f)e = e\kappa(f) = \lambda_e\kappa(f).$$

Thus $\kappa \in E_5$ and $E_4 \subseteq E_5$.

If $\kappa \in E_5$ and $e, f \in E$, then

$$\kappa\lambda_e(f) = \kappa(ef) = \kappa(e)f = \lambda_{\kappa(e)}(f).$$

Hence $\lambda_e\kappa = \kappa\lambda_e = \lambda_{\kappa(e)}$ and $\kappa \in E_6$. Thus $E_5 \subseteq E_6$.

Now let $\kappa \in E_6$ and $e \in E$. Then, for some $f \in E$, $\kappa\lambda_e = \lambda_f$. Then

$$\kappa(e) = \kappa(ee) = \kappa\lambda_e(e) = \lambda_f(e) = fe \in E.$$

and

$$\kappa^2(e) = \kappa(\kappa(e)) = \kappa(fe) = \kappa(ef) = \kappa(e)f = (fe)f = fe = \kappa(e).$$

Thus $\kappa \in E_1$, $E_6 \subseteq E_1$ and $E_1 = E_2 = \ldots = E_6$.

Clearly any commutative subsemigroup of $\Lambda(S)$ consisting of idempotents and containing $\Gamma(E)$ is contained in $E_5$. On the other hand $E_5$ is clearly a semi-

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group of idempotents containing $\Gamma(E)$ and for $\kappa, \lambda \in E_5 = \Lambda(E)$, and for any $e \in E$,

$$\kappa\lambda(e) = \kappa(\lambda(e)e) = \kappa(e\lambda(e)) = \kappa(e)\lambda(e) = \lambda(e)\kappa(e) = \lambda\kappa(e).$$

Thus the elements of $E_5$ commute and $E_5 = E_7$.

If $T$ is a semigroup and $X$ is a subsemigroup of commuting idempotents then by [**9**, Corollary 1.6], there is a unique maximal inverse subsemigroup $X^C$ of $T$ with $X$ as its set of idempotents. This may be described as follows. For $a, b \in T$ we say that $(a, b)$ is a *regular pair* if $aba = a$ and $bab = b$. Then $X^C = \{a \in T$: for some $b$, $(a, b)$ is a regular pair, $ab, ba \in X, aXb \subseteq X$ and $bXa \subseteq X\}$.

Since, by Lemma 2.2, $E_1$ is a subsemigroup of $\Lambda(S)$ of commuting idempotents there is a unique maximal inverse subsemigroup of $\Lambda(S)$ with $E_1$ as its set of idempotents. Let $\Gamma_1 = E_1{}^C$, and let

$$\Gamma_2 = \{\lambda \in \Lambda(S): \text{for some } \lambda' \in \Lambda(S),\ (\lambda, \lambda') \text{ is a regular pair with } \lambda\lambda', \lambda'\lambda \in E_1\}.$$

LEMMA 2.3. *$\Gamma_1 = \Gamma_2$ and $\Gamma_1$ is the unique maximal inverse subsemigroup of $\Lambda(S)$ which contains $\Gamma(S)$.*

*Proof.* From the definition of $\Gamma_1 = E_1{}^C$ it is clear that $\Gamma_1 \subseteq \Gamma_2$. Let $\lambda \in \Gamma_2$ and $\lambda'$ be such that $\lambda\lambda', \lambda'\lambda \in E_1$. From the definition of $E_1{}^C$ it is clear that in order to prove that $\lambda \in \Gamma_1$ it suffices to show that $\lambda'\kappa\lambda \in E_1$, for all $\kappa \in E_1$ (the requirement being symmetric in $\lambda$ and $\lambda'$). Clearly $\lambda'\kappa\lambda \in \Lambda(S)$.

Then, for any $e \in E$,

$$\lambda'\kappa\lambda(e) = \lambda'\kappa\lambda(e^2) = \lambda'\kappa(\lambda(e)e) = \lambda'\kappa\lambda_{\lambda(e)}(e) = \lambda'\lambda_{\lambda(e)}\kappa(e)$$
$$= \lambda'(\lambda(e)\kappa(e)) = \lambda'\lambda(e)\kappa(e).$$

Since $\kappa$ and $\lambda'\lambda$ are both elements of $E_1$, $\lambda'\lambda(e)$ and $\kappa(e)$ are both idempotents and hence $\lambda'\kappa\lambda(e)$ is also an idempotent. Hence $\lambda'\kappa\lambda \in E_4 = E_1$. Hence $\Gamma_1 = \Gamma_2$.

Suppose now that $T$ is any inverse subsemigroup of $\Lambda(S)$ containing $\Gamma(S)$. Let $t \in T$. Since $T$ is an inverse semigroup $t$ has an inverse $t'$ in $T$ and so $(t, t')$ is a regular pair. Furthermore $tt'$ and $t't$ are idempotents of $T$ and so commute with all the idempotents of $T$ and hence, in particular, commute with $\lambda_e$, for all $e \in E$. Therefore $tt', t't \in E_5 = E_1$ and so $t \in \Gamma_2 = \Gamma_1$. Thus $\Gamma_1 \supseteq T$ and $\Gamma_1$ is the unique maximal inverse subsemigroup of $\Lambda(S)$ containing $\Gamma(S)$.

For any $\lambda \in \Lambda(S)$ let $\theta_\lambda: E \rightarrow E$ be the mapping defined by $\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1}$. The mapping $\theta: \lambda \rightarrow \theta_\lambda$ will be vital to our subsequent work. The mappings $\theta_\lambda$ were introduced by J. Ault [**1**] while investigating the unit group of $\Omega(S)$.

The observations in the following two lemmas will be used frequently.

LEMMA 2.4. *Let $\lambda \in \Lambda(S)$.*

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(1) *For any* $e \in E$, $\lambda(\lambda(e)^{-1}\lambda(e)) = \lambda(e)$.
(2) *For any* $e \in E$, $\theta_\lambda(\lambda(e)^{-1}\lambda(e)) = \theta_\lambda(e)$.
(3) $\{e \in E : e = \lambda(f)^{-1}\lambda(f)$, *for some* $f \in E\} = \{e \in E : e = \lambda(e)^{-1}\lambda(e)\}$.

*Proof.* (1) We have

$$\begin{aligned}\lambda(\lambda(e)^{-1}\lambda(e)) &= \lambda(\lambda(e)^{-1}\lambda(e)e)\\ &= \lambda(e\lambda(e)^{-1}\lambda(e))\\ &= \lambda(e)\lambda(e)^{-1}\lambda(e) = \lambda(e).\end{aligned}$$

(2) By (1),

$$\begin{aligned}\theta_\lambda(\lambda(e)^{-1}\lambda(e)) &= \lambda(\lambda(e)^{-1}\lambda(e))(\lambda(\lambda(e)^{-1}\lambda(e)))^{-1}\\ &= \lambda(e)\lambda(e)^{-1} = \theta_\lambda(e).\end{aligned}$$

(3) Let $e = \lambda(f)^{-1}\lambda(f)$, for $f \in E$. Then, by (1),

$$\lambda(e) = \lambda(\lambda(f)^{-1}\lambda(f)) = \lambda(f).$$

Hence $e = \lambda(f)^{-1}\lambda(f) = \lambda(e)^{-1}\lambda(e)$ and (3) then follows.

*Notation.* We shall write

$$\Delta_\lambda = \{e : e = \lambda(e)^{-1}\lambda(e)\} = \{e : e = \lambda(f)^{-1}\lambda(f), \text{ for some } f \in E\}.$$

LEMMA 2.5. *Let* $\lambda \in \Gamma_1$, $\lambda'$ *be the inverse of* $\lambda$ *in* $\Gamma_1$ *and* $e \in E$. *Then*
(1) $\lambda(e)^{-1} = \lambda'(\lambda(e)\lambda(e)^{-1})$;
(2) $\lambda'\lambda(e) = \lambda(e)^{-1}\lambda(e)$;
(3) $Ee \cap \Delta_\lambda = E\lambda'\lambda(e) = E\lambda(e)^{-1}\lambda(e)$;
(4) *For any* $e \in \Delta_\lambda$, $e = \lambda'\lambda(e) = \lambda(e)^{-1}\lambda(e)$.

*Proof.* (1) We have

$$\begin{aligned}\lambda(e)\lambda'(\lambda(e)\lambda(e)^{-1})\lambda(e) &= \lambda(e)\lambda'(\lambda(e)\lambda(e)^{-1}\lambda(e)) = \lambda(e)\lambda'\lambda(e)\\ &= \lambda(e\lambda'\lambda(e)) = \lambda\lambda'\lambda(e) = \lambda(e)\end{aligned}$$

and

$$\begin{aligned}\lambda'(\lambda(e)\lambda(e)^{-1})\lambda(e)\lambda'(\lambda(e)\lambda(e)^{-1}) &= \lambda'\ (\lambda(e)\lambda(e)^{-1}\lambda(e))\lambda'\lambda(e)\lambda(e)^{-1}\\ &= (\lambda'\lambda(e))^2\lambda(e)^{-1} = \lambda'\lambda(e)\lambda(e)^{-1}\\ &= \lambda'(\lambda(e)\lambda(e)^{-1}).\end{aligned}$$

(2) From (1), we have

$$\lambda(e)^{-1}\lambda(e) = \lambda'(\lambda(e)\lambda(e)^{-1})\lambda(e) = \lambda'\lambda(e).$$

(3) Clearly, by (2), we have $E\lambda'\lambda(e) \subseteq Ee \cap \Delta_\lambda$. Let $f \in Ee \cap \Delta_\lambda$. Then, by Lemma 2.4,

$$f = fe = \lambda(f)^{-1}\lambda(f)e = \lambda'\lambda(f)e = \lambda'\lambda(fe) = \lambda'\lambda(e)f \leqq \lambda'\lambda(e).$$

Thus $f \in E\lambda'\lambda(e)$ and we have $E\lambda'\lambda(e) = Ee \cap \Delta_\lambda$.
(4) Part (4) follows from (2) and Lemma 2.4 (3).

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For any mapping $\alpha: A \to B$, ($A$, $B$ sets) we shall write $\Delta(\alpha) = A$, $\nabla(\alpha) = \{\alpha(a): a \in A\}$.

For any semilattice $X$ we shall be interested in several semigroups related to $X$. First we shall denote by $F_X$ the semigroup of order preserving mappings $\alpha$ for which $\Delta(\alpha) = X$ and $\nabla(\alpha)$ is an ideal of $X$.

LEMMA 2.6. *The mapping $\theta: \lambda \to \theta_\lambda$ is a homomorphism of $\Lambda(S)$ into $F_E$. Moreover,*

(1) *$\Delta_\lambda$ is a P-ideal and $\nabla(\theta_\lambda) = \theta_\lambda(\Delta_\lambda)$;*

(2) *$\theta_\lambda$ is an isomorphism when restricted to any principal ideal of $\Delta_\lambda$.*

*Proof.* Let $\lambda \in \Lambda(S)$, $e, f \in E$ and $e \leqq f$. Then

$$\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1} = \lambda(fe)\lambda(fe)^{-1} = \lambda(f)e\lambda(f)^{-1} \leqq \lambda(f)\lambda(f)^{-1} = \theta_\lambda(f).$$

Hence, $\theta_\lambda$ is order preserving. Now suppose that $f \leqq \theta_\lambda(e)$. Then

$$f = f\theta_\lambda(e) = f\lambda(e)\lambda(e)^{-1} = \lambda(e)\lambda(e)^{-1}f\lambda(e)\lambda(e)^{-1} = \theta_\lambda(\lambda(e)^{-1}f\lambda(e)).$$

Thus $f \in \nabla(\theta_\lambda)$ and $\nabla(\theta_\lambda)$ is an ideal in $X$. Hence $\theta_\lambda \in F_E$. Now, for $\lambda, \lambda' \in \Lambda(S)$ and $e \in E$, we have

$$\begin{aligned}\theta_{\lambda'}\theta_\lambda(e) &= \theta_{\lambda'}(\lambda(e)\lambda(e)^{-1}) = \lambda'(\lambda(e)\lambda(e)^{-1})(\lambda'(\lambda(e)\lambda(e)^{-1}))^{-1}\\ &= \lambda'(\lambda(e)\lambda(e)^{-1})(\lambda'\lambda(e)\lambda(e)^{-1})^{-1} = \lambda'(\lambda(e)\lambda(e)^{-1})\lambda(e)(\lambda'\lambda(e))^{-1}\\ &= \lambda'(\lambda(e)\lambda(e)^{-1}\lambda(e))(\lambda'\lambda(e))^{-1} = \lambda'\lambda(e)(\lambda'\lambda(e))^{-1} = \theta_{\lambda'\lambda}(e).\end{aligned}$$

Thus $\theta_{\lambda'}\theta_\lambda = \theta_{\lambda'\lambda}$ and $\theta$ is a homomorphism of $\Lambda(S)$ into $F_E$.

(1) Now let $f \in E$ and $e = \lambda(f)^{-1}\lambda(f)$. Then $e \leqq f$ and $e \in \Delta_\lambda$. Conversely, let $g \in Ef \cap \Delta_\lambda$. Then $g \leqq f$ and, by Lemma 2.4,

$$g = \lambda(g)^{-1}\lambda(g) = \lambda(fg)^{-1}\lambda(fg) = \lambda(f)^{-1}\lambda(f)g = eg \leqq e.$$

Thus $g \in Ee$ and $Ef \cap \Delta_\lambda = Ee$. In other words $\Delta_\lambda$ is a $P$-ideal. In addition, by Lemma 2.4, $\theta_\lambda(e) = \theta_\lambda(f)$. Thus $\theta_\lambda(\Delta_\lambda) = \nabla(\theta_\lambda)$ and (1) is verified.

(2) Consider any principal ideal of $\Delta_\lambda$, say $Ef$, where $f = \lambda(f)^{-1}\lambda(f)$ and consider any $g, h \in Ef$. Let $\theta_\lambda(g) \leqq \theta_\lambda(h)$. Then $\lambda(g)\lambda(g)^{-1} \leqq \lambda(h)\lambda(h)^{-1}$. Now $\lambda(g)\lambda(g)^{-1} = \lambda(fg)\lambda(fg)^{-1} = \lambda(f)g\lambda(f)^{-1}$ and similarly $\lambda(h)\lambda(h)^{-1} = \lambda(f)h\lambda(f)^{-1}$. Hence

$$\begin{aligned}g = fg &= \lambda(f)^{-1}\lambda(f)g\\ &= \lambda(f)^{-1}\lambda(f)g\lambda(f)^{-1}\lambda(f) \leqq \lambda(f)^{-1}\lambda(f)h\lambda(f)^{-1}\lambda(f)\\ &= \lambda(f)^{-1}\lambda(f)h = fh = h.\end{aligned}$$

Since we know that $\theta_\lambda$ is order preserving this proves (2).

LEMMA 2.7. *Let $\lambda \in \Lambda(S)$. Then the following statements are equivalent.*

(1) *$\theta_\lambda$ is a homomorphism;*

(2) *the restriction of $\theta_\lambda$ to $\Delta_\lambda$ is a homomorphism;*

(3) *the restriction of $\theta_\lambda$ to $\Delta_\lambda$ is an isomorphism.*

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*Proof.* Clearly (1) implies (2). Assume that (2) holds and that for some $e, f \in \Delta_\lambda$, $e \neq f$, we have $\theta_\lambda(e) = \theta_\lambda(f)$. Then, since $\theta_\lambda$ is a homomorphism on $\Delta_\lambda$, $\theta_\lambda(e) = \theta_\lambda(f) = \theta_\lambda(ef)$ where either $ef < e$ or $ef < f$. But, by Lemma 2.6, $\theta_\lambda$ is an isomorphism on principal ideals of $\Delta_\lambda$. Hence we have a contradiction and $\theta_\lambda$ is an isomorphism on $\Delta_\lambda$.

Now assume that (3) holds and let $e, f \in E$. Then

$$\begin{aligned}\theta_\lambda(e)\theta_\lambda(f) &= \theta_\lambda(\lambda(e)^{-1}\lambda(e))\theta_\lambda(\lambda(f)^{-1}\lambda(f)) = \theta_\lambda(\lambda(e)^{-1}\lambda(e)\lambda(f)^{-1}\lambda(f))\\ &= \theta_\lambda(\lambda(e)^{-1}\lambda(e)ef\lambda(f)^{-1}\lambda(f)) = \theta_\lambda(\lambda(ef)^{-1}\lambda(ef)\lambda(ef)^{-1}\lambda(ef))\\ &= \theta_\lambda(\lambda(ef)^{-1}\lambda(ef)) = \theta_\lambda(ef).\end{aligned}$$

Hence (1) holds.

We can now characterize the elements of $\Gamma_1$, in terms of the mappings $\theta_\lambda$, as follows:

PROPOSITION 2.8. *Let* $\lambda \in \Lambda(S)$. *Then* $\lambda \in \Gamma_1$ *if and only if*

(1) $\nabla(\theta_\lambda)$ *is a P-ideal, and*

(2) $\theta_\lambda$ *is a homomorphism.* (Clearly condition (2) may be replaced by the equivalent conditions of Lemma 2.7.)

*Proof.* (1) Let $\lambda \in \Gamma_1$. Let $\lambda^{-1}$ be the inverse of $\lambda$ in $\Gamma_1$ and $e$ be any element of $E$. Let $f = \lambda\lambda^{-1}(e)$. Since $\lambda\lambda^{-1} \in E_1$, $f \in E$ and

$$f = ff^{-1} = \theta_{\lambda\lambda^{-1}}(e) = \theta_\lambda\theta_{\lambda^{-1}}(e) \in \nabla(\theta_\lambda).$$

Since $\lambda\lambda^{-1}(e) = \lambda\lambda^{-1}(e)e$, we have $f \leqq e$ and so $f \in Ee \cap \nabla(\theta_\lambda)$. Let $g \in Ee \cap \nabla(\theta_\lambda)$, say $g = \theta_\lambda(h)$, for some $h \in E$. Then, since $\lambda\lambda^{-1} \in E_1$,

$$\begin{aligned}fg &= \lambda\lambda^{-1}(e)g = \lambda\lambda^{-1}(eg) = \lambda\lambda^{-1}(g) = \theta_{\lambda\lambda^{-1}}(g) = \theta_{\lambda\lambda^{-1}}(\theta_\lambda(h)) = \theta_{\lambda\lambda^{-1}\lambda}(h)\\ &= \theta_\lambda(h) = g.\end{aligned}$$

Thus $g \leqq f$ and so $g \in Ef$. Hence $Ef = Ee \cap \Delta(\theta_\lambda)$ and (1) is satisfied.

(2) Let $\lambda \in \Gamma_1$, $\lambda^{-1}$ be the inverse of $\lambda$ in $\Gamma_1$ and $e, f \in \Delta_\lambda$ be such that $\theta_\lambda(e) \leqq \theta_\lambda(f)$. Then, by Lemma 2.5 (4),

$$e = \lambda^{-1}\lambda(e) = \lambda^{-1}\lambda(e)\lambda^{-1}\lambda(e) = \theta_{\lambda^{-1}\lambda}(e).$$

Similarly, $f = \theta_{\lambda^{-1}\lambda}(f)$ and so

$$e = \theta_{\lambda^{-1}\lambda}(e) = \theta_{\lambda^{-1}}\theta_\lambda(e) \leqq \theta_{\lambda^{-1}}\theta_\lambda(f) = \theta_{\lambda^{-1}\lambda}(f) = f.$$

Hence, $\theta_\lambda$ is an isomorphism when restricted to $\Delta_\lambda$ and so, by Lemma 2.7, $\theta_\lambda$ is a homomorphism.

Conversely, let $\lambda \in \Lambda(S)$ satisfy conditions (1) and (2). We define a mapping $\lambda': S \to S$. For any $a \in S$ we have $Eaa^{-1} \cap \nabla(\theta_\lambda) = Ef$ where $f = \theta_\lambda(g)$, for some $f, g \in E$, by (1). Let $\lambda'(a) = \lambda(g)^{-1}a$. Suppose that we also have $f = \theta_\lambda(e)$. Then $\theta_\lambda(\lambda(e)^{-1}\lambda(e)) = f = \theta_\lambda(\lambda(g)^{-1}\lambda(g))$ and, since $\theta_\lambda$ is an isomorphism when restricted to $\Delta_\lambda$, $\lambda(e)^{-1}\lambda(e) = \lambda(g)^{-1}\lambda(g)$. Hence

$$\lambda(e) = \lambda(e)\lambda(e)^{-1}\lambda(e) = \lambda(e)\lambda(g)^{-1}\lambda(g) = \lambda(eg)\lambda(eg)^{-1}\lambda(eg) = \lambda(eg).$$

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Similarly, $\lambda(g) = \lambda(eg)$ and so $\lambda(e) = \lambda(g)$. Therefore $\lambda'$ is a well-defined mapping.

Now let $a, b \in S$ with $f, g$ as above and let $h, k$ be such that

$$Eabb^{-1}a^{-1} \cap \nabla(\theta_\lambda) = Eh \text{ where } h = \theta_\lambda(k).$$

Then clearly $h \leqq f$. Hence

$$h = hf = \theta_\lambda(k)\theta_\lambda(g) = \theta_\lambda(kg)$$

since $\theta_\lambda$ is a homomorphism. Therefore

$$\begin{aligned} h\lambda(g) &= \theta_\lambda(kg)\lambda(g) = \lambda(kg)\lambda(kg)^{-1}\lambda(g) \\ &= \lambda(kg)k\lambda(g)^{-1}\lambda(g) = \lambda(kg)k\lambda(g)^{-1}\lambda(g)k \\ &= \lambda(kg)\lambda(kg)^{-1}\lambda(kg) = \lambda(kg)g\lambda(k)^{-1}\lambda(k)g \\ &= \lambda(kg)g\lambda(k)^{-1}\lambda(k) = \lambda(kg)\lambda(kg)^{-1}\lambda(k) = h\lambda(k) = \lambda(k). \end{aligned}$$

Hence

$$\lambda'(a)b = \lambda(g)^{-1}ab = \lambda(g)^{-1}hab = \lambda(k)^{-1}ab = \lambda'(ab).$$

Thus, $\lambda' \in \Lambda(S)$.

For any $e \in E$, $E\lambda(e)\lambda(e)^{-1} \cap \nabla(\theta_\lambda) = E\lambda(e)\lambda(e)^{-1}$ and so $\lambda'(\lambda(e)) = \lambda(e)^{-1}\lambda(e)$. Hence

$$\lambda\lambda'\lambda(e) = \lambda(\lambda(e)^{-1}\lambda(e)) = \lambda(e\lambda(e)^{-1}\lambda(e)) = \lambda(e)\lambda(e)^{-1}\lambda(e) = \lambda(e).$$

On the other hand, let $Ee \cap \nabla(\theta_\lambda) = Ef$ where $f = \theta_\lambda(g)$, $f, g \in E$. Then

$$\begin{aligned} \lambda'\lambda\lambda'(e) &= \lambda'\lambda(\lambda(g)^{-1}e) = \lambda'(\lambda(g)\lambda(g)^{-1}e) = \lambda'(e)\lambda(g)\lambda(g)^{-1} \\ &= \lambda(g)^{-1}e\lambda(g)\lambda(g)^{-1} = \lambda(g)^{-1}e = \lambda'(e). \end{aligned}$$

Thus $(\lambda, \lambda')$ is a regular pair. To show that $\lambda \in \Gamma_1$ it remains to show that $\lambda\lambda'$ and $\lambda'\lambda$ are elements of $E_1$. To do this it suffices to show that $\lambda\lambda'(E) \subseteq E$ and $\lambda'\lambda(E) \subseteq E$. Let $e \in E$, $Ee \cap \nabla(\theta_\lambda) = Ef$ and $f = \theta_\lambda(g)$. Then

$$\lambda\lambda'(e) = \lambda(\lambda(g)^{-1}e) = \lambda(g\lambda(g)^{-1}e) = \lambda(g)\lambda(g)^{-1}e.$$

which is an element of $E$ and

$$\lambda'\lambda(e) = \lambda(e)^{-1}\lambda(e)$$

which is also an element of $E$. Thus $\lambda \in \Gamma_1$.

From Lemma 2.3, we already have two descriptions of the relationship between $\Gamma_1$ and $\Gamma(S)$. In the following proposition we give a third.

PROPOSITION 2.9. *$\Gamma_1$ is the idealizer of $\Gamma(S)$ in $\Lambda(S)$.*

*Proof.* Let $I$ denote the idealizer of $\Gamma(S)$. First we show that $\Gamma_1 \subseteq I$. Since $\Gamma(S)$ is a left ideal in $\Lambda(S)$, $\Gamma(S)$ is certainly a left ideal of $\Gamma_1$. Hence, we wish to show, for any $a \in S$, $\kappa \in \Gamma_1$, that $\lambda_a\kappa \in \Gamma(S)$. It is sufficient to do so for

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$a = e \in E$. Let $Ee \cap \nabla(\theta_\kappa) = Eg$ where $g = \theta_\kappa(h)$, $g, h \in E$. Then for any $f \in E$, $e\theta_\kappa(f) \in Ee \cap \nabla(\theta_\kappa)$, so that $e\kappa(f) = ge\kappa(f)$ and we have

$$\begin{aligned}\lambda_e\kappa(f) &= e\kappa(f) = ge\kappa(f) = g\kappa(f) = \theta_\kappa(h)\kappa(f)\kappa(f)^{-1}\kappa(f)\\ &= \theta_\kappa(h)\theta_\kappa(f)\kappa(f) = \theta_\kappa(hf)\kappa(f), \text{ since } \theta_\kappa \text{ is a homomorphism}\\ &= \kappa(hf)\kappa(hf)^{-1}\kappa(f) = \kappa(hf)h\kappa(f)^{-1}\kappa(f)h\\ &= \kappa(hf)\kappa(hf)^{-1}\kappa(hf) = \kappa(hf) = \kappa(hf)f = \lambda_{\kappa(h)}f.\end{aligned}$$

Thus $\lambda_e\kappa = \lambda_{\kappa(h)} \in \Gamma(S)$ and $\Gamma_1 \subseteq I$.

Suppose now that $\kappa \in I$. Let $e \in E$. Then $\lambda_e\kappa = \lambda_a$, for some $a \in S$. Hence

$$Ee \cap \nabla(\theta_\kappa) = e\nabla(\theta_\kappa) = \nabla(\theta_{\lambda_e}\theta_\kappa) = \nabla(\theta_{\lambda_e\kappa}) = \nabla(\theta_{\lambda_a}) = Eaa^{-1},$$

where, for the last equality, it is clear that $\nabla(\theta_{\lambda_a}) \subseteq Eaa^{-1}$. On the other hand, for any $e \in E$, $eaa^{-1} = \theta_{\lambda_a}(a^{-1}e) \in \nabla(\theta_{\lambda_a})$. Hence $\nabla(\theta_\kappa)$ is a $P$-ideal. We complete the proof by showing that $\theta_\kappa$ is an isomorphism of $\Delta_\kappa$ onto $\nabla(\theta_\kappa)$.

Let $e$, $f$ be any elements of $\Delta_\kappa$. Suppose that $\theta_\kappa(e) \leqq \theta_\kappa(f)$. Then $\kappa(e)\kappa(e)^{-1} \leqq \kappa(f)\kappa(f)^{-1} = k$, say. Let $a$ be such that $\lambda_k\kappa = \lambda_a$. Now $\kappa(e)\kappa(e)^{-1} = k\kappa(e)(k\kappa(e))^{-1} = \lambda_k\kappa(e)(\lambda_k\kappa(e))^{-1} = \lambda_a(e)\lambda_a(e)^{-1}$. Likewise $\kappa(f)\kappa(f)^{-1} = \lambda_a(f)\lambda_a(f)^{-1}$. Hence $\lambda_a(e)\lambda_a(e)^{-1} \leqq \lambda_a(f)\lambda_a(f)^{-1}$ and so

$$\begin{aligned}a^{-1}aea^{-1}a &= a^{-1}\lambda_a(e)\lambda_a(e)^{-1}a \leqq a^{-1}\lambda_a(f)\lambda_a(f)^{-1}a\\ &= a^{-1}afa^{-1}a.\end{aligned}$$

Thus $ea^{-1}ae \leqq fa^{-1}af$ or $(\lambda_a(e))^{-1}\lambda_a(e) \leqq (\lambda_a(f))^{-1}\lambda_a(f)$. But

$$(\lambda_a(e))^{-1}\lambda_a(e) = (\lambda_k\kappa(e))^{-1}\lambda_k\kappa(e) = \kappa(e)^{-1}k\kappa(e) = \kappa(e)^{-1}\kappa(e) = e.$$

Similarly $(\lambda_a(f))^{-1}\lambda_a(f) = f$ and so $e \leqq f$. Therefore $\theta_k$ is an isomorphism of $\Delta_\kappa$ onto $\nabla(\theta_\kappa)$. Hence $\theta_\kappa$ is a homomorphism and $\kappa \in \Gamma_1$. Thus $\Gamma_1 = I$.

The following result relates $\Gamma_1$, the idealizer of $\Gamma(S)$ in $\Lambda(S)$, to $\Omega(S)$. The statement that we give here is the dual of [**6**, Proposition 5, Section 2].

PROPOSITION 2.10. *$\Pi_\Lambda(\Omega(S))$ is the idealizer of $\Gamma(S)$ in $\Lambda(S)$.*

Summing up the main results in this section we have.

THEOREM 2.11. *For an inverse semigroup $S$, $\Pi_\Lambda(\Omega(S))$ can variously be described as:*

(1) *the idealizer of $\Gamma(S)$ in $\Lambda(S)$;*

(2) *the unique maximal inverse subsemigroup of $\Lambda(S)$ containing $\Gamma(S)$;*

(3) *the unique maximal inverse subsemigroup of $\Lambda(S)$ with the idealizer of $\Gamma(E)$ in $\Lambda(S)$ as its set of idempotents;*

(4) *the unique maximal inverse subsemigroup of $\Lambda(S)$ with $\Lambda(E)$ as its set of idempotents;*

(5) *the set of all $\lambda \in \Lambda(S)$ such that*

(a) *$\nabla(\theta_\lambda)$ is a $P$-ideal, and*

(b) *$\theta_\lambda$ is a homomorphism.*

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*Proof.* The first characterization follows from Proposition 2.10. The characterizations (2), (3) and (4) then follow from Lemmas 2.2, 2.3, while the fifth characterization follows from Lemmas 2.8 and 2.9.

The usefulness of Theorem 2.11 lies in the fact that it gives various characterizations of $\Omega(S)$ in terms of left translations only, eliminating the necessity, while working with $\Omega(S)$, of continually manipulating pairs of mappings (recall that elements of $\Omega(S)$ are defined as linked pairs of translations). This feature will be used in later sections to characterize the translational hull of certain standard inverse semigroups and is used by the author elsewhere when considering the problem of extending homomorphisms between inverse semigroups to homomorphisms between their translational hulls.

Since the mappings $\theta_\lambda$ have played such a key role in the above discussions the temptation to investigate the homomorphism $\theta$ a little further is irresistible. This we do at the beginning of the next section.

In general, for an inverse semigroup $S$, $\Lambda(S)$ need not even be a regular semigroup as the following example illustrates. For any semilattice $X$, let $T_X$ denote the set of mappings $\alpha$ such that $\Delta(\alpha)$ and $\nabla(\alpha)$ are both principal ideals of $X$ and $\alpha$ is an isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$.

*Example.* Let $R_1$ and $R_2$ be two disjoint copies of the real numbers $R$. Let $X = R_1 \cup R_2 \cup \{z\}$ where $z \notin R_1 \cup R_2$. Denote by $x_1$ $(x_2)$ the element of $R_1$ $(R_2)$ corresponding to the real number $x$. For $a, b \in X$, let $a \leqq b$ if and only if either $a, b \in R_i (i = 1, 2)$ and $a \leqq b$ in $R_i$ or $a = z$. Let $S = T_X$, and for $x \in X$ let $\epsilon_x$ denote the identity mapping on $Xx$. Let $\delta$ be an order isomorphism of $R$ onto the negative real numbers and let $\gamma \in F_X$ be defined by:

$$\gamma(y) = \begin{cases} z, \text{ if } y \in R_1 \cup \{z\}, \\ (\delta x)_1, \text{ if } y = x_2 \in R_2. \end{cases}$$

Now define the mapping $\lambda$ of $T_X$ by

$$\lambda(\alpha) = \gamma \circ \alpha.$$

Then $\lambda(\alpha) \in T_X$ and $\lambda$ is a left translation of $T_X$. Moreover, $\lambda$ has no inverse in $\Lambda(T_X)$ and consequently, $\Lambda(T_X)$ is not regular.

Now let $\xi \in F_X$ be such that

$$\xi(y) = \begin{cases} y, \text{ if } y = x_1 \in R_1 \text{ or } y = z, \\ (\delta x)_1, \text{ if } y = x_2 \in R_2, \end{cases}$$

and let $l$ be the mapping of $T_X$ such that $l(\alpha) = \xi \circ \alpha$ for all $\alpha \in T_X$. Then $l \in \Lambda(T_X)$ and $l^2 = l$. However, $\theta_l$ is not an isomorphism of $\Delta_l$ onto $\nabla(\theta_l)$ and so $l \notin \Gamma_1$. This illustrates that, in general, $\Gamma_1$ does not contain all regular pairs in $\Lambda(S)$.

**3. The homomorphism $\theta$.** Let $T$ be an inverse semigroup. Then a congruence $\tau$ on $T$ is said to be *idempotent separating* if $a^2 = a$, $b^2 = b$ and

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$(a, b) \in \tau$ implies that $a = b$. Any inverse semigroup $T$ has a unique maximum idempotent separating congruence that has been characterized by Howie [**4**] as follows:

LEMMA 3.1. *Let $T$ be an inverse semigroup and $\mu$ be the maximum idempotent separating congruence on $T$. Then*

$$\mu = \{(a, b): aea^{-1} = beb^{-1} \text{ for all } e^2 = e \in T\}.$$

We can now characterize the congruence

$$\theta \circ \theta^{-1} = \{\lambda, l): \theta(\lambda) = \theta(l)\}$$

induced on $\Gamma_1$ by $\theta$. (For the purposes of this and the following two sections we consider $\theta$ as a homomorphism of $\Gamma_1$ into $F_E$.)

THEOREM 3.2. *The congruence $\theta \circ \theta^{-1}$ induced by $\theta$ on $\Gamma_1$ is the maximum idempotent separating congruence $\mu$ on $\Gamma_1$.*

*Proof.* Let $\lambda \in E_1$. Then, for any $e \in E$, $\lambda(e) \in E$, and so

$$\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1} = \lambda(e).$$

Hence, if $\theta_\lambda = \theta_l$ for $\lambda$, $l \in E_1$ then necessarily $\lambda = l$. Therefore $\theta \circ \theta^{-1}$ is idempotent separating and so $\theta \circ \theta^{-1} \subseteq \mu$.

Now suppose that $(\lambda, l) \in \mu$ and let $e \in E$. Let $\lambda'$ and $l'$ be inverses for $\lambda$ and $l$, respectively, in $\Gamma_1$. Then

$$\begin{aligned}\theta_\lambda(e) &= \lambda(e)\lambda(e)^{-1} = \lambda(e)\lambda'(\lambda(e)\lambda(e)^{-1})\\ &= \lambda\lambda_e(e)\lambda'\lambda(e)\lambda(e)^{-1} = \lambda\lambda_e(e\lambda'\lambda(e))\lambda(e)^{-1}\\ &= (\lambda\lambda_e\lambda'\lambda(e))\lambda(e)^{-1} = \lambda\lambda_e\lambda'(\lambda(e)\lambda(e)^{-1}),\end{aligned}$$

by Lemma 2.5. Hence, by Lemma 3.1, since $(\lambda, l) \in \mu$ and $\lambda_e \in E_1$, we have,

$$\theta_\lambda(e) = l\lambda_e l'(\lambda(e)\lambda(e)^{-1}) = l(el'(\lambda(e)\lambda(e)^{-1})) = l(e)l'(\lambda(e)\lambda(e)^{-1}).$$

Therefore

$$\theta_\lambda(e) \leqq l(e)l(e)^{-1} = \theta_l(e).$$

Similarly, $\theta_l(e) \leqq \theta_\lambda(e)$. Hence $\theta_\lambda(e) = \theta_l(e)$, for all $e \in E$ and so $\theta_\lambda = \theta_l$. Therefore $\mu \subseteq \theta \circ \theta^{-1}$ and the proof is complete.

For an inverse semigroup $T$ let $\mu_T$ denote the maximum idempotent separating congruence on $T$. Since the mapping $a \to \lambda_a$ is an isomorphism of $S$ onto $\Gamma(S)$ we clearly have

$$\mu_{\Gamma(S)} = \{(\lambda_a, \lambda_b): (a, b) \in \mu_S\}.$$

LEMMA 3.3. *The congruence $\theta \circ \theta^{-1}$ is given by*

$$\theta \circ \theta^{-1} = \{(\lambda, l): (\lambda(e), l(e)) \in \mu_S, \text{ for all } e \in E\}.$$

*Proof.* Let $\theta_\lambda = \theta_l$ and $e \in E$. Then

$$\theta(\lambda_{\lambda(e)}) = \theta(\lambda\lambda_e) = \theta(\lambda)\theta(\lambda_e) = \theta(l)\theta(\lambda_e) = \theta(l\lambda_e) = \theta(\lambda_{l(e)}).$$

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Hence

$$(\lambda_{\lambda(e)}, \lambda_{l(e)}) \in \theta \circ \theta^{-1} \cap \Gamma(S) \times \Gamma(S) \subseteq \mu_{\Gamma(S)}.$$

Therefore $(\lambda(e), l(e)) \in \mu_S$.

Conversely, let $(\lambda(e), l(e)) \in \mu_S$, for all $e$. Then by Lemma 3.1,

$$\theta_\lambda(e) = \lambda(e)\lambda(e)^{-1} = \lambda(e)e\lambda(e)^{-1} = l(e)el(e)^{-1} = l(e)l(e)^{-1} = \theta_l(e).$$

Thus $\theta_\lambda = \theta_l$.

COROLLARY 3.4. $\mu_{\Gamma_1} \cap \Gamma(S) \times \Gamma(S) = \mu_{\Gamma(S)}$.

The term *fundamental* has been introduced by Munn for those inverse semigroups for which the maximum idempotent separating congruence is the identity congruence.

COROLLARY 3.5. *$S$ is fundamental if and only if $\Gamma_1$ (and therefore $\Omega(S)$) is fundamental.*†

COROLLARY 3.6. *$S$ is fundamental if and only if $\theta$ is an isomorphism.*

Although not directly relevant to the rest of our discussions we mention in passing the following observation.

LEMMA 3.7. *Let $\sigma$ $(\tau, \nu)$ denote the minimum group congruence on $\Omega(S)$ $(\Pi(S), S)$. Then $\sigma \cap \Pi(S) \times \Pi(S) = \tau$ and every $\sigma$-class of $\Omega(S)$ has non-empty intersection with $\Pi(S)$. Thus $\Pi(S)/\sigma \cong \Pi(S)/\tau \cong S/\nu$.*

**4. The extension of the Howie-Munn and Vagner-Preston representations of $S$ to $\Gamma_1$.** For any semilattice $X$ let $W_X$ denote the set of order preserving mappings of ideals of $X$ onto ideals of $X$. Let $V_X$ denote the set of order preserving mappings of $P$-ideals onto ideals of $X$ which are isomorphisms when restricted to principal ideals. Let $U_X$ denote the set of those mappings which are isomorphisms of P-ideals of $X$ onto $P$-ideals of $X$. Then it is easily seen that $V_X$ and $W_X$ are semigroups, that $U_X$ is an inverse semigroup and that $T_X \subseteq U_X \subseteq V_X \subseteq W_X$. One easily verifies the following result.

LEMMA 4.1. (1) *$V_X$ is the left idealizer of $T_X$ in $W_X$.*

(2) *$U_X$ is the idealizer of $T_X$ in $W_X$.*

The right idealizer of $T_X$ in $W_X$ can similarly be described as the set of isomorphisms of ideals of $X$ onto $P$-ideals of $X$.

The following representation of an inverse semigroup is due to Howie and Munn (see [**5**]). Here we have the mappings on the left rather than the right.

LEMMA 4.2. *Let $S$ be an inverse semigroup with semilattice of idempotents $E$. Let $\theta'$ be the mapping of $S$ into $T_E$ defined by $\theta'(a) = \theta'_a$ where*

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*Added in proof.* This result has been proved independently by B.N. Schein in *Completions, translational hulls and ideal extensions of inverse semigroups*, Czechoslovak Math. J. *23* (1973), 575–610.

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(1) $\Delta(\theta_a') = Ea^{-1}a$, *and*
(2) $\theta_a'(e) = aea^{-1}$, *for all* $e \in \Delta(\theta_a')$.
*Then* $\theta'$ *is a homomorphism of* $S$ *into* $T_E$ *such that* $\theta' \circ (\theta')^{-1}$ *is the maximum idempotent separating congruence on* $S$.

Our first objective is to show that this representation of $S$ extends naturally to a homomorphism of $\Gamma_1$ into $U_E$.

We shall find the following observation useful.

LEMMA 4.3. *Let* $\lambda, l \in \Lambda(S)$. *Then*

$$\Delta_{\lambda l} = \{e \in \Delta_l : \theta_l(e) \in \Delta_\lambda\}.$$

*Proof.* Let $e \in \Delta_{\lambda l}$. Then

$$e = (\lambda l(e))^{-1}\lambda l(e) = (\lambda l(e))^{-1}(\lambda l(e))l(e)^{-1}l(e) \leqq l(e)^{-1}l(e).$$

Hence $e \in \Delta_l$. Now

$$\begin{aligned}\theta_l(e) &= l(e)l(e)^{-1} = l(e)el(e)^{-1} = l(e)(\lambda l(e))^{-1}\lambda l(e)l(e)^{-1}\\ &= (\lambda(l(e)l(e)^{-1}))^{-1}\lambda(l(e)l(e)^{-1}).\end{aligned}$$

Thus $\theta_l(e) \in \Delta_\lambda$.

Conversely, let $e \in \Delta_l$ and $\theta_l(e) \in \Delta_\lambda$. Let $f = \theta_l(e) = l(e)l(e)^{-1}$. Then $\lambda l(e) = \lambda(l(e)l(e)^{-1})l(e) = \lambda(f)l(e)$ and

$$\begin{aligned}e &= l(e)^{-1}l(e) = l(e)^{-1}l(e)l(e)^{-1}l(e) = l(e)^{-1}fl(e) = l(e)^{-1}\lambda(f)^{-1}\lambda(f)l(e)\\ &= (\lambda l(e))^{-1}\lambda l(e).\end{aligned}$$

Therefore $e \in \Delta_{\lambda l}$ and the proof of the lemma is complete.

We have already seen that, for any $\lambda \in \Gamma_1$, the restriction $\psi_\lambda$ of $\theta_\lambda$ to $\Delta_\lambda$ is an isomorphism of $\Delta_\lambda$ onto $\nabla(\theta_\lambda)$. Thus we have a mapping $\theta_\lambda \to \psi_\lambda$ of $\theta(\Gamma_1)$ into $U_E$.

THEOREM 4.4. *The mapping* $\psi : \lambda \to \psi_\lambda$ *is a homomorphism of* $\Gamma_1$ *into* $U_E$ *such that the composition of the mappings* $a \to \lambda_a$ *and* $\psi$ *is the Howie-Munn representation* $\theta'$ *of Lemma* 4.2. *Moreover, the congruence* $\psi \circ \psi^{-1}$ *induced by* $\psi$ *on* $\Gamma_1$ *is the maximum idempotent separating congruence on* $\Gamma_1$.

*Proof.* Let $\lambda, l \in \Gamma_1$. Then $\Delta(\psi_{\lambda l}) = \Delta_{\lambda l}$. On the other hand, $\Delta(\psi_\lambda \psi_l) = \{e : e \in \Delta(\psi_l)$ and $\psi_l(e) \in \Delta(\psi_\lambda)\} = \{e : e \in \Delta_l$ and $\theta_l(e) \in \Delta_\lambda\}$. Therefore, by Lemma 4.3, we have $\Delta(\psi_{\lambda l}) = \Delta(\psi_\lambda\psi_l)$. If $e \in \Delta(\psi_{\lambda l})$ then

$$\psi_{\lambda l}(e) = \theta_{\lambda l}(e) = \theta_\lambda\theta_l(e) = \psi_\lambda\psi_l(e).$$

Hence $\psi$ is a homomorphism.

Now suppose that, for $\lambda, l \in \Lambda(S)$, $\psi_\lambda = \psi_l$. Then $\Delta_\lambda = \Delta_l$. Let $e \in E$. Then, by Lemma 2.5 (3),

$$E\lambda(e)^{-1}\lambda(e) = Ee \cap \Delta_\lambda = Ee \cap \Delta_l = El(e)^{-1}l(e).$$

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Hence $\lambda(e)^{-1}\lambda(e) = l(e)^{-1}l(e) = f$, say. Now $\lambda(e) = \lambda(e)\lambda(e)^{-1}\lambda(e) = \lambda(e\lambda(e)^{-1}\lambda(e)) = \lambda(ef) = \lambda(f)$. Similarly, $l(e) = l(f)$. Hence

$$\theta_\lambda(e) = \theta_\lambda(f) = \psi_\lambda(f) = \psi_l(f) = \theta_l(f) = \theta_l(e).$$

Thus $\theta_\lambda = \theta_l$ and $\psi \circ \psi^{-1} \subseteq \theta \circ \theta^{-1}$. Since each $\psi_\lambda$ is the restriction of the corresponding $\theta_\lambda$ to $\Delta_\lambda$, it is clear that $\theta \circ \theta^{-1} \subseteq \psi \circ \psi^{-1}$. Hence $\psi \circ \psi^{-1} = \theta \circ \theta^{-1}$, the maximum idempotent separating congruence, by Theorem 3.2.

Let $\psi_{\lambda_a}$ be denoted by $\psi_a$. Then

$$\begin{aligned}\Delta(\psi_a) &= \{e : e = \lambda_a(f)^{-1}\lambda_a(f), \text{ for some } f \in E\} \\ &= \{e : e = fa^{-1}a, \text{ for some } f \in E\} = Ea^{-1}a = \Delta(\theta_a').\end{aligned}$$

Finally, for $e \in a^{-1}a$,

$$\psi_a(e) = \lambda_a(e)\lambda_a(e)^{-1} = ae(ae)^{-1} = aea^{-1} = \theta_a'(e).$$

Thus $\psi_a = \theta_a'$ and the composition of the mappings $a \to \lambda_a$ and $\psi$ is $\theta'$.

Let $\mathscr{I}_X$ denote the symmetric inverse semigroup on a set $X$ (cf. [**2**]). Then the Vagner-Preston representation of an inverse semigroup $S$ by one-to-one partial transformations of $S$ is described in the following lemma.

LEMMA 4.5. [**2**, Theorem 1.20]. *Let $S$ be an inverse semigroup and for each $a \in S$ define the element $\alpha_a'$ of $\mathscr{I}_S$ by*
(1) $\Delta(\alpha_a') = a^{-1}S(= a^{-1}aS)$;
(2) $\alpha_a'(x) = ax$ *for any* $x \in \Delta(\alpha_a')$.
*Then the mapping $\alpha' : a \to \alpha_a'$ is an isomorphism of $S$ into $\mathscr{I}_S$.*

We now extend $\alpha'$ to $\Gamma_1$. For any element $\lambda \in \Gamma_1$, we define a mapping $\alpha_\lambda$ by
(1) $\Delta(\alpha_\lambda) = \Delta_\lambda S = \{es : e \in \Delta_\lambda, s \in S\} = \{x \in S : xx^{-1} \in \Delta_\lambda\}$;
(2) $\alpha_\lambda(x) = \lambda(x)$, for any $x \in \Delta(\alpha_\lambda)$.
Let $x, y \in \Delta(\alpha_\lambda)$, $e = xx^{-1}$, $f = yy^{-1}$ and $\alpha_\lambda(x) = \alpha_\lambda(y)$. Then $e, f \in \Delta_\lambda$ and, by Lemma 3.1 (4), (with $\lambda'$ the inverse of $\lambda$ in $\Gamma_1$)

$$x = ex = \lambda'\lambda(e)x = \lambda'\lambda(x) = \lambda'\alpha_\lambda(x) = \lambda'\alpha_\lambda(y) = \ldots = y.$$

Thus $\alpha_\lambda \in \mathscr{I}_S$.

THEOREM 4.6. *The mapping $\alpha : \lambda \to \alpha_\lambda$ is an embedding of $\Gamma_1$ into $\mathscr{I}_S$ such that the composition of the mappings $a \to \lambda_a$ and $\alpha$ is the Vagner-Preston representation of $S$.*

*Proof.* Let $\lambda, l \in \Gamma_1$. Then $x \in \Delta(\alpha_\lambda\alpha_l)$ if and only if $xx^{-1} \in \Delta_l$ and $l(x)l(x)^{-1} = \alpha_l(x)\alpha_l(x)^{-1} \in \Delta_\lambda$. But $l(x)l(x)^{-1} = l(xx^{-1})x(l(xx^{-1})x)^{-1} = l(xx^{-1})(xx^{-1})l(xx^{-1}) = l(xx^{-1})l(xx^{-1})^{-1} = \theta_l(xx^{-1})$. Thus

$$\Delta(\alpha_\lambda\alpha_l) = \{x : xx^{-1} \in \Delta_l \text{ and } \theta_l(xx^{-1}) \in \Delta_\lambda\}$$

and

$$\Delta(\alpha_{\lambda l}) = \{x : xx^{-1} \in \Delta_{\lambda l}\}.$$

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By Lemma 4.3, $\Delta(\alpha_{\lambda l}) = \Delta(\alpha_\lambda \alpha_l)$. For $x \in \Delta(\alpha_{\lambda l})$ we have $\alpha_\lambda \alpha_l(x) = \alpha_\lambda(l(x)) = \lambda(l(x)) = \alpha_{\lambda l}(x)$. Thus $\alpha_\lambda \alpha_l = \alpha_{\lambda l}$ and $\alpha$ is a homomorphism.

Now suppose that $\alpha_\lambda = \alpha_l$ and let $x \in S$. Then $\Delta_\lambda = \Delta_l$. So let $f \in E$ be such that $Ef = Exx^{-1} \cap \Delta_\lambda = Exx^{-1} \cap \Delta_l$. Then $\lambda(xx^{-1}) = \lambda(f)$ and $l(xx^{-1}) = l(f)$. Therefore,

$$\lambda(x) = \lambda(xx^{-1})x = \lambda(f)x = \alpha_\lambda(f)x = \alpha_l(f)x$$
$$= l(f)x = l(xx^{-1})x = l(x),$$

and $\lambda = l$. Hence $\alpha$ is an isomorphism.

Let the image of $a \in S$ under the mappings $a \to \lambda_a$ and $\lambda \to \alpha_\lambda$ be denoted by $\alpha_a$ (rather than $\alpha_{\lambda_a}$) and let $\Delta_{\lambda_a} = \Delta_a$. Then $\Delta_a = \{\lambda_a(e)^{-1}\lambda_a(e) : e \in E\} = Ea^{-1}a$ and so $\Delta(\alpha_a) = \Delta_a S = a^{-1}aS$ and, for $x \in a^{-1}aS$, $\alpha_a(x) = \lambda_a(x) = ax$. Thus $\alpha_a = \alpha_a'$ (where $\alpha_a'$ is as in Lemma 4.5).

**5.** $\Lambda(T_X)$**.** Throughout this section let $X$ denote a semilattice and $S$ denote a *full* inverse subsemigroup of $T_X$, that is, an inverse subsemigroup of $T_X$ which contains all the idempotents of $T_X$. It has been shown by Munn [**5**] that such an inverse semigroup is fundamental, an observation that we shall require below.

Let $E$ denote the semilattice of idempotents of $S$. For any $x \in X$, let $\epsilon(x)$ denote the identity mapping on $Xx$. For any $e \in E$, the domain of $e$ is a principal ideal of $X$. Denote this by $X\delta(e)$, say. Since $S$ is a full inverse subsemigroup of $T_X$ the mappings $\epsilon : x \to \epsilon(x)$ and $\delta : e \to \delta(e)$ are then inverse isomorphisms of $X$ onto $E$ and $E$ onto $X$, respectively. For each $\lambda \in \Lambda(S)$, we define a mapping $\psi_\lambda$ with domain $\Delta(\psi_\lambda) = \delta(\Delta_\lambda) = \{x : \epsilon(x) \in \Delta_\lambda\} = \{x : \epsilon(x) = \lambda(e)^{-1}\lambda(e)$, for some $e \in E\} = \{x : \epsilon(x) = \lambda(\epsilon(x))^{-1}\lambda(\epsilon(x))\}$. For any $x \in \Delta(\psi_\lambda)$, let $\psi_\lambda(x) = \delta\theta_\lambda\epsilon(x)$. Since $\epsilon$ and $\delta$ are isomorphisms and from the properties of $\theta_\lambda$ it follows that $\Delta(\psi_\lambda)$ is a $P$-ideal, that $\nabla(\psi_\lambda)$ is an ideal and that $\psi_\lambda$ is an order-preserving mapping of $\Delta(\psi_\lambda)$ onto $\nabla(\psi_\lambda)$ which is an isomorphism when restricted to principal ideals.

THEOREM 5.1. *The mapping* $\psi : \lambda \to \psi_\lambda$ *is an isomorphism of* $\Lambda(S)$ *into* $V_X$ *such that*

(1) $\psi(\lambda_a) = a$, *for all* $a \in S$;
(2) $\psi(\Gamma_1) \subseteq U_X$;
(3) $\psi(\Lambda(S))$ *is the left idealizer of* $S$ *in* $W_X$;
(4) $\psi(\Gamma_1)$ *is the idealizer of* $S$ *in* $W_X$.

*If* $S = T_X$, *then*

(5) $\psi(\Lambda(S)) = V_X$;
(6) $\psi(\Gamma_1) = U_X$.

*Proof.* It is clear from the definition of $\psi_\lambda$ that $\psi_\lambda \in V_X$, for each $\lambda \in \Lambda(S)$. We first show that $\Delta(\psi_{\lambda l}) = \Delta(\psi_\lambda \psi_l)$.

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We have

$$\begin{aligned}\Delta(\psi_\lambda\psi_l) &= \{x : x \in \Delta(\psi_l) \text{ and } \psi_l(x) \in \Delta(\psi_\lambda)\}\\ &= \{x : \epsilon(x) \in \Delta_l \text{ and } \delta\theta_l\epsilon(x) \in \Delta(\psi_\lambda)\}\\ &= \{x : \epsilon(x) \in \Delta_l \text{ and } \theta_l\epsilon(x) \in \Delta_\lambda\}.\end{aligned}$$

Hence

$$\epsilon(\Delta(\psi_\lambda\psi_l)) = \{\epsilon(x) : \epsilon(x) \in \Delta_l \text{ and } \theta_l\epsilon(x) \in \Delta_l\}$$

while

$$\epsilon(\Delta(\psi_{\lambda l})) = \{\epsilon(x) : \epsilon(x) \in \Delta_{\lambda l}\}.$$

By Lemma 4.3, $\epsilon\ (\Delta(\psi_\lambda\psi_l)) = \epsilon(\Delta(\psi_{\lambda l}))$ and hence $\Delta(\psi_\lambda\psi_l) = \Delta(\psi_{\lambda l})$.

For any $x \in \Delta(\psi_{\lambda l}) = \Delta(\psi_\lambda\psi_l)$,

$$\psi_\lambda\psi_l(x) = \delta\theta_\lambda\epsilon\,\delta\theta_l\epsilon(x) = \delta\theta_\lambda\theta_l\epsilon(x) = \delta\theta_{\lambda l}\epsilon(x) = \psi_{\lambda l}(x).$$

Thus $\psi$ is a homomorphism.

Suppose that, for $\lambda,\ l \in \Lambda(S)$, $\psi_\lambda = \psi_l$. Then $\Delta_\lambda = \Delta_l$. Let $e \in E$. Then $f = \lambda(e)^{-1}\lambda(e)$ is such that $Ee \cap \Delta_\lambda = Ef$ and $g = l(e)^{-1}l(e)$ is such that $Ee \cap \Delta_l = Eg$. Since $\Delta_\lambda = \Delta_l$, we must have $\lambda(e)^{-1}\lambda(e) = l(e)^{-1}l(e)$, for all $e \in E$. Furthermore, since $\psi_\lambda(\delta(f)) = \psi_l(\delta(f))$, we have that $\theta_\lambda(f) = \theta_l(f)$. Hence

$$\lambda(e)\lambda(e)^{-1} = \theta_\lambda(e) = \theta_\lambda(f) = \theta_l(f) = \theta_l(e) = l(e)l(e)^{-1}.$$

Thus $(\lambda(e), l(e)) \in \mathscr{H}$, for any $e \in E$ (where $\mathscr{H}$ denotes Green's relation $\mathscr{H}$; cf. [**2**]). Now consider $a = l(e)^{-1}\lambda(e)$. We have $aa^{-1} = a^{-1}a = l(e)^{-1}l(e) = \lambda(e)^{-1}\lambda(e)$. Let $f$ be any idempotent $\leqq aa^{-1}$. Then

$$\begin{aligned}a^{-1}fa &= \lambda(e)^{-1}l(e)fl(e)^{-1}\lambda(e) = \lambda(e)^{-1}l(ef)l(ef)^{-1}\lambda(e)\\ &= \lambda(e)^{-1}\lambda(ef)\lambda(ef)^{-1}\lambda(e).\end{aligned}$$

Consequently

$$\begin{aligned}fa^{-1}fa &= \lambda(ef)^{-1}\lambda(ef)\lambda(ef)^{-1}\lambda(ef)\\ &= \lambda(ef)^{-1}\lambda(ef) = f\lambda(e)^{-1}\lambda(e) = f.\end{aligned}$$

Hence $f \leqq a^{-1}fa$. Similarly, $f \leqq afa^{-1}$. Thus

$$f = aa^{-1}faa^{-1} \geqq afa^{-1} \geqq f,$$

and $f = afa^{-1}$, for all $f \leqq aa^{-1}$. Hence $(a, aa^{-1}) \in \mu$, the maximum idempotent separating congruence on $S$. Since $S$ is fundamental, $a = aa^{-1} = a^{-1}a$ and so $\lambda(e) = l(e)$, for all $e \in E$, and $\lambda = l$. Therefore $\psi$ is an isomorphism.

(1) Let $a \in S$. Then $\Delta(a) = \Delta(a^{-1}a) = X\delta(a^{-1}a)$. On the other hand

$$\begin{aligned}\Delta(\psi_{\lambda_a}) &= \{x : \epsilon(x) \in \Delta_{\lambda_a}\} = \{x : \epsilon(x) = \lambda_a(\epsilon(x))^{-1}\lambda_a(\epsilon(x))\}\\ &= \{x : \epsilon(x) = \epsilon(x)a^{-1}a\} = \{x : \epsilon(x) \leqq a^{-1}a\} = \Delta(a).\end{aligned}$$

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For any $x \in \Delta(a)$, therefore,

$$\psi_{\lambda_a}(x) = \delta\theta_{\lambda_a}\epsilon(x) = \delta a\epsilon(x)a^{-1} = \delta\epsilon(a(x)) = a(x).$$

Thus $\psi_{\lambda_a} = a$.

(2) Since $\epsilon$ and $\delta$ are isomorphisms (2) follows directly from Proposition 2.8 and Lemma 2.7.

(3) Since $\Gamma(S)$ is a left ideal of $\Lambda(S)$ it follows that $S = \psi(\Gamma(S))$ is a left ideal of $\psi(\Lambda(S))$. Conversely, suppose that $\alpha \in W_X$ is such that $\alpha S \subseteq S$. Since $\alpha S \subseteq S$, $\alpha$ induces a left translation, $\lambda$ say, on $S$; that is, $\lambda(S) = \alpha s$, for all $s \in S$. Consider $\psi_\lambda$. If $x \in \Delta(\alpha)$, then $x \in \Delta(\alpha\epsilon(x)) = \Delta(\epsilon(x))$ and $(\alpha\epsilon(x))^{-1}(\alpha\epsilon(x)) = \epsilon(x)$. Thus $(\lambda(\epsilon(x)))^{-1}\lambda(\epsilon(x)) = (\alpha\epsilon(x))^{-1}(\alpha\epsilon(x)) = \epsilon(x)$. Thus $\epsilon(x) \in \Delta_\lambda$ and $x \in \Delta(\psi_\lambda)$. Conversely, if $x \in \Delta(\psi_\lambda)$ then $\epsilon(x) \in \Delta_\lambda$ and $\epsilon(x) = (\lambda\epsilon(x))^{-1}(\lambda\epsilon(x)) = (\alpha\epsilon(x))^{-1}(\alpha\epsilon(x))$. Hence $\Delta(\epsilon(x)) = \Delta(\alpha\epsilon(x)) \subseteq \Delta(\alpha)$. Hence $x \in \Delta(\alpha)$ and $\Delta(\alpha) = \Delta(\psi_\lambda)$. Finally, for $x \in \Delta(\alpha) = \Delta(\psi_\lambda)$,

$$\begin{aligned}\psi_\lambda(x) &= \delta\theta_\lambda\epsilon(x) = \delta(\lambda\epsilon(x))(\lambda\epsilon(x))^{-1} = \delta(\alpha\epsilon(x))(\alpha\epsilon(x))^{-1} \\ &= \delta\epsilon(\alpha(x)) = \alpha(x).\end{aligned}$$

Thus $\alpha = \psi_\lambda \in \psi(\Lambda(S))$.

(4) By (3), if $\alpha \in W_X$ is in the idealizer of $S$, then $\alpha \in \psi(\Lambda(S))$. But, by Proposition 2.9, the idealizer of $\Gamma(S)$ in $\Lambda(S)$ is $\Gamma_1$ and $\psi$ is an isomorphism. Hence, the idealizer of $S$ in $W_X$ is $\psi(\Gamma_1)$.

Parts (5) and (6) now follow from parts (1), (3) and (4) and Lemma 4.1.

For any semilattice $X$, let $A(X)$ denote the automorphism group of $X$.

COROLLARY 5.2. *In the notation of Theorem* 5.1, *if* $S = T_X$ *then the unit group of* $\psi(\Gamma_1)$ *is* $A(X)$.

*Proof.* Since $X$ is a $P$-ideal of $X$, the unit group of $U_X$ is $A(X)$. The corollary then follows from Theorem 5.1 (6).

**6. Brandt semigroups.** In this section, let $S = \mathscr{M}^0(G, I, I)$ be a Brandt semigroup, where $G$ is a group and $I$ is some set (see [**2**]). In [**7**] Petrich has characterized the translational hull of any completely 0-simple semigroup and by specializing his results to Brandt semigroups one could obtain the results that we obtain below by applying the techniques developed above.

If $S = \mathscr{M}^0(G, I, I)$ then $E = E_S = \{(1, i, i) : i \in I\} \cup \{0\}$. Hence any $P$-ideal of $E$ is of the form $\{(1, i, i) : i \in J\} \cup \{0\}$ for some arbitrary subset $J$ of $I$. For each $i$, let $e_i = (1, i, i)$ and let $\lambda \in \Gamma_1$. Since $\Delta_\lambda$ and $\nabla(\theta_\lambda)$ are both $P$-ideals of $E$, we have $\Delta_\lambda = \{e_i : i \in J_1\} \cup \{0\}$ and $\nabla(\theta_\lambda) = \{e_i : i \in J_2\} \cup \{0\}$, for some subsets $J_1, J_2$ of $I$.

Then the restriction $\psi_\lambda$ of $\theta_\lambda$ to $\Delta_\lambda$ is an isomorphism of $\Delta_\lambda$ onto $\nabla(\theta_\lambda)$ and so determines a bijection, which we also denote by $\psi_\lambda$, of $J_1 \to J_2$ (that is, $\psi_\lambda(1, i, i) = (1, \psi_\lambda(i), \psi_\lambda(i))$). For any $i \notin J_1$, $Ee_i \cap \Delta_\lambda = \{0\}$ and hence $\theta_\lambda(e_i) = \theta_\lambda(0) = 0$. Thus $\lambda(e_i)\lambda(e_i)^{-1} = 0$ and consequently $\lambda(e_i) = 0$. Let

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$x = (a, i, j) \in S$. Then $x = e_i x$ and $\lambda(x) = \lambda(e_i)x$. If $i \notin J_1$, then $\lambda(x) = \lambda(e_i)x = 0x = 0$. So suppose that $i \in J_1$ and that $\theta_\lambda(e_i) = e_k$, say. Then $\lambda(e_i)\lambda(e_i)^{-1} = e_k$ and, since $e_i \in \Delta_\lambda$, $\lambda(e_i)^{-1}\lambda(e_i) = e_i$. Therefore $\lambda(e_i) = (g_i, k, i)$, for some $g_i \in G$, and $\lambda(x) = (g_i a, k, j) = (g_i a, \psi_\lambda(i), i)$.

Following Petrich [7] we define the left wreath product $L = L(\mathcal{I}_I, G)$ of the symmetric inverse semigroup on $I$ with $G$ as follows. Let

$$L = \{(\psi, f) : \psi \in \mathcal{I}_I, \psi \neq 0, f : \Delta(\psi) \to G\} \cap \{0\}$$

with multiplication defined by

$$(\psi, f)(\psi', f') = (\psi\psi', f'') \text{ if } \psi\psi' \neq 0 \text{ and } 0 \text{ otherwise,}$$
$$0(\psi, f) = (\psi, f)0 = 0,$$

where $f''(i) = (f\psi'(i))(f'(i))$, if $i \in \Delta(\psi\psi')$.

From the above discussion, we have a mapping $\phi : \Gamma_1 \to L$ given by $\lambda \to (\psi_\lambda, f_\lambda)$ where, for $i \in \Delta(\psi_\lambda)$, $f_\lambda(i)$ is defined by $\lambda(e_i) = (f_\lambda(i), \psi_\lambda(i), i)$. Since, by Theorem 4.4, the mapping $\lambda \to \psi_\lambda$ is a homomorphism it is straightforward to verify that $\phi$ is a homomorphism. On the other hand, once the mappings $\psi_\lambda$ and $f_\lambda$ are known $\lambda$ is completely determined. Hence $\phi$ is a monomorphism. Finally, for any $(\psi, f) \in L$ let $\lambda$ be defined by

$$\lambda(a, i, j) = (f(i)a, \psi(i), j), \text{ for any } (a, i, j) \in S.$$

Then $\lambda$ is a left translation and $(\psi_\lambda, f_\lambda) = (\psi, f)$. Thus $\phi$ is an isomorphism. Hence we have the following result.

THEOREM 6.1. *Let $S = \mathcal{M}^0(G, I, I)$ be a Brandt semigroup. Then the mapping $\phi : \lambda \to (\psi_\lambda, f_\lambda)$ where $\psi_\lambda$ and $f_\lambda$ are defined by*

$$\lambda(a, i, j) = (f_\lambda(i)a, \psi_\lambda(i), j)$$

*is an isomorphism of $\Gamma_1$ onto the left wreath product $L(\mathcal{I}_I, G)$ of the symmetric inverse semigroup on $I$ and $G$.*

In particular, the unit group of $\phi(\Gamma_1)$ is the wreath product $L(S(I), G)$ of the group of all permutations $S(I)$ of $I$ with $G$, where the wreath product is now the usual wreath product of groups (with functions acting on the left).

The description of the unit group of $\phi(\Gamma_1)$ given in Theorem 6.1 is a special case of a theorem of Ault's [**1**].

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