1

Lectures on Vertex Algebras

Atsushi Matsuo

Abstract

The purpose of the present chapter is to explain the basics of vertex algebras, as well as some more advanced topics on vertex operator algebras, to the reader mainly in the fields of group theory and algebraic combinatorics.

CONTENTS

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Introduction

The Monster, the largest sporadic finite simple group of order

$$
\begin{aligned}
&2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71\\
&=\underbrace{808017424794512875886459904961710757005754368000000000}_{54\text{ digits}},
\end{aligned}
$$

is known to be realized as the automorphism group of the moonshine module $\mathbf{V}^{\natural}$, a distinguished example of a vertex operator algebra, equipped with a grading of the shape

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\begin{array}{ccccccccccc} \mathbf{V}^\natural & = & \mathbb{C}\mathbf{1} & \oplus & 0 & \oplus & \mathbf{B}^\natural & \oplus & \mathbf{V}_3^\natural \oplus \mathbf{V}_4^\natural \oplus \cdots, \\ \dim & & 1 & & 0 & & 196884 & & \end{array}$$

of which the dimensions of the homogeneous subspaces satisfy

$$\begin{aligned} q^{-1}\sum_{n=0}^{\infty} \dim \mathbf{V}_n^\natural q^n &= j(\tau) - 744 \\ &= q^{-1} + 0 + 196884q + 21493760q^2 + \cdots, \end{aligned} \tag{1}$$

where $j(\tau)$ is the elliptic modular function and $q = e^{2\pi\sqrt{-1}\tau}$.

The 196884-dimensional subspace $\mathbf{B}^\natural$ of degree 2 inherits a structure of a commutative nonassociative algebra with unity equipped with a nondegenerate symmetric invariant bilinear form, which we call the *Griess–Conway algebra*, as suggested by S. P. Norton. The algebra $\mathbf{B}^\natural$ is a variant of the algebras constructed by R. L. Griess in [61] to prove the existence of the Monster, and it is indeed the same as the algebra constructed by J. H. Conway in [38].

The notion of vertex algebras was introduced by R. E. Borcherds in the seminal paper [32] in 1986 by axiomatizing properties of infinite sequences of operators constructed from even lattices that generalize those considered for the root lattices of ADE type in the famous *Frenkel–Kac construction,* achieved by I. B. Frenkel and V. G. Kac in [57], to realize representations of affine Kac–Moody algebras associated with simple Lie algebras of the corresponding type. Such sequences of operators are related to the *vertex operators* in string theory, whence the term *vertex algebra*. The vertex operator is actually not a single operator but an infinite series with operator coefficients. The concept of vertex algebras can be seen to be a mathematical formulation of what is called the *operator product algebra* or the *chiral algebra* in physics.

Borcherds then applied vertex algebras to the study of the Monster via the moonshine module $\mathbf{V}^\natural$, which was previously introduced by I. B. Frenkel, J. Lepowsky, and A. Meurman [59] as a vector space equipped with some structures, and achieved in [33], with numerous outstanding ideas and works, the proof of the Conway–Norton conjecture, the conjecture that states the famous moonshine phenomena relating representations of the Monster and certain modular functions, the simplest among which is (1).

The concepts of vertex operator algebras (VOA) and their modules, in turn, were formulated by I. B. Frenkel, J. Lepowsky, and A. Meurman in [1] in order to set up appropriate "algebras" and "modules" by modifying those for vertex algebras. More precisely, a VOA is not just a vertex algebra, but a pair consisting of a vertex algebra and its element generating a representation of the Virasoro algebra satisfying a number of conditions that would make it suitable for applications.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 1 Codes, lattices and VOAs

Doubly even codes	Postive-definite even lattices	VOAs
Length	Rank	Central charge
Weight enumerator	Theta function	Conformal character
Self-dual	Unimodular	Holomorphic
Extended Hamming code H_8	Gosset lattice E_8	Lattice VOA $\mathbf{V}_{E_8}$
Extended Golay code G_{24}	Leech lattice Λ	Moonshine module $\mathbf{V}^\natural$
Mathieu group M_{24}	Conway group C_0	Monster $M = F_1$

For example, VOAs are assumed to be graded by integers with the homogeneous subspaces being finite-dimensional, so that one may consider the conformal character, the generating series of dimensions such as (1).

In fact, important applications of vertex algebras are often based on the properties of the Virasoro algebra, thus justifying the definition of VOAs.

The moonshine module $\mathbf{V}^\natural$ indeed carries a natural structure of a VOA. It possesses a distinguished position among VOAs when viewed through the famous analogies of binary codes, lattices, and VOAs as indicated in Table 1, although the uniqueness of $\mathbf{V}^\natural$ conjectured in [1], which is an analogue of the uniqueness of the extended Golay code G_{24} and the Leech lattice Λ, is yet to be settled. Thus the concept of VOAs is as natural as those of binary codes and lattices. However, even constructing a single example of a VOA is not so easy.

In Section 1.1, we will describe the definition of vertex algebras after preliminary sections, and then proceed to realization of vertex algebras by formal series with operator coefficients in Section 1.2, where the concept of modules over vertex algebras will also be introduced. Such realization enables us to state and prove the existence of vertex algebra structures under certain circumstances. Standard examples of vertex algebras will be described in Section 1.3.

Section 1.4 is devoted to construction of the vertex algebras associated with even lattices, where commutation relations of vertex operators play fundamental roles. In Section 1.5, we will explain the definition and construction of what are called *twisted modules* over vertex algebras by repeating the arguments of the previous sections in slightly more general settings, which enables one to construct the moonshine module $\mathbf{V}^\natural$ as a module over a fixed-point subalgebra of the Leech lattice vertex algebra by a lift of the (-1)-involution.

In Section 1.6, we will give brief accounts of theory of VOAs including fusion rules and modular invariance. We will then finish the sections by mentioning properties of the moonshine module and their variants that opened ways to new research directions.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The author is grateful to Professors Alexander A. Ivanov and Elena V. Konstantinova for inviting him to give the series of lectures in G2G2 2021 at Rogla. It was a hard task, to be honest, but very much fruitful indeed. The lectures were actually given online from Tokyo, and the author wishes to visit Rogla sometime in the future.

The author thanks Takuro Abe, Tomoyuki Arakawa, Hiroki Shimakura and Hiroshi Yamauchi for useful conversations in preparation of the manuscripts for the lectures and the referee for useful comments. The present sections are partly based on the author's past lectures at Nagoya Institute of Technology, National Taiwan University, University of the Ryukyus in 2003 etc.

1.1 Axioms for Vertex Algebras

A vertex algebra is a vector space equipped with countably many binary operations indexed by integers satisfying a number of axioms.

In Section 1.1, we start with preliminary sections on algebras and formal series and then describe the definition of vertex algebras and some consequences of the axioms. We will give a few examples: the commutative vertex algebras, the Heisenberg vertex algebra, and a Virasoro vertex algebra as a vertex subalgebra of the Heisenberg vertex algebra.

We will work over a field $\mathbb{F}$ of any characteristic not 2, thus vector spaces and linear maps are always over such a field $\mathbb{F}$, unless otherwise stated. We denote the set of integers by $\mathbb{Z}$ and that of nonnegative integers by $\mathbb{N}$.

1.1.1 Preliminaries on Algebras

For a vector space $\mathbf{M}$, consider the set $\operatorname{End}\mathbf{M}$ of all operators (endomorphisms) acting on $\mathbf{M}$. The symbol $I = I_{\mathbf{M}}$ refers to the identity operator.

For an operator $A \in \operatorname{End}\mathbf{M}$, we will denote the value of A at $v \in \mathbf{M}$ by juxtaposition:

$$A\colon \mathbf{M} \longrightarrow \mathbf{M}, \quad v \mapsto Av.$$

Compositions of operators, also written by juxtaposition, are taken from right to left unless specified by parentheses: for $A, B, C \in \operatorname{End}\mathbf{M}$ and $v \in \mathbf{M}$,

$$ABC = A(BC), \quad ABCv = A(B(Cv)), \quad \text{etc.}$$

The commutator of operators is denoted by the bracket as

$$[A, B] = AB - BA$$

for $A, B \in \operatorname{End}\mathbf{M}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.1.1.1 Associative Algebras

Let us first recall the definition of associative algebras. We will always assume that associative algebras are unital.

An *associative algebra* is a vector space $\mathbf{A}$ equipped with a bilinear map

$$\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}, \quad (a, b) \mapsto ab,$$

called *multiplication* or the *product operation*, satisfying the following axioms:

(A1) Associativity. For all $a, b, c \in \mathbf{A}$:

$$(ab)c = a(bc).$$

(A2) Unity. There exists an element $\mathbf{1} \in \mathbf{A}$ such that for all $a \in \mathbf{A}$:

$$\mathbf{1}a = a \quad \text{and} \quad a\mathbf{1} = a.$$

The element $\mathbf{1} \in \mathbf{A}$ in (A2) is uniquely determined by the conditions therein and called the *unity* of $\mathbf{A}$,

For a vector space $\mathbf{M}$, the set $\operatorname{End}\mathbf{M}$ of all operators acting on $\mathbf{M}$ becomes an associative algebra by composition of operators, of which the unity is the identity operator.

1.1.1.2 Modules over Associative Algebras

A *module* over $\mathbf{A}$, or an $\mathbf{A}$-*module,* is a vector space $\mathbf{M}$ equipped with a bilinear map

$$\mathbf{A} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (a, v) \mapsto av,$$

called an *action* of $\mathbf{A}$ on $\mathbf{M}$, satisfying

(AM1) Associativity. For all $a, b \in \mathbf{A}$ and $v \in \mathbf{M}$:

$$(ab)v = a(bv).$$

(AM2) Identity. For all $v \in \mathbf{M}$: $\qquad \mathbf{1}v = v.$

For $a \in \mathbf{A}$, the operator on $\mathbf{M}$ sending v to av is called the *action* of a on $\mathbf{M}$.

For an $\mathbf{A}$-module $\mathbf{M}$, consider the map assigning the action on $\mathbf{M}$ to each element of $\mathbf{A}$:

$$\rho_{\mathbf{M}} : \mathbf{A} \longrightarrow \operatorname{End}\mathbf{M}, \quad a \mapsto [v \mapsto av].$$

Then this map is a homomorphism of algebras. Such a homomorphism is called a *representation* of $\mathbf{A}$ on $\mathbf{M}$. The concepts of modules over $\mathbf{A}$ and representations of $\mathbf{A}$ are essentially the same.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The algebra $\mathbf{A}$ itself becomes an $\mathbf{A}$-module by the product operation, for which the *left action* of $a \in \mathbf{A}$ sending x to ax is called *left multiplication* by a. The corresponding representation

$$\rho_{\mathbf{A}} : \mathbf{A} \longrightarrow \operatorname{End}\mathbf{A}, \quad a \mapsto [x \mapsto ax]$$

is an isomorphism of algebras onto its image.

1.1.1.3 Lie Algebras

A *Lie algebra* is a vector space $\mathbf{L}$ equipped with a bilinear map

$$[\ ,\ ] : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}, \quad (X, Y) \mapsto [X, Y],$$

called the *bracket operation*, satisfying

(1) For all $X, Y, Z \in \mathbf{L}$:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

(2) For all $X \in \mathbf{L}$:

$$[X, X] = 0.$$

As the base field is assumed to be not of characteristic 2, the set of the two conditions is equivalently replaced by

(L1) Jacobi identity. For all $X, Y, Z \in \mathbf{L}$:

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]].$$

(L2) Antisymmetry. For all $X, Y \in \mathbf{L}$:

$$[X,Y] = -[Y,X].$$

Throughout the sections, we will take the latter conditions (L1) and (L2) as the axioms for Lie algebras and call the identity in (L1) the *Jacobi identity*, although this term usually refers to (1) rather than (L1).

For a vector space $\mathbf{M}$, the space $\operatorname{End}\mathbf{M}$ becomes a Lie algebra by the commutator of operators, for which the Jacobi identity

$$[[A,B],C] = [A,[B,C]] - [B,[A,C]], \quad A, B, C \in \operatorname{End}\mathbf{M}$$

trivially holds by cancellation of terms in

$$\begin{aligned}
&(ABC - BAC) - (CAB - CBA) \\
&\quad = \big((ABC - ACB) - (BCA - CBA)\big) \\
&\qquad - \big((BAC - BCA) - (ACB - CAB)\big).
\end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

A variant of this simple observation will serve as a basis for the Borcherds identity, the main identity for vertex algebras, where A, B, C are replaced by series with operator coefficients. (See Subsection 1.2.3.1.)

Similarly, any associative algebra $\mathbf{A}$ is regarded as a Lie algebra by the commutator

$$[a,b] = ab - ba, \quad a,b \in \mathbf{A}.$$

We will denote this Lie algebra by $\mathbf{L}(\mathbf{A})$.

Note 1.1. A vector space $\mathbf{L}$ equipped with a bracket operation satisfying (L1) but not necessarily (L2) is called a (left) *Leibniz algebra* and the property (L1) is called the (left) *Leibniz identity*. Note that (L1) is equivalently written as

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]],$$

which says that the operations of taking the brackets by elements of $\mathbf{L}$ are derivations with respect to the bracket operation itself.

1.1.1.4 Modules over Lie Algebras

An $\mathbf{L}$-*module*, or a *module* over $\mathbf{L}$, is a vector space $\mathbf{M}$ equipped with a bilinear map

$$\mathbf{L} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (X,v) \mapsto Xv,$$

satisfying

(LM) For all $X, Y \in \mathbf{L}$ and $v \in \mathbf{M}$:

$$[X,Y]v = X(Yv) - Y(Xv).$$

For an $\mathbf{L}$-module $\mathbf{M}$, consider the map assigning the corresponding action on $\mathbf{M}$ to each element of $\mathbf{L}$:

$$\rho_{\mathbf{M}} : \mathbf{L} \longrightarrow \operatorname{End} \mathbf{M}, \quad X \mapsto [v \mapsto Xv].$$

Then this map is a homomorphism of Lie algebras. Such a homomorphism is called a *representation* of $\mathbf{L}$ on $\mathbf{M}$. The concepts of modules over $\mathbf{L}$ and representations of $\mathbf{L}$ are essentially the same.

The Lie algebra $\mathbf{L}$ itself becomes an $\mathbf{L}$-module by the bracket operation, for which the action of $X \in \mathbf{L}$ sending Y to $[X,Y]$ is called the *adjoint action* of X, and the corresponding representation

$$\rho_{\mathbf{L}} : \mathbf{L} \longrightarrow \operatorname{End} \mathbf{L}, \quad X \mapsto [Y \mapsto [X,Y]],$$

is called the *adjoint representation* of $\mathbf{L}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let ρ be a representation of $\mathbf{L}$ on a vector space $\mathbf{M}$:

$$\rho : \mathbf{L} \longrightarrow \operatorname{End}\mathbf{M}, \quad \rho(X) : v \mapsto Xv.$$

For $X_1, X_2, \ldots, X_d \in \mathbf{L}$, the product $X_1X_2 \cdots X_d$ makes sense in $\operatorname{End}\mathbf{M}$ as

$$(X_1X_2 \cdots X_d)v = X_1X_2 \cdots X_d v,$$

but such a product $X_1X_2 \cdots X_d$ does not make sense in the Lie algebra $\mathbf{L}$.

The universal enveloping algebra resolves this inconvenience by collecting expressions of the form $X_1X_2 \cdots X_d$ subject to appropriate relations. We will give the precise formulation in the next subsection.

1.1.1.5 Universal Enveloping Algebras

Let $\mathbf{L}$ be a Lie algebra and consider the tensor algebra over $\mathbf{L}$,

$$\mathbf{T}(\mathbf{L}) = \bigoplus_{d=0}^{\infty} \mathbf{T}^d(\mathbf{L}), \quad \mathbf{T}^d(\mathbf{L}) = \underbrace{\mathbf{L} \otimes \cdots \otimes \mathbf{L}}_{d \text{ times}}.$$

We will identify the elements of $\mathbf{T}^0(\mathbf{L})$ with the scalars.

Let $\mathbf{U}(\mathbf{L})$ be the quotient of $\mathbf{T}(\mathbf{L})$ by the two-sided ideal $\mathbf{J}(\mathbf{L})$ generated by the elements of the form

$$X \otimes Y - Y \otimes X - [X,Y], \quad X, Y \in \mathbf{L}.$$

We will denote the image of $X_1 \otimes \cdots \otimes X_d$ in $\mathbf{U}(\mathbf{L})$ by $X_1 \cdots X_d$.

Let j be the canonical map which sends $X \in \mathbf{L}$ to its image in $\mathbf{U}(\mathbf{L})$:

$$j : \mathbf{L} \longrightarrow \mathbf{U}(\mathbf{L}) = \mathbf{T}(\mathbf{L})/\mathbf{J}(\mathbf{L}).$$

The associative algebra $\mathbf{U}(\mathbf{L})$ equipped with the map j is called the *universal enveloping algebra* of the Lie algebra $\mathbf{L}$, which is characterized by the following universal property:

For any associative algebra $\mathbf{A}$ and any homomorphism $\varphi : \mathbf{L} \longrightarrow \mathbf{L}(\mathbf{A})$ of Lie algebras, there exists a unique homomorphism of associative algebras $\psi : \mathbf{U}(\mathbf{L}) \longrightarrow \mathbf{A}$ such that the diagram

$$\begin{tikzcd}
\mathbf{L} \arrow[r, "\varphi"] \arrow[d, "j"'] & \mathbf{L}(\mathbf{A}) \arrow[d, equal] \\
\mathbf{U}(\mathbf{L}) \arrow[r, "\psi"'] & \mathbf{A}
\end{tikzcd}$$

commutes.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Considering the case when $\mathbf{A} = \operatorname{End}\mathbf{M}$ for a vector space $\mathbf{M}$, we see that giving an $\mathbf{L}$-module structure on $\mathbf{M}$ is equivalent to giving a $\mathbf{U}(\mathbf{L})$-module structure on $\mathbf{M}$:

$$\mathbf{L}\text{-modules} \longleftrightarrow \mathbf{U}(\mathbf{L})\text{-modules}.$$

The structure of the universal enveloping algebra as a vector space is described by the following theorem, called *Poincaré–Birkhoff–Witt theorem*, or PBW for short.

Theorem 1.2 (PBW) *Let $\mathbf{L}$ be a Lie algebra and $\mathbf{B}$ a totally ordered basis of $\mathbf{L}$. Then the elements of the set*

$$\{X_1 \cdots X_k \mid k \in \mathbb{N},\ X_1, \ldots, X_k \in \mathbf{B},\ X_1 \leq \cdots \leq X_k\}$$

form a basis of $\mathbf{U}(\mathbf{L})$.

In particular, it follows that the canonical map $j : \mathbf{L} \longrightarrow \mathbf{U}(\mathbf{L})$ is injective, and the representation

$$\mathbf{L} \longrightarrow \operatorname{End}\mathbf{U}(\mathbf{L}),$$

given by left multiplication, is an isomorphism of Lie algebras onto its image.

When the Lie algebra $\mathbf{L}$ is commutative, the algebra $\mathbf{U}(\mathbf{L})$ reduces to the symmetric algebra $\mathbf{S}(\mathbf{L})$ over the vector space $\mathbf{L}$.

1.1.2 Preliminaries on Formal Series

We will substantially work with formal series with operator coefficients. Let us summarize notations and basic properties of formal series in advance.

The formal series we will be dealing with are series consisting of infinitely many terms of both positive and negative degrees. We will simply call such a formal series a *series* for short.

1.1.2.1 Spaces of Formal Series

Let z be an indeterminate, V a vector space, and $v(z)$ a series with coefficients in V. Throughout the text, unless otherwise stated, the coefficients of a series $v(z)$ are indexed as in

$$v(z) = \sum_n v_n z^{-n-1},$$

where the summation is over all $n \in \mathbb{Z}$. The set of such series is denoted as

$$V[[z, z^{-1}]] = \Big\{ \sum_n v_n z^{-n-1} \,\Big|\, v_n \in V \text{ for all } n \in \mathbb{Z} \Big\}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Recall the following spaces of series of specific types:

$$V[[z]] = \Big\{ \sum_n v_n z^{-n-1} \Big| v_n = 0 \text{ for all } n \geq 0 \Big\},$$
$$V((z)) = \Big\{ \sum_n v_n z^{-n-1} \Big| \exists N \in \mathbb{N}\colon v_n = 0 \text{ for all } n \geq N \Big\},$$
$$V[z, z^{-1}] = \Big\{ \sum_n v_n z^{-n-1} \Big| \exists N \in \mathbb{N}\colon v_n = 0 \text{ unless } -N \leq n \leq N \Big\},$$
$$V[z] = \Big\{ \sum_n v_n z^{-n-1} \Big| \exists N \in \mathbb{N}\colon v_n = 0 \text{ unless } -N \leq n < 0 \Big\}.$$

Their elements are, respectively, called *formal power series*, *formal Laurent series*, *Laurent polynomials*, and *polynomials*. We may write

$$V((z)) = V[[z]][z^{-1}], \quad V[z, z^{-1}] = V[z][z^{-1}], \quad \text{etc.}$$

A bilinear map $U \times V \longrightarrow W$ induces bilinear maps on series such as

$$U((z)) \times V((z)) \longrightarrow W((z)), \quad U[z, z^{-1}] \times V[[z, z^{-1}]] \longrightarrow W[[z, z^{-1}]],$$

by the product

$$\sum_m u_m z^{-m-1} \sum_n v_n z^{-n-1} = \sum_{m,n} u_m v_n z^{-m-n-2}.$$

We will also consider series in many indeterminates. For example,

$$V((y))((z)) = \left\{ \sum_n v_n(y) z^{-n-1} \;\middle|\; \begin{array}{l} v_n(y) \in V((y)) \text{ for all } n \text{ and} \\ \exists N \geq 0\colon v_n(y) = 0 \text{ for all } n \geq N \end{array} \right\},$$

$$V((z))((y)) = \left\{ \sum_m v_m(z) y^{-m-1} \;\middle|\; \begin{array}{l} v_m(z) \in V((z)) \text{ for all } m \text{ and} \\ \exists M \geq 0\colon v_m(z) = 0 \text{ for all } m \geq M \end{array} \right\}.$$

We will write

$$V((y, z)) = V((y))((z)) \cap V((z))((y)).$$

The three spaces do not agree unless $V = 0$.

Note 1.3. A series of the form $v(z) = \sum_n v_n z^{-n-1}$ is seen to be a formal *Fourier series* by substitution $z = e^{2\pi i t}$. In this regard, the coefficients are sometimes called the *Fourier coefficients* or even the *Fourier modes* of the series.

1.1.2.2 Binomial Expansions

Let x, y, z, etc. be indeterminates. The binomial theorem states that, for non-negative integer n,

$$(x + z)^n = \sum_{i=0}^{n} \binom{n}{i} x^i z^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} z^i.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

We may consider similar expansions for negative powers, but then the two expansions become different. We will distinguish them by writing

$$(x+z)^n\big|_{|x|>|z|} = \sum_{i=0}^{\infty} \binom{n}{i} x^{n-i} z^i,$$
$$(x+z)^n\big|_{|x|<|z|} = \sum_{i=0}^{\infty} \binom{n}{i} x^{i} z^{n-i}.$$

We understand that the left-hand sides denote the formal series given by the right-hand sides, respectively. We thus have

$$(x+z)^n\big|_{|x|>|z|} \in \mathbb{F}[x, x^{-1}][[z]] \subset \mathbb{F}((x))((z)),$$
$$(x+z)^n\big|_{|x|<|z|} \in \mathbb{F}[z, z^{-1}][[x]] \subset \mathbb{F}((z))((x)).$$

The regions attached signify where the series are convergent when working over $\mathbb{C}$. Similarly, we write

$$(y-z)^n\big|_{|y|>|z|} = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} y^{n-i} z^i,$$
$$(y-z)^n\big|_{|y|<|z|} = \sum_{i=0}^{\infty} (-1)^{n-i} \binom{n}{i} y^{i} z^{n-i},$$

which belong to $\mathbb{F}((y))((z))$ and $\mathbb{F}((z))((y))$, respectively.

Note 1.4. These expansions are often written in the literatures as

$$\iota_{y,z}(y-z)^n = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} y^{n-i} z^i,$$
$$\iota_{z,y}(y-z)^n = \sum_{i=0}^{\infty} (-1)^{n-i} \binom{n}{i} y^{i} z^{n-i}.$$

It is also common to distinguish them by the order of the summands in the argument as $(y-z)^n$ and $(-z+y)^n$, although sometimes confusing.

1.1.2.3 Divided Derivatives of Series

Consider the operators $\partial_z^{(k)}$ acting on series in z defined for $k \in \mathbb{N}$ as

$$\partial_z^{(k)} \colon V[[z, z^{-1}]] \longrightarrow V[[z, z^{-1}]], \quad v(z) \mapsto \partial_z^{(k)} v(z),$$

where $\partial_z^{(k)} v(z)$ denotes the kth *divided derivative*:

$$\partial_z^{(k)} \sum_n v_n z^{-n-1} = \sum_n \binom{-n-1}{k} v_n z^{-n-k-1}$$
$$= \sum_n \binom{-n+k-1}{k} v_{n-k} z^{-n-1}.$$

We will omit the subscript z in $\partial_z^{(k)}$ if there is no danger of confusion.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The operators $\partial^{(k)}$ are iterative in the following sense:

$$\partial^{(i)}\partial^{(j)} = \binom{i+j}{i}\partial^{(i+j)} \text{ for all } i, j \in \mathbb{N}.$$

They annihilate constants:

$$\partial^{(k)}v = 0 \text{ for all } k \geq 1 \text{ and } v \in V.$$

They satisfy the Leibniz rule

$$\partial^{(k)}(u(z)v(z)) = \sum_{i+j=k}(\partial^{(i)}u(z))(\partial^{(j)}v(z)),$$

as long as the products make sense.

For an indeterminate x, define a formal power series $e^{x\partial_z}$ with operator coefficients by

$$e^{x\partial_z} = \sum_{k=0}^{\infty} x^k \partial_z^{(k)}.$$

Then it acts on a series $v(z)$ in the following sense:

$$e^{x\partial_z}v(z) = \sum_{k=0}^{\infty} x^k \partial_z^{(k)}v(z) = v(x+z)\big|_{|x|<|z|} \in V[[z, z^{-1}]][[x]]. \tag{1.1}$$

This is seen to be a formal analogue of Taylor expansion of $v(y)$ at $y = z$ giving a power series in $x = y - z$. Note that the Leibniz rule can be restated as

$$e^{x\partial_z}(u(z)v(z)) = (e^{x\partial_z}u(z))(e^{x\partial_z}v(z)).$$

Over a field of characteristic zero, we have

$$e^{x\partial_z} = \sum_{k=0}^{\infty}\frac{(x\partial_z)^k}{k!}, \text{ where } \partial_z = \frac{\partial}{\partial z},$$

which justifies the notation.

1.1.2.4 Formal Delta Functions

Let us next consider the *formal delta function*, which is a series defined by

$$\delta(x) = \sum_n x^n.$$

We will often encounter the following series of two indeterminates:

$$\delta(y, z) = z^{-1}\delta(y/z) = \sum_n y^n z^{-n-1} = \sum_n y^{-n-1}z^n.$$

Notice the following formula:

$$\delta(y, z) = \sum_{i=0}^{\infty} y^{-i-1}z^i + \sum_{i=0}^{\infty} y^i z^{-1-i} = \frac{1}{y-z}\bigg|_{|y|>|z|} - \frac{1}{y-z}\bigg|_{|y|<|z|}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let $\delta^{(k)}(y,z)$ denote the kth divided derivative of $\delta(y,z)$ with respect to z:

$$\delta^{(k)}(y,z) = \partial_z^{(k)}\delta(y,z) = \frac{1}{(y-z)^{k+1}}\Big|_{|y|>|z|} - \frac{1}{(y-z)^{k+1}}\Big|_{|y|<|z|}.$$

The relation

$$(y-z)^{k+1}\delta^{(k)}(y,z) = 0$$

holds for all $k \in \mathbb{N}$.

Note 1.5. The formal delta function $\delta(x)$ is seen to be the Fourier series expansion of a periodic analogue of the Dirac delta function up to a scalar factor.

1.1.3 Vertex Algebras

There are many equivalent ways to define vertex algebras. Here we pick up the one given by Borcherds in [33] and include the identity property. The resulting set of axioms is, in the author's opinion, the most natural.

To begin with, let us recall that an associative algebra $\mathbf{C}$ is said to be *commutative* if the multiplication is *symmetric*, that is, $ab = ba$ holds for all $a, b \in \mathbf{C}$. Since associative algebras are unital by assumption, we may replace symmetry by commutativity of left multiplication, that is, $a(bc) = b(ac)$ for $a, b, c \in \mathbf{C}$.

We may therefore define a commutative associative algebra alternatively by saying that it is a vector space $\mathbf{C}$ equipped with a bilinear map

$$\mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}, \quad (a,b) \mapsto ab,$$

satisfying

(C1) Commutativity and associativity. For all $a, b, c \in \mathbf{C}$:

$$a(bc) = b(ac) \quad \text{and} \quad (ab)c = a(bc).$$

(C2) Unity. There exists an element $\mathbf{1} \in \mathbf{C}$ such that, for all $a \in \mathbf{C}$,

$$\mathbf{1}a = a \quad \text{and} \quad a\mathbf{1} = a\,.$$

Note that associativity follows from commutativity under the presence of unity.

1.1.3.1 Definition of Vertex Algebras

Let $\mathbf{V}$ be a vector space equipped with countably many bilinear maps indexed by integers n as

$$\mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}, \quad (a,b) \mapsto a_{(n)}b.$$

We will call $a_{(n)}b$ the *nth product* for each n.

A *vertex algebra* is a vector space $\mathbf{V}$ equipped with such product operations satisfying the following axioms:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

(V0) Local truncation. For any $a, b \in \mathbf{V}$, there exists an $N \in \mathbb{N}$ such that

$$a_{(N+i)}b = 0 \text{ for all } i \geq 0.$$

(V1) Borcherds identity. For all $a, b, c \in \mathbf{V}$ and $p, q, r \in \mathbb{Z}$:

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a_{(p+r-i)}(b_{(q+i)}c) - \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} b_{(q+r-i)}(a_{(p+i)}c).$$

(V2) Vacuum. There exists an element $\mathbf{1} \in \mathbf{V}$ satisfying

1. Identity. For any $a \in \mathbf{V}$ and $n \in \mathbb{Z}$:

$$\mathbf{1}_{(n)}a = \begin{cases} 0 & (n \neq -1), \\ a & (n = -1). \end{cases}$$

2. Creation. For any $a \in \mathbf{V}$ and $n \in \mathbb{Z}_{\geq -1}$:

$$a_{(n)}\mathbf{1} = \begin{cases} 0 & (n \geq 0), \\ a & (n = -1). \end{cases}$$

Here are remarks on the axioms.

1. The three sums in the Borcherds identity in (V1) are finite sums by (V0). We therefore assume (V0) without a mention in referring to (V1).
2. The element $\mathbf{1}$ in (V2) is called the *vacuum* of $\mathbf{V}$, which is uniquely determined as it is a unity with respect to the (-1)st product.
3. The products $a_{(n)}\mathbf{1}$ with $n \leq -2$ are not specified in (V2), but their properties are encoded in the operators $T^{(k)} : \mathbf{V} \to \mathbf{V}$ defined by $T^{(k)}a = a_{(-k-1)}\mathbf{1}$ for $k = 0, 1, 2, \ldots$, called the *translation operators*, as described later in Subsection 1.1.3.4.
4. The identity property in (V2) in fact follows from the other axioms.

The concepts of subalgebras, homomorphisms, isomorphisms, ideals, and quotients, etc. are defined in obvious ways. A subalgebra of a vertex algebra is called a *vertex subalgebra*. For modules over vertex algebras, see Section 1.2.5.

The axioms (V1) and (V2) can be seen to be modelled on properties of series with operator coefficients under some assumptions on the set of series. See Section 1.2.3 for details.

Note 1.6. 1. Local truncation is usually called *truncation* in the literatures. 2. For each $a \in \mathbf{V}$ and $n \in \mathbb{Z}$, consider the action $a_{(n)} : x \mapsto a_{(n)}x$. Then the countably many product operations are collectively treated by the generating series $Y(a, z) = \sum_n a_{(n)}z^{-n-1}$. See Section 1.1.5 for details.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.1.3.2 Structure of the Borcherds Identity

Although the Borcherds identity looks extremely complicated, it includes important properties as special cases. Here are a few instances.

1. The Borcherds identity (V1) with $(p,q,r) = (0,0,0)$ reads
$$(a_{(0)}b)_{(0)}c = a_{(0)}(b_{(0)}c) - b_{(0)}(a_{(0)}c),$$
which is the Jacobi identity (L1) for Lie algebras with respect to the bracket given by the 0th product as $[a,b] = a_{(0)}b$.
2. If $a_{(n)}b = 0$ and $a_{(n)}c = 0$ hold for all $n \geq 0$, then the Borcherds identities with $(p,q,r) = (0,-1,-1)$ and $(p,q,r) = (-1,-1,0)$, respectively, read
$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c), \quad a_{(-1)}(b_{(-1)}c) = b_{(-1)}(a_{(-1)}c),$$
which is the axiom (C1) for commutative associative algebras with respect to the (-1)st product.

Thus the Borcherds identity can be viewed as an "enhancement" of the Jacobi identity for Lie algebras and an "extension" of associativity and commutativity for commutative associative algebras.

There are redundancies in the Borcherds identity. Let $B(p,q,r)$ be either of the three sums in (V1):
$$B(p,q,r) = \sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c, \quad \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a_{(p+r-i)}(b_{(q+i)}c),$$
$$\text{or} \quad \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} b_{(q+r-i)}(a_{(p+i)}c).$$
Then the following recurrence relation holds for all $p,q,r \in \mathbb{Z}$:
$$B(p+1,q,r) = B(p,q+1,r) + B(p,q,r+1). \tag{1.2}$$
This implies the following lemma.

Lemma 1.7 *If the Borcherds identity holds for some p and all q,r and for some r and all p,q, then it holds for all p,q,r.*

Note 1.8. A vertex algebra equipped with the 0th product need not be a Lie algebra since skew-symmetry (L2) may not hold (cf. Notes 1.1 and 1.10).

1.1.3.3 Commutator and Associativity Formulas

The Borcherds identity with $(p,q,r) = (m,n,0)$ reads
$$\sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}c = a_{(m)}(b_{(n)}c) - b_{(n)}(a_{(m)}c).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Therefore, the following property holds in any vertex algebra $\mathbf{V}$.

(VC) Commutator formula. For all $a, b \in \mathbf{V}$ and $m, n \in \mathbb{Z}$:

$$[a_{(m)}, b_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{(m+n-i)} .$$

The Borcherds identity with $(p, q, r) = (0, n, m)$ is as follows.

(VA) Associativity formula. For all $a, b, c \in \mathbf{V}$ and $m, n \in \mathbb{Z}$:

$$(a_{(m)}b)_{(n)}c = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a_{(m-i)}b_{(n+i)}c - \sum_{i=0}^{\infty} (-1)^{m-i} \binom{m}{i} b_{(m+n-i)}a_{(i)}c.$$

By Lemma 1.7, the Borcherds identity holds if and only if both the commutator formula and the associativity formula hold:

$$\text{(V1)} \iff \text{(VC)} + \text{(VA)}.$$

As an application of the commutator formula, consider the subspace of End $\mathbf{V}$ spanned by the left actions of elements of a vertex algebra $\mathbf{V}$:

$$\operatorname{Span} \left\{ a_{(n)} \middle| a \in \mathbf{V}, n \in \mathbb{Z} \right\} \subset \operatorname{End} \mathbf{V}.$$

Then (VC) implies that this space is closed under taking commutators, thus forms a Lie subalgebra of End $\mathbf{V}$.

As for the associativity formula, let $\langle \mathbf{S} \rangle_{\mathrm{VA}}$ denote the vertex subalgebra *generated* by a subset $\mathbf{S}$ of a vertex algebra, that is, the span of the elements obtained by repeatedly applying the product operations to elements of $\mathbf{S}$ in arbitrary order. Then, by (VA) and (V2), it is actually given by left actions as

$$\langle \mathbf{S} \rangle_{\mathrm{VA}} = \operatorname{Span} \left\{ a^1_{(n_1)} \cdots a^k_{(n_k)} \mathbf{1} \middle| k \in \mathbb{N}, a^1, \ldots, a^k \in \mathbf{S}, n_1, \ldots, n_k \in \mathbb{Z} \right\}.$$

We understand that application zero time gives the vacuum $\mathbf{1}$.

Note 1.9. 1. The associativity formula in the sense above is called the *iterate formula* or the *associator formula* in the literatures. 2. The Lie algebra spanned by the left actions of a vertex algebra is often called the *Lie algebra of Fourier modes* (cf. Subsection 1.6.3.1).

1.1.3.4 Translation Operators

For any vertex algebra $\mathbf{V}$, canonically associated are the operators

$$T^{(k)} : \mathbf{V} \longrightarrow \mathbf{V}, \quad k = 0, 1, 2, \cdots$$

defined by setting, for $a \in \mathbf{V}$,

$$T^{(k)}a = a_{(-k-1)}\mathbf{1}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

These operators are called the *translation operators* or the *derivations* of the vertex algebra $\mathbf{V}$. Note that $T^{(0)} = I$ is just the identity operator.

By these operators, the creation property in (V2) is completed as

$$a_{(n)}\mathbf{1} = \begin{cases} 0 & (n \geq 0), \\ T^{(k)}a & (n = -k-1 < 0). \end{cases}$$

On the other hand, the identity property in (V2) implies

$$T^{(k)}\mathbf{1} = \begin{cases} 0 & (k \geq 1), \\ \mathbf{1} & (k = 0). \end{cases}$$

The following properties are consequences of the Borcherds identity.

(VT) Translation. For all $a, b \in \mathbf{V}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$:

$$(T^{(k)}a)_{(n)}b = (-1)^k \binom{n}{k} a_{(n-k)}b.$$

(VL) Leibniz rule. For all $a, b \in \mathbf{V}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$:

$$T^{(k)}(a_{(n)}b) = \sum_{i+j=k} (T^{(i)}a)_{(n)}(T^{(j)}b).$$

(VI) Iterativity. For all $i, j \in \mathbb{N}$:

$$T^{(i)}T^{(j)} = \binom{i+j}{i} T^{(i+j)}.$$

Over a field of characteristic zero, (VI) implies

$$T^{(k)} = \frac{T^k}{k!} \quad \text{for} \quad T = T^{(1)},$$

and the properties (VT) and (VL) for all $k \in \mathbb{N}$ follow from those with $k = 1$.

1.1.3.5 Skew-Symmetry

The following property follows from the axioms (V1) and (V2).

(VS) Skew-symmetry. For all $a, b \in \mathbf{V}$ and $n \in \mathbb{Z}$:

$$a_{(n)}b = (-1)^{n+1} \sum_{i=0}^{\infty} (-1)^i T^{(i)}(b_{(n+i)}a).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

We may view this property as a counterpart in vertex algebras of symmetry and antisymmetry for commutative associative algebras and Lie algebras, respectively, depending on the parity of n.

To be more precise, consider the following subspace of $\mathbf{V}$:

$$T^{(\geq 1)}\mathbf{V} = \sum_{k=1}^{\infty} T^{(k)}\mathbf{V}.$$

Then, picking up the term with $i = 0$ in (VS), we have

$$a_{(n)}b \equiv (-1)^{n+1} b_{(n)}a \mod T^{(\geq 1)}\mathbf{V}.$$

Therefore, modulo the subspace $T^{(\geq 1)}\mathbf{V}$, the nth product of a vertex algebra $\mathbf{V}$ is symmetric for odd n and antisymmetric for even n.

Note 1.10. 1. The 0th product of a vertex algebra $\mathbf{V}$ satisfies the Jacobi identity (L1) on $\mathbf{V}$, while the skew-symmetry (L2) holds modulo $T^{(\geq 1)}\mathbf{V}$, and the quotient $\mathbf{V}/T^{(\geq 1)}\mathbf{V}$ indeed becomes a Lie algebra. 2. If the base field is of characteristic zero, then the subspace $T^{(\geq 1)}\mathbf{V}$ agrees with the image of the translation operator $T = T^{(1)}$. Moreover, if T agrees with the left action of an element of $\mathbf{V}$, then, by skew-symmetry, left ideals of $\mathbf{V}$ become two-sided ideals. This remark indeed applies to vertex operator algebras. See Sections 1.6.1 and 1.6.2 for details.

1.1.4 A Few Examples

It is not at all easy to construct examples of vertex algebras. The easiest is the commutative vertex algebra, but it is not really a new object since it is just a commutative associative algebra with iterative derivations.

The second easiest, the simplest noncommutative example of a vertex algebra, is supplied by *free boson theory* in physics. It is called the *Heisenberg vertex algebra* and contains another example of a vertex algebra, a *Virasoro vertex algebra,* as a vertex subalgebra.

1.1.4.1 Commutative Vertex Algebras

In a vertex algebra $\mathbf{V}$, the following conditions for elements $a, b \in \mathbf{V}$ are equivalent to each other by (VC) and (V2).

(1) $a_{(k)}b = 0$ for all $k \geq 0$. (2) $[a_{(m)}, b_{(n)}] = 0$ for all $m, n \in \mathbb{Z}$.

A vertex algebra $\mathbf{V}$ is said to be *commutative* if the equivalent conditions hold for all $a, b \in \mathbf{V}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Regard such a vertex algebra as a vector space and equip it with the product given by the (-1)st product:

$$ab = a_{(-1)}b.$$

Then it becomes a commutative associative algebra, for which the vacuum $\mathbf{1}$ is the unity. The translation operators $T^{(k)}$ act as iterative derivations with respect to the product, by which the nth products are written as

$$a_{(n)}b = \begin{cases} 0 & (n \geq 0), \\ (T^{(k)}a)b & (n = -k-1 < 0). \end{cases}$$

Thus the commutative vertex algebras fall into the concept of commutative associative algebras with iterative derivations. In this regard, vertex algebras are essentially infinite-dimensional objects, as we see by the following proposition.

Proposition 1.11 *Any finite-dimensional vertex algebra is commutative.*

Proof. Assume $a_{(n)}b \neq 0$ for some $n \geq 0$ and take the minimal n among such. Then the matrix with entries $(T^{(i)}a)_{(n+j)}b$ indexed by $i, j \in \mathbb{N}$ is upper triangular with the diagonals $(-1)^i \binom{n+i}{i} a_{(n)}b$, which are nonzero for infinitely many $i \in \mathbb{N}$. Therefore, the sequence $a, Ta, T^{(2)}a, \cdots$ contains infinitely many linearly independent elements of $\mathbf{V}$. □

1.1.4.2 Heisenberg Vertex Algebra

Let h be an element of a vertex algebra $\mathbf{V}$ satisfying

$$h_{(n)}h = \begin{cases} 0 & (n \geq 2), \\ \mathbf{1} & (n = 1). \end{cases} \tag{1.3}$$

Then $h_{(0)}h = 0$ follows by skew-symmetry (VS), and the commutator formula (VC) implies that the operators $a_n = h_{(n)}$ satisfy

$$[a_m, a_n] = m\delta_{m+n,0}, \tag{1.4}$$

the commutation relation for the *Heisenberg algebra*.

The Heisenberg commutation relation (1.4) can be realized in a vertex algebra. To see it, consider the polynomial ring $\mathbb{F}[x_1, x_2, \cdots]$ with countably many indeterminates. Identify the scalar multiple of the unity of the ring with the scalars and denote the multiplication operator for a polynomial by the same symbol.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let a_n, $n \in \mathbb{Z}$ denote the operators acting on $\mathbb{F}[x_1, x_2, \cdots]$ defined by

$$a_n = \begin{cases} n\frac{\partial}{\partial x_n} & (n > 0), \\ 0 & (n = 0), \\ x_k & (n = -k < 0). \end{cases} \tag{1.5}$$

Then they satisfy the commutation relation (1.4).

The vertex algebra given in the following proposition is called the *Heisenberg vertex algebra*.

Proposition 1.12 *The vector space $\mathbb{F}[x_1, x_2, \cdots]$ carries a unique structure of a vertex algebra such that*

$$x_{1(n)} = a_n \ \textit{for all } n \in \mathbb{Z}, \tag{1.6}$$

with the vacuum **1** *being the unity of the polynomial ring.*

By the definition (1.5) of the actions a_n, the condition (1.6) implies, for example:

$$x_{1(n)}x_1 = \begin{cases} 0 & (n \geq 2), \\ 1 & (n = 1), \\ 0 & (\mathrm{n} = 0), \\ x_k x_1 & (n = -k < 0). \end{cases}$$

Thus the element $h = x_1$ satisfies (1.3). Repeated use of the associativity formula (VA) allows us to calculate the nth products for all polynomials.

The difficulty lies in guaranteeing consistency of the elements arising from the use of the Borcherds identity. We will see in the Section 1.2 that we can avoid this difficulty by identifying the nth products with certain products defined on series with operator coefficients, for which the Borcherds identity automatically holds under certain circumstances.

We will describe the details of this example including the proof of the proposition in Section 1.3, where the underlying polynomial ring $\mathbb{F}[x_1, x_2, \cdots]$ will be naturally identified with what is called the *Fock module* of *charge* 0 over the Heisenberg algebra (cf. Proposition 3.1).

Note 1.13. The vertex algebra described above is the Heisenberg vertex algebra of *rank one*. It is also called the *free bosonic vertex algebra* in the literatures.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.1.4.3 Virasoro Vectors

Let ω be an element of a vertex algebra $\mathbf{V}$ satisfying the following condition with a scalar c:

$$\omega_{(n)}\omega = \begin{cases} 0 & (n \geq 4), \\ (c/2)\mathbf{1} & (n = 3), \\ 2\omega & (n = 1). \end{cases} \tag{1.7}$$

Then $\omega_{(0)}\omega = T^{(1)}\omega$ and $\omega_{(2)}\omega = 0$ follow by skew-symmetry (VS).

Such an ω is called a *Virasoro vector* since the operators $L_n = \omega_{(n+1)}$ satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} c\delta_{m+n,0},$$

the commutation relation for the *Virasoro algebra* of central charge c, where

$$\frac{m^3 - m}{12} = \frac{1}{2}\binom{m+1}{3}$$

is a half integer.

The Heisenberg vertex algebra $\mathbb{F}[x_1, x_2, \cdots]$ actually contains a Virasoro vector. Indeed, consider the following element:

$$u = x_1^2 = x_{1(-1)}x_1 = a_{-1}a_{-1}1.$$

By the associativity formula (VA) with $m = -1$, we have

$$u_{(n)}u = (x_{1(-1)}x_1)_{(n)}u = \sum_{i=0}^{\infty} a_{-i-1}a_{n+i}a_{-1}a_{-1}1 + \sum_{i=0}^{\infty} a_{n-1-i}a_i a_{-1}a_{-1}1.$$

After some algebra,

$$\begin{aligned} u_{(1)}u &= 2a_{-1}a_1a_{-1}a_{-1}1 &&= 4u, \\ u_{(3)}u &= 2a_1a_{-1}1 &&= 2, \\ u_{(4)}u &= 2a_2a_{-1}1 &&= 0, \ \cdots . \end{aligned}$$

Therefore, the following element satisfies (1.7) as desired with $c = 1$:

$$\omega = \frac{u}{2} = \frac{1}{2}x_1^2. \tag{1.8}$$

We will call it the *standard* Virasoro vector of the Heisenberg vertex algebra to distinguish it from other Virasoro vectors in the same vertex algebra.

We may consider the vertex subalgebra generated by ω, which is actually spanned by the elements of the form

$$L_{n_1} \cdots L_{n_k}\mathbf{1} \ \ (k \in \mathbb{N},\ n_1, \ldots, n_k \in \mathbb{Z}).$$

This is an example of what is called a *Virasoro vertex algebra*.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Note 1.14. 1. Over the field $\mathbb{C}$ of complex numbers, the Virasoro vertex algebra obtained above is actually isomorphic to the simple vertex algebra denoted $\mathbf{L}(1,0)$ See Subsection 1.3.3.3 for details. 2. The construction of ω as above is a particular case of the process called the *Sugawara construction* for affine Lie algebras (cf. Subsection 1.3.2.4). 3. For any scalar λ, the vector $\omega_\lambda = x_1^2/2 + \lambda x_2$ is a Virasoro vector of central charge $c = 1 - 12\lambda^2$. This construction is called the *Feigin–Fuchs construction*.

1.1.5 Description by Generating Series

Let $\mathbf{V}$ be a vertex algebra. For each $a \in \mathbf{V}$ and $n \in \mathbb{Z}$, consider the left actions with respect to the nth product:

$$a_{(n)} : \mathbf{V} \longrightarrow \mathbf{V}, \quad x \mapsto a_{(n)}x.$$

Then the countably many product operations are collectively expressed in the generating series

$$Y(a,z) = \sum_n a_{(n)} z^{-n-1},$$

and the axioms (V0) and (V2), for instance, are expressed as follows:

(V0) Local truncation. For all $a, b \in \mathbf{V}$: $Y(a,z)b \in \mathbf{V}((z))$.
(V2) Vacuum. There exists an element $\mathbf{1} \in \mathbf{V}$ satisfying, for all $a \in \mathbf{V}$,

$$Y(\mathbf{1},z)a = a \quad \text{and} \quad Y(a,z)\mathbf{1} \in a + \mathbf{V}[[z]]z.$$

In this section, we will describe various properties of vertex algebras in terms of generating series. The expression of the Borcherds identity (V1) by generating series in its full form, called the *Cauchy–Jacobi identity,* will be given in Subsection 1.1.5.3.

1.1.5.1 Local Commutativity and Associativity

Recall that the conjunction of the commutator formula (VC) and the associativity formulas (VA) is equivalent to the Borcherds identity (V1). In this subsection, we will consider another pair of conditions whose conjunction is equivalent to the Borcherds identity.

By local truncation (V0), there exists an N such that $a_{(N+i)}b = 0$ for all $i \geq 0$. For such an N, the Borcherds identity with $(p,q,r) = (m,n,N)$ reads:

$$\begin{aligned} 0 = &\sum_{i=0}^{\infty} (-1)^i \binom{N}{i} a_{(m+N-i)}(b_{(n+i)}c) \\ &- \sum_{i=0}^{\infty} (-1)^{N-i} \binom{N}{i} b_{(n+N-i)}(a_{(m+i)}c). \end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Similarly, there exists an L such that $a_{(L+i)}c = 0$ for all $i \geq 0$. For such an L, the Borcherds identity with $(p,q,r) = (L,n,m)$ reads:

$$\sum_{i=0}^{\infty} \binom{L}{i} (a_{(m+i)}b)_{(n+L-i)}c = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a_{(m+L-i)}(b_{(n+i)}c).$$

These properties are better described by generating series as follows:

(VLC) Local commutativity. For any $a, b \in \mathbf{V}$, there exists an $N \in \mathbb{N}$ such that

$$(y-z)^N Y(a,y)Y(b,z) = (y-z)^N Y(b,z)Y(a,y).$$

(VLA) Local associativity. For any $a, c \in \mathbf{V}$, there exists an $L \in \mathbb{N}$ such that, for all $b \in \mathbf{V}$,

$$(x+z)^L Y(Y(a,x)b,z)c = (x+z)^L Y(a,x+z)\big|_{|x|>|z|} Y(b,z)c.$$

By the recurrence relation (1.2), we have the following implications:

$$(\text{VC}) \implies (\text{VLC}), \quad (\text{VA}) \implies (\text{VLA}),$$

$$(\text{V1}) \iff (\text{VLC}) + (\text{VLA}).$$

Note 1.15. 1. In the literatures, local commutativity is usually called *locality* or *weak commutativity,* while local associativity is called *weak associativity.* They are sometimes called *commutativity* and *associativity,* respectively.
2. The commutator formula (VC) and the associativity formula (VA) are written respectively in terms of generating series as

$$[Y(a,y), Y(a,z)] = \sum_{i=0}^{\infty} Y(a_{(i)}b, z)\delta^{(i)}(y,z),$$
$$Y(a_{(m)}b, z) = Y(a,z)_{(m)}Y(b,z) \quad (m \in \mathbb{Z}).$$

These formulas, as well as local commutativity and local associativity, have clear meanings in the language of *operator product expansion.* See Section 1.2 for details.

1.1.5.2 Translation Covariance

Let us next consider the translation operators defined for each $k \in \mathbb{N}$ by

$$T^{(k)} \colon \mathbf{V} \longrightarrow \mathbf{V}, \quad a \mapsto T^{(k)}a = a_{(-k-1)}\mathbf{1}.$$

For an indeterminate x, we formally write

$$e^{xT} = \sum_{k=0}^{\infty} x^k T^{(k)}.$$

Here are properties related to the translation operators.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

(VT) Translation. For all $a \in \mathbf{V}$ and $k \in \mathbb{N}$:

$$Y(T^{(k)}a, z) = \partial_z^{(k)} Y(a, z).$$

(VL) Leibniz rule. For all $a \in \mathbf{V}$:

$$e^{xT} Y(a, z) = Y(e^{xT} a, z) e^{xT}.$$

(VS) Skew-symmetry. For all $a, b \in \mathbf{V}$:

$$Y(a, z)b = e^{zT} Y(b, -z)a.$$

The translation property (VT) implies

$$Y(e^{xT} a, z) = e^{x\partial_z} Y(a, z) = Y(a, x + z)\big|_{|x|<|z|}.$$

Therefore, the operators $T^{(k)}$ are seen to generate *translation.*

Combining it with (VL), we have the *translation covariance*,

$$e^{xT} Y(a, z) e^{-xT} = Y(a, x + z)|_{|x|<|z|},$$

from which $Y(a, z)\mathbf{1} = e^{zT} a$ follows by (V2).

Note 1.16. Over a field of characteristic zero, we have

$$e^{xT} = \sum_{k=0}^{\infty} \frac{(xT)^k}{k!}, \quad \text{where } T = T^{(1)}.$$

The properties (VT) and (VL) for all k follow from those with $k = 1$:

$$Y(Ta, z) = \partial_z Y(a, z) \ \text{ and } \ [T, Y(a, z)] = Y(Ta, z),$$

whence $\partial_z Y(a, z) = [T, Y(a, z)]$, the "equation of motion" of $Y(a, z)$.

1.1.5.3 Cauchy–Jacobi Identity

For convenience of readers in consulting the literatures, we will describe the Borcherds identity (V1) in terms of generating series.

Recall the formal delta function $\delta(z) = \sum_n z^n$, where n runs over the integers, and consider the following expressions:

$$y^{-1}\delta\Big(\frac{z+x}{y}\Big) = \sum_n \sum_{i=0}^{\infty} \binom{n}{i} x^i y^{-n-1} z^{n-i} \qquad (|x| < |z|),$$

$$x^{-1}\delta\Big(\frac{y-z}{x}\Big) = \sum_n \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} x^{-n-1} y^{n-i} z^i \qquad (|y| > |z|),$$

$$x^{-1}\delta\Big(\frac{z-y}{-x}\Big) = \sum_n \sum_{i=0}^{\infty} (-1)^{n-i} \binom{n}{i} x^{-n-1} y^i z^{n-i} \quad (|y| < |z|).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The region of expansion is signified by the order of variables in the numerator of the argument of delta for each.

The Borcherds identity (V1) is now expressed in terms of generating series.

(V1) Cauchy–Jacobi identity. For all $a, b, c \in \mathbf{V}$:

$$\begin{aligned} &y^{-1}\delta\Big(\frac{z+x}{y}\Big)Y(Y(a,x)b,z) \\ &\quad= x^{-1}\delta\Big(\frac{y-z}{x}\Big)Y(a,y)Y(b,z) - x^{-1}\delta\Big(\frac{z-y}{-x}\Big)Y(b,z)Y(a,y). \end{aligned} \tag{1.9}$$

Indeed, the coefficients to $x^{-r-1}y^{-p-1}z^{-q-1}$ in the Cauchy–Jacobi identity form the Borcherds identity as described in Subsection 1.1.3.1.

Note 1.17. 1. The Cauchy–Jacobi identity is usually called the *Jacobi identity* for vertex algebras in the literatures. 2. The left-hand side of (1.9) is equivalently rewritten by the relation

$$y^{-1}\delta\Big(\frac{z+x}{y}\Big) = z^{-1}\delta\Big(\frac{y-x}{z}\Big),$$

which can be easily verified by direct calculations.

1.1.5.4 Tensor Product of Vertex Algebras

As an application of the description by generating series, let us briefly explain the *tensor product* of vertex algebras, which produces a new vertex algebra from a pair of given vertex algebras.

Let $\mathbf{V}$ and $\mathbf{W}$ be vertex algebras with the vacuums $\mathbf{1}_{\mathbf{V}}$ and $\mathbf{1}_{\mathbf{W}}$. Consider the tensor product $\mathbf{V} \otimes \mathbf{W}$ of vector spaces and set

$$Y(a \otimes b, z) = Y(a,z) \otimes Y(b,z). \tag{1.10}$$

Then it equips $\mathbf{V} \otimes \mathbf{W}$ with a structure of a vertex algebra, called the *tensor product* of vertex algebras, for local commutativity and local associativity for $\mathbf{V}$ and $\mathbf{W}$ imply those for $\mathbf{V} \otimes \mathbf{W}$, and the vacuum properties holds with $\mathbf{1}_{\mathbf{V}\otimes\mathbf{W}} = \mathbf{1}_{\mathbf{V}} \otimes \mathbf{1}_{\mathbf{W}}$.

As the right-hand side of (1.10) equals

$$\sum_{i,j} a_{(i)}z^{-i-1} \otimes b_{(j)}z^{-j-1} = \sum_{i,j} a_{(i)} \otimes b_{(j)}z^{-i-j-2},$$

the nth product of the tensor product is given by

$$(a \otimes b)_{(n)} = \sum_{i+j+1=n} a_{(i)} \otimes b_{(j)}.$$

The same process works for constructing the tensor product of a finite number of vertex algebras.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Bibliographic Notes

The main reference for Section 1.1 is the monograph [7] by K. Nagatomo and the author, where we worked over a field of characteristic zero. It is more or less straightforward to describe most materials covered in Section 1.1 over commutative rings (cf. Borcherds [32], [34] and Borcherds and Ryba [35]). See Mason [78] for accounts over $\mathbb{Z}$ under the name "vertex ring," including the proof that the creation property implies the identity property under the Borcherds identity.

For more information on the Borcherds identity and its formulation by generating series, see Frenkel, Lepowsky, and Meurman [1], Feingold, Frenkel, and Ries [2], Frenkel, Huang, and Lepowsky [3], or Lepowsky and Li [10]. The formulation of local commutativity as in (VLA) is due to Dong and Lepowsky [4]. Some textbooks such as Kac [6] or Frenkel and Ben-Zvi [8] are based on an equivalent but apparently different formulation of vertex algebras, where the translation operator is taken as a part of the structure. See Rosellen [11] for various formulations and their relation to other algebraic concepts.

For geometric interpretation of vertex algebras, see Frenkel and Ben-Zvi [8]. See Bakalov and Kac [29], Etingof and Kazhdan [52], and Li [75], for *noncommutative* or *nonlocal* analogues of vertex algebras, that is, objects satisfying local associativity but not necessarily local commutativity in our terminology. For more general frameworks, see Beilinson and Drinfeld [9] and Borcherds [34].

1.2 Vertex Algebras of Series

In order to construct an example of a group, it is often convenient to realize it as a set of bijective transformations of a set. If such a set **G** of transformations is closed under composition and inversion and contains the identity transformation, then it becomes a group by composition of transformations. The advantage of such construction lies in that associativity automatically holds for composition of maps. The set on which **G** acts then carries a structure of a *permutation representation* of **G**.

Analogously, we can construct a commutative associative algebra **C** by realizing it as a vector space consisting of commuting operators on a vector space **M**, for which associativity is again automatic, and **M** becomes a *representation* of **C** or equivalently a *module* over **C**.

In Section 1.2, we will explain a way to construct vertex algebras along the same line. That is, we will realize a vertex algebra **V** as a vector space consisting of series with operator coefficients equipped with product operations

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

for which the Borcherds identity is automatic under certain conditions. We will also introduce the concepts of a *representation* of $\mathbf{V}$ or a *module* over $\mathbf{V}$.

We will continue to work over a field $\mathbb{F}$ of characteristic not 2 unless otherwise stated.

1.2.1 Residue Products of Series

In this section, we introduce a sequence of operations

$$\begin{aligned} \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))) \times \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))) &\longrightarrow \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))) \\ (A(z), B(z)) &\longmapsto A(z)_{(m)}B(z), \end{aligned}$$

indexed by $m \in \mathbb{Z}$, called the *residue products*, which associate a series to a pair of series for each m.

1.2.1.1 Expansions in Various Regions

Let V be a vector space and x, y, z indeterminates, and consider the space

$$V((x, y, z)) = V[[x, y, z]][x^{-1}, y^{-1}, z^{-1}],$$

whose elements are written in the following form with some $L, M, N \in \mathbb{N}$:

$$w(x, y, z) = \frac{w_0(x, y, z)}{x^N y^L z^M}, \quad w_0(x, y, z) \in V[[x, y, z]].$$

For such an element, substitute $x = y - z$, and apply the binomial expansions to each term. Then we obtain

$$\begin{aligned} w(y - z, y, z)\big|_{|y|>|z|} &\in V((y))((z)), \\ w(y - z, y, z)\big|_{|y|<|z|} &\in V((z))((y)). \end{aligned}$$

Similarly, we obtain the following series in x and z:

$$\begin{aligned} w(x, x + z, z)\big|_{|x|>|z|} &\in V((x))((z)), \\ w(x, x + z, z)\big|_{|x|<|z|} &\in V((z))((x)). \end{aligned}$$

Since $x^N w(x, y, z) \in V[[x]]((y, z))$ and $y^L w(x, y, z) \in V[[y]]((x, z))$, we have

$$(y - z)^N w(y - z, y, z)\big|_{|y|>|z|} = (y - z)^N w(y - z, y, z)\big|_{|y|<|z|}, \tag{2.1}$$

$$(x + z)^L w(x, x + z, z)\big|_{|x|>|z|} = (x + z)^L w(x, x + z, z)\big|_{|x|<|z|}. \tag{2.2}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The expansions fit in the diagram

$$
\begin{tikzcd}
w(y-z,y,z)\big|_{|y|>|z|} \arrow[rr, leftrightarrow, bend left, "(2.1)"] \arrow[dd, equal, "(2.3)"'] & & w(y-z,y,z)\big|_{|y|<|z|} \\
& w(x,y,z) \arrow[ul, mapsto, "|y|>|z|"'] \arrow[ur, mapsto, "|y|<|z|"] \arrow[dl, mapsto, "|x|>|z|"] \arrow[dd, mapsto, "|x|<|z|"] & \\
w(x,x+z,z)\big|_{|x|>|z|} \arrow[dr, leftrightarrow, bend right, "(2.2)"'] & & \\
& w(x,x+z,z)\big|_{|x|<|z|}, &
\end{tikzcd}
$$

where the bent double-headed arrows indicate relations (2.1) and (2.2), and the vertical equality is given by the identification

$$w(y-z,y,z)\big|_{|y|>|z|} = w(x,x+z,z)\big|_{|x|>|z|} \tag{2.3}$$

via the isomorphisms

$$V((y))((z)) \underset{\psi}{\overset{\phi}{\rightleftarrows}} V((x))((z))$$

inverse to each other, defined by

$$\phi\colon v(y,z) \mapsto v(x+z,z)\big|_{|x|>|z|}, \quad \psi\colon v(x,z) \mapsto v(y-z,z)\big|_{|y|>|z|}.$$

In this sense, we may think of the three series

$$w(y-z,y,z)\big|_{|y|>|z|}, \quad w(y-z,y,z)\big|_{|y|<|z|}, \quad w(x,x+z,z)\big|_{|x|<|z|}$$

as being "analytically continued" to each other under $x = y - z$.

1.2.1.2 Extracting Coefficients by Formal Residues

Let Res_z denote the operation of taking the formal residue in z:

$$\mathrm{Res}_z : V[[z,z^{-1}]] \longrightarrow V, \quad v(z) = \sum_n v_n z^{-n-1} \mapsto \mathrm{Res}_z\, v(z) = v_0.$$

By this operation, the coefficients of a series are extracted as

$$v_n = \mathrm{Res}_z\, v(z) z^n.$$

If a series $v(y,z)$ in y and z belongs to $V((y))((z))$ or $V((z))((y))$, then Res_y $v(y,z)$ sits in $V((z))$, thus the operation Res_y gives rise to maps

$$\mathrm{Res}_y : V((y))((z)) \longrightarrow V((z)), \quad \mathrm{Res}_y : V((z))((y)) \longrightarrow V((z)).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let $s(y,z)$ and $t(y,z)$ be series in $V((y))((z))$ and $V((z))((y))$, respectively, such that there exists a series $w(x,y,z)$ in $V((x,y,z))$ satisfying

$$s(y,z) = w(y-z,y,z)\big|_{|y|>|z|} \quad \text{and} \quad t(y,z) = w(y-z,y,z)\big|_{|y|<|z|}. \tag{2.4}$$

We are interested in finding the expansion

$$u(x,z) = w(x,x+z,z)\big|_{|x|<|z|}$$

as in the following diagram:

$$\begin{tikzcd}
s(y,z) & & t(y,z) \\
& w(x,y,z) \arrow[ul, mapsto, "|y|>|z|"'] \arrow[ur, mapsto, "|y|<|z|"] \arrow[d, mapsto, "|x|<|z|"] & \\
& u(x,z). &
\end{tikzcd}$$

Let us expand the series $u(x,z)$ as follows:

$$u(x,z) = \sum_m u_m(z)x^{-m-1} = \sum_{m,n} u_{m,n}x^{-m-1}z^{-n-1}.$$

Then the series $u_m(z)$ are expressed as

$$u_m(z) = \mathrm{Res}_y (y-z)^m\big|_{|y|>|z|} s(y,z) - \mathrm{Res}_y (y-z)^m\big|_{|y|<|z|} t(y,z), \tag{2.5}$$

and their coefficients by

$$u_{m,n} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} s_{m-i,n+i} - \sum_{i=0}^{\infty} (-1)^{m-i} \binom{m}{i} t_{i,m+n-i}.$$

In particular, $u(x,z)$ does not depend on the choice of $w(x,y,z)$ as in (2.4), although it is clear from the beginning.

The formula (2.5) is actually equivalent to the following identity valid for any series $v(x,y,z)$ in $V((x,y,z))$:

$$\begin{aligned}
\mathrm{Res}_x\, v(x,x+z,z)\big|_{|x|<|z|} = {} & \mathrm{Res}_y\, v(y-z,y,z)\big|_{|y|>|z|} \\
& - \mathrm{Res}_y\, v(y-z,y,z)\big|_{|y|<|z|}.
\end{aligned} \tag{2.6}$$

Indeed, (2.5) is obtained by substituting $x^m w(x,y,z)$ for $v(x,y,z)$ in (2.6).

Note 2.1. 1. Heuristically, take the series $s(y,z)$ and $t(y,z)$ as if they were expansions of a meromorphic function $v(y,z)$ of y with only poles at $y = 0, z, \infty$. Then the coefficients to $(y-z)^{-m-1}$ in the expansion of $v(y,z)$ at $y = z$ are

$$\frac{1}{2\pi\sqrt{-1}} \oint_{C_z} (y-z)^m v(y,z)dy,$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

where C_z is a small circle surrounding z with $|y - z| < |z|$. As this becomes

$$\frac{1}{2\pi\sqrt{-1}} \oint_{C_{0,z}} (y-z)^m v(y,z)dy - \frac{1}{2\pi\sqrt{-1}} \oint_{C_0} (y-z)^m v(y,z)dy$$

by deformation of contour, where $C_{0,z}$ is a circle surrounding 0 and z with $|y| > |z|$ and C_0 surrounding 0 with $|y| < |z|$. Thus formula (2.5) is seen to describe $u_m(z)$ by the latter expression (cf. [8], [13], etc.). 2. Substituting $x^r y^p z^q w(x, y, z)$ for $v(x, y, z)$ in (2.6), we have

$$\begin{aligned} \sum_{i=0}^{\infty} \binom{p}{i} u_{r+i,p+q-i} = & \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} s_{p+r-i,q+i} \\ & - \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} t_{q+r-i,p+i}. \end{aligned}$$

The resemblance with the Borcherds identity (V1) is not an accident, as we will see in the sequel.

1.2.1.3 Series Acting on Vector Spaces

Let $\mathbf{M}$ be a vector space and consider formal series in $(\mathrm{End}\,\mathbf{M})[[z, z^{-1}]]$. We will call such a series a *series acting on* $\mathbf{M}$, or just a *series on* $\mathbf{M}$ for short.

For such a series $A(z)$, set

$$A(z) = \sum_{n} A_n z^{-n-1},$$

where the summation is over all $n \in \mathbb{Z}$ and A_n are operators on $\mathbf{M}$. For an element $v \in \mathbf{M}$, we write

$$A(z)v = \sum_{n} A_n v z^{-n-1}.$$

In particular, consider the series $I(z)$ such that the only nonzero term is the constant term being the identity operator I:

$$I(z) = \sum_{n} I_n z^{-n-1}, \quad \text{where} \quad I_n v = \begin{cases} 0 & (n \neq -1), \\ v & (n = -1). \end{cases}$$

We will call it the *identity series* and often identify it with the scalar 1.

For a series $A(z)$, split it into the sum of series with nonnegative and negative powers as

$$\sum_{n} A_n z^{-n-1} = \underbrace{\sum_{n<0} A_n z^{-n-1}}_{\substack{\text{nonnegative} \\ \text{powers in } z}} + \underbrace{\sum_{n\geq 0} A_n z^{-n-1}}_{\substack{\text{negative} \\ \text{powers in } z}}. \tag{2.7}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let us denote the resulting series by $A(z)^{\geq 0}$ and $A(z)^{<0}$, respectively:

$$A(z)^{\geq 0} = \sum_{n<0} A_n z^{-n-1} = \sum_{n\geq 0} A_{-n-1} z^n,$$
$$A(z)^{<0} = \sum_{n\geq 0} A_n z^{-n-1} = \sum_{n<0} A_{-n-1} z^n.$$

We will also denote them by $A(z)_{<0}$ and $A(z)_{\geq 0}$, respectively.

1.2.1.4 Locally Truncated Series

We will say that a series $A(z)$ on a vector space $\mathbf{M}$ is *locally truncated* if $A(z)v \in \mathbf{M}((z))$ for all $v \in \mathbf{M}$, thus

$$A(z) \text{ is locally truncated} \iff A(x) \in \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z))).$$

In other words, $A(z)$ is locally truncated if and only if for any $v \in \mathbf{M}$, there exists an $N \in \mathbb{N}$ such that $A_{N+i}v = 0$ for all $i \in \mathbb{N}$.

Consider series $A(z)$ and $B(z)$, split $A(y)$ as in (2.7), and set

$$\mathop{\circ}\limits^{\circ} A(y)B(z) \mathop{\circ}\limits^{\circ} = A(y)^{\geq 0}B(z) + B(z)A(y)^{<0}.$$

Assume that $A(z)$ and $B(z)$ are locally truncated. Then, for $v \in \mathbf{M}$,

$$\mathop{\circ}\limits^{\circ} A(y)B(z) \mathop{\circ}\limits^{\circ} v \in \mathbf{M}((y, z)).$$

Therefore, the following expression gives rise to a locally truncated series:

$$\mathop{\circ}\limits^{\circ} A(z)B(z) \mathop{\circ}\limits^{\circ} = A(z)^{\geq 0}B(z) + B(z)A(z)^{<0}. \tag{2.8}$$

Such an expression is called the *normally ordered product* of $A(z)$ and $B(z)$.

Note 2.2. A series on a vector space is also called a *formal distribution* and a locally truncated series a *field* in the literatures following [6].

1.2.1.5 Residue products

Let $A(z)$ and $B(z)$ be series on a vector space $\mathbf{M}$ and m an integer:

$$A(z), B(z) \in (\operatorname{End}\mathbf{M})[[z, z^{-1}]], \quad m \in \mathbb{Z}.$$

If $m \geq 0$, then we may consider the following expression as a series with operator coefficients:

$$\begin{aligned} A(z)_{(m)}B(z) &= \operatorname{Res}_y (y-z)^m [A(y), B(z)] \\ &= \operatorname{Res}_y (y-z)^m A(y)B(z) - \operatorname{Res}_y (y-z)^m B(z)A(y). \end{aligned}$$

For $m < 0$, assume that $A(z)$ and $B(z)$ are locally truncated. Then, for $v \in \mathbf{M}$,

$$A(y)B(z)v \in \mathbf{M}((y))((z)), \quad B(z)A(y)v \in \mathbf{M}((z))((y)),$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

and the following expressions make sense as series in y and z:

$$(y-z)^m\big|_{|y|>|z|}A(y)B(z)v,\quad (y-z)^m\big|_{|y|<|z|}B(z)A(y)v.$$

We may therefore consider the expression

$$\begin{aligned}A(z)_{(m)}B(z) = \mathrm{Res}_y(y-z)^m\big|_{|y|>|z|}A(y)B(z)\\ - \mathrm{Res}_y(y-z)^m\big|_{|y|<|z|}B(z)A(y)\end{aligned}$$

as a series with operator coefficients.

We will call the series $A(z)_{(m)}B(z)$ thus obtained the mth *residue product* of $A(z)$ and $B(z)$ for each $m \in \mathbb{Z}$.

Lemma 2.3 *If series $A(z)$ and $B(z)$ on a vector space are locally truncated, then so is the residue product $A(z)_{(m)}B(z)$ for all $m \in \mathbb{Z}$.*

To describe the coefficients explicitly, set

$$A(z)_{(m)}B(z) = (A_{(m)}B)(z) = \sum_n (A_{(m)}B)_n z^{-n-1}.$$

Then we have

$$(A_{(m)}B)_n = \sum_{i=0}^{\infty}(-1)^m\binom{m}{i}A_{m-i}B_{n+i} - \sum_{i=0}^{\infty}(-1)^{m-i}\binom{m}{i}B_{m+n-i}A_i. \tag{2.9}$$

For $m=-1$, the residue product $A(z)_{(-1)}B(z)$ agrees with the normally ordered product ${}^\circ_\circ A(z)B(z){}^\circ_\circ$ defined by (2.8), and, for $k \in \mathbb{N}$,

$$A(z)_{(-k-1)}B(z) = {}^\circ_\circ\big(\partial^{(k)}A(z)\big)B(z){}^\circ_\circ.$$

In particular, for the identity series $I(z)$,

$$I(z)_{(m)}A(z) = \begin{cases}0 & (m \neq -1),\\ A(z) & (m=-1),\end{cases}$$

$$A(z)_{(m)}I(z) = \begin{cases}0 & (m \geq 0),\\ \partial^{(k)}A(z) & (m=-k-1<0),\end{cases}$$

which is the vacuum property (V2) completed by the divided derivatives.

1.2.2 Operator Product Expansions

In Section 1.2.2, we will explain a rigorous formulation in certain circumstances of what is called *operator product expansion* (OPE) in physics and its relation to residue products.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let us briefly outline the concept by example. Let a_n, $n \in \mathbb{Z}$, be operators on a vector space $\mathbf{M}$ satisfying the Heisenberg commutation relation (1.4). By the generating series $a(z) = \sum_n a_n z^{-n-1}$, the relation is written as

$$[a(y), a(z)] = \sum_n n y^{-n-1} z^{n-1} \ (= \delta^{(1)}(y,z)). \tag{2.10}$$

Note that the equality $(y-z)^2 a(y)a(z) = (y-z)^2 a(z)a(y)$ follows.

Split (2.10) into two equalities by collecting terms with nonnegative and negative powers in y. Then, after some algebra, we arrive at

$$\begin{cases} a(y)a(z) = \left.\dfrac{1}{(y-z)^2}\right|_{|y|>|z|} + {}^{\circ}_{\circ} a(y)a(z) {}^{\circ}_{\circ}, \\ a(z)a(y) = \left.\dfrac{1}{(y-z)^2}\right|_{|y|<|z|} + {}^{\circ}_{\circ} a(y)a(z) {}^{\circ}_{\circ}. \end{cases} \tag{2.11}$$

The two equalities in (2.11) are written as

$$a(y)a(z) \simeq a(z)a(y) \sim \frac{1}{(y-z)^2} \tag{2.12}$$

and called the OPE. The mth residue products $a(z)_{(m)}a(z)$ for $m \geq 0$ are then read off from the OPE (2.12) as

$$a(z)_{(m)}a(z) = \begin{cases} 0 & (m \geq 2), \\ 1 & (m = 1), \\ 0 & (m = 0), \end{cases}$$

where 1 for $m = 1$ is the numerator in (2.12), that is, the identity series $I(z)$.

1.2.2.1 OPE of Locally Commutative Series

Let $A(z)$ and $B(z)$ be series on a vector space $\mathbf{M}$. We will say that they are *locally commutative* if the following holds for some $N \in \mathbb{N}$:

$$(y-z)^N A(y)B(z) = (y-z)^N B(z)A(y). \tag{2.13}$$

In other words,

$$\sum_{i=0}^{\infty} (-1)^i \binom{N}{i} A_{m+N-i} B_{n+i} v = \sum_{i=0}^{\infty} (-1)^{N-i} \binom{N}{i} B_{n+N-i} A_{m+i} v \tag{2.14}$$

for some $N \in \mathbb{N}$ and all $m, n \in \mathbb{Z}$.

Let $A(z)$ and $B(z)$ be locally commutative series on a vector space $\mathbf{M}$ and take $N \in \mathbb{N}$ such that (2.13) holds. Split $A(y)$ into the sum of series with nonnegative and negative powers as in (2.7). Then local commutativity becomes

$$(y-z)^N \left[A(y)_{\geq 0}, B(z)\right] = -(y-z)^N \left[A(y)_{<0}, B(z)\right].$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Comparing the degrees in y, we see that there exist series $C_0(z), \ldots, C_{N-1}(z)$ in z such that

$$\begin{cases} (y-z)^N \left[A(y)^{<0}, B(z)\right] = \sum_{k=0}^{N-1} (y-z)^{N-1-k} C_k(z), \\ -(y-z)^N \left[A(y)^{\geq 0}, B(z)\right] = \sum_{k=0}^{N-1} (y-z)^{N-1-k} C_k(z). \end{cases}$$

Multiplying them by $(y-z)^{-N}\big|_{|y|>|z|}$ and $(y-z)^{-N}\big|_{|y|<|z|}$, respectively, and adding ${}^\circ_\circ A(y)B(z){}^\circ_\circ = A(y)^{\geq 0}B(z) + B(z)A(y)^{<0}$, we have

$$\begin{cases} A(y)B(z) = \sum_{k=0}^{N-1} \dfrac{C_k(z)}{(y-z)^{k+1}}\Big|_{|y|>|z|} + {}^\circ_\circ A(y)B(z){}^\circ_\circ, \\ B(z)A(y) = \underbrace{\sum_{k=0}^{N-1} \dfrac{C_k(z)}{(y-z)^{k+1}}\Big|_{|y|<|z|}}_{\text{singular part}} + \underbrace{{}^\circ_\circ A(y)B(z){}^\circ_\circ}_{\text{regular part}}. \end{cases} \tag{2.15}$$

The two equalities are written at once as

$$A(y)B(z) \simeq B(z)A(y) \sim \sum_{k=0}^{N-1} \frac{C_k(z)}{(y-z)^{k+1}}, \tag{2.16}$$

and it is called (the singular part of) the OPE.

As the difference of the left-hand sides of (2.15) becomes the commutators $[A(y), B(z)]$, the commutation relations of the coefficients are encoded in the series $C_0(z), \ldots, C_{N-1}(z)$ in the OPE (2.16), which are related to the residue products as

$$A(z)_{(m)}B(z) = \begin{cases} 0 & (N \leq m), \\ C_m(z) & (0 \leq m < N). \end{cases}$$

The OPE as described above gives a rigorous formulation of what is called by the same term in physics.

1.2.2.2 Expansion of Regular Parts

Let us further expand the regular part ${}^\circ_\circ A(y)B(z){}^\circ_\circ$ of the equalities in (2.15) under the assumption that $A(z)$ and $B(z)$ are locally truncated. For $v \in \mathbf{M}$, set

$$w(x, y, z) = \sum_{k=0}^{N-1} \frac{C_k(z)v}{x^{k+1}} + {}^\circ_\circ A(y)B(z){}^\circ_\circ v.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

By local truncation, we have ${}^{\circ}_{\circ}A(y)B(z){}^{\circ}_{\circ}v \in \mathbf{M}((y,z))$, thus

$$w(x,y,z) \in \mathbf{M}((y,z))[x^{-1}] \subset \mathbf{M}((x,y,z)).$$

The OPE (2.15) is now restated as

$$\begin{cases} A(y)B(z)v = w(y-z,y,z)\big|_{|y|>|z|}, \\ B(z)A(y)v = w(y-z,y,z)\big|_{|y|<|z|}. \end{cases}$$

Expand $w(x,y,z)$ in the region $|x| < |z|$ by substitution $y = x + z$ and denote the resulting series by $(A \circ B)(x,z)$. Then, by (1.1), we have

$$(A \circ B)(x,z)v = \sum_{k=0}^{N-1} \frac{C_k(z)v}{x^{k+1}} + \sum_{k=0}^{\infty} x^k {}^{\circ}_{\circ}\big(\partial^{(k)}A(z)\big)B(z){}^{\circ}_{\circ}v,$$

or equivalently

$$(A \circ B)(x,z)v = \sum_m x^{-m-1} A(z)_{(m)}B(z).$$

The situation is summarized in the following diagram as in Subsection 1.2.1.1:

$$\begin{tikzcd}
A(y)B(z)v \arrow[rr, leftrightarrow, bend left, "(2.17)"] \arrow[dd, equal] & & B(z)A(y)v \\
& w(x,y,z) \arrow[ul, mapsto, "|y|>|z|"'] \arrow[ur, mapsto, "|y|<|z|"] \arrow[dl, mapsto, "|x|>|z|"'] \arrow[dd, mapsto, "|x|<|z|"] & \\
A(x+z)B(z)v\big|_{|x|>|z|} \arrow[dr, leftrightarrow, bend right, "(2.18)"'] & & \\
& (A \circ B)(x,z)v. &
\end{tikzcd}$$

Here the bent double-headed arrows signify the following relations for sufficiently large L and N:

$$(y-z)^N A(y)B(z)v = (y-z)^N B(z)A(y)v, \tag{2.17}$$

$$(x+z)^L A(x+z)B(z)v\big|_{|x|>|z|} = (x+z)^L (A \circ B)(x,z)v. \tag{2.18}$$

The relation (2.17) is local commutativity which we have assumed, whereas (2.18) is a consequence, which is a form of local associativity.

In this sense, the series

$$A(y)B(z)v,\ B(z)A(y)v,\ (A \circ B)(x,z)v$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

are thought of as being “analytically continued” to each other, and the OPE (2.16) is formally completed as

$$A(y)B(z) \simeq B(z)A(y) \simeq \sum_{k=0}^{N-1} \frac{C_k(z)}{(y-z)^{k+1}} + \sum_{k=0}^{\infty} (y-z)^k \,{}^{\circ}_{\circ}\big(\partial^{(k)} A(z)\big) B(z)\,{}^{\circ}_{\circ}$$

by including the expansion of the regular part and formally substituting $y - z$ for x, although the result does not make sense in general as series in y and z.

Note 2.4. Let $A(z)$ and $B(z)$ be locally commutative and locally truncated. Then the formula in Note 2.1 implies the Borcherds identity in the form

$$\begin{aligned}\sum_{i=0}^{\infty} \binom{p}{i} (A_{(r+i)}B)_{p+q-i} &= \textstyle\sum_{i=0}^{\infty} (-1)^r \displaystyle\binom{r}{i} A_{p+r-i} B_{q+i} \\ &\quad - \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} B_{q+r-i} A_{p+i}\end{aligned}$$

for all $p, q, r \in \mathbb{Z}$, as noted by Tuite [22] in a different method, which is to be identified with the Borcherds identity (M1) for modules in Subsection 1.2.5.2.

1.2.2.3 Skew-Symmetry of Residue Products

Let us now consider the series $(B \circ A)(-x, y)$ obtained by switching the roles of $A(y)$ and $B(z)$ as in the following diagram:

$$\begin{tikzcd}
A(y)B(z)v \arrow[rrrr, leftrightarrow, bend left=10] \arrow[dd, equal] & & & & B(z)A(y)v \arrow[dd, equal] \\
& & w(x,y,z) \arrow[ull, mapsto, "|y|>|z|"'] \arrow[urr, mapsto, "|z|>|y|"] \arrow[dll, mapsto, "|x|>|z|"'] \arrow[drr, mapsto, "|x|>|y|"] \arrow[ddl, mapsto, "|x|<|z|"'] \arrow[ddr, mapsto, "|x|<|y|"] & & \\
A(x+z)B(z)v\big|_{|x|>|z|} \arrow[dr, leftrightarrow, bend right] & & & & B(-x+y)A(z)v\big|_{|x|>|y|} \arrow[dl, leftrightarrow, bend left] \\
& (A \circ B)(x,z)v \arrow[rr, equal] & & (B \circ A)(-x,y)v. &
\end{tikzcd}$$

Then we have

$$\begin{aligned}(A \circ B)(x,z)v &= (B \circ A)(-x, x+z)v\big|_{|x|<|z|} \\ &= \sum_{m,n} (B_{(m)}A)_n (-x)^{-m-1} (x+z)^{-n-1}\big|_{|x|<|z|} \\ &= e^{x\partial_z} \sum_{m,n} (B_{(m)}A)_n (-x)^{-m-1} z^{-n-1} \\ &= e^{x\partial_z} (B \circ A)(-x, z)v.\end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Therefore,

$$A(z)_{(m)}B(z) = (-1)^{m+1}\sum_{i=0}^{\infty}(-1)^i\partial^{(i)}\big(B(z)_{(m+i)}A(z)\big),$$

which is the skew-symmetry for the residue products (cf. Subsection 1.1.3.5).

1.2.3 Vertex Algebras of Series

Recall that a linear space consisting of commuting operators on a vector space becomes a commutative associative algebra by composition of operators if it is closed under composition and contains the identity operator.

In this section, we will pursue such consideration for vertex algebras, where the analogue of operators is given by series with operator coefficients and that of composition is the residue products.

1.2.3.1 Borcherds Identity for Residue Products

Recall that the set of locally truncated series on $\mathbf{M}$ in an indeterminate z can be identified with the set $\mathrm{Hom}(\mathbf{M},\mathbf{M}((z)))$, and the residue products of locally truncated series are again locally truncated. Therefore, the residue products equip $\mathrm{Hom}(\mathbf{M},\mathbf{M}((z)))$ with a sequence of binary operations:

$$\begin{aligned}\mathrm{Hom}(\mathbf{M},\mathbf{M}((z)))\times\mathrm{Hom}(\mathbf{M},\mathbf{M}((z))) &\longrightarrow \mathrm{Hom}(\mathbf{M},\mathbf{M}((z)))\\ (A(z),B(z)) &\longmapsto A(z)_{(n)}B(z).\end{aligned}$$

Let $A(z)$, $B(z)$, and $C(z)$ be locally truncated series on a vector space $\mathbf{M}$. Let us further assume that they are locally commutative with each other, that is, $A(z)$ and $B(z)$, $B(z)$ and $C(z)$, and $A(z)$ and $C(z)$ are locally commutative separately.

Theorem 2.5 *Let $A(z)$, $B(z)$, and $C(z)$ be locally truncated series on a vector space locally commutative with each other. Then the Borcherds identity*

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{p}{i}(A(z)_{(r+i)}B(z))_{(p+q-i)}C(z)\\ &\quad=\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}A(z)_{(p+r-i)}(B(z)_{(q+i)}C(z))\\ &\qquad-\sum_{i=0}^{\infty}(-1)^{r-i}\binom{r}{i}B(z)_{(q+r-i)}(A(z)_{(p+i)}C(z))\end{aligned}\tag{2.19}$$

holds for all $p,q,r\in\mathbb{Z}$ with respect to the residue products.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

To see it, let us first consider the case when the integers p, q, r are nonnegative. The Jacobi identity for the commutators in the following form obviously holds by cancellation:

$$[[A(x), B(y)], C(z)] = [A(x), [B(y), C(z)]] - [B(y), [A(x), C(z)]]. \tag{2.20}$$

Multiply both sides by $D(x, y, z) = (x - z)^p (y - z)^q (x - y)^r$:

$$\begin{aligned}\underline{(x - z)^p}(y - z)^q (x - y)^r &[[A(x), B(y)], C(z)] \\ &= (x - z)^p (y - z)^q \, \underline{(x - y)^r} \, [A(x), [B(y), C(z)]] \\ &- (x - z)^p (y - z)^q \, \underline{(x - y)^r} \, [B(y), [A(x), C(z)]].\end{aligned}$$

To the underlined factors, apply the following expansions, respectively:

$$\begin{aligned}(x - z)^p &= \big((x - y) + (y - z)\big)^p = \sum_{i=0}^{p} \binom{p}{i} (x - y)^i (y - z)^{p-i}, \\ (x - y)^r &= \big((x - z) - (y - z)\big)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x - z)^{r-i} (y - z)^i, \\ (x - y)^r &= \big((x - z) - (y - z)\big)^r = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (x - z)^i (y - z)^{r-i}.\end{aligned}$$

Then we have

$$\begin{aligned}\sum_{i=0}^{p} \binom{p}{i} &(x - y)^{r+i} (y - z)^{p+q-i} [[A(x), B(y)], C(z)] \\ &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x - z)^{p+r-i} (y - z)^{q+i} [A(x), [B(y), C(z)]] \\ &\quad - \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (x - z)^{p+i} (y - z)^{q+r-i} [B(y), [A(x), C(z)]].\end{aligned}$$

Taking $\operatorname{Res}_x \operatorname{Res}_y$, we arrive at the Borcherds identity (2.19).

For general p, q, r, the situation is much more complicated, for the factor $D(x, y, z)$ is no longer a polynomial.

In such a case, multiply the terms of the Jacobi identity (2.20) by the expansions of $D(x, y, z) = (x - z)^p (y - z)^q (x - y)^r$ in regions depending on the order of $A(x), B(y), C(z)$ as

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\begin{aligned}
Q_{123} &= D(x,y,z)\big|_{|x|>|y|>|z|}A(x)B(y)C(z),\\
Q_{132} &= D(x,y,z)\big|_{|x|>|z|>|y|}A(x)C(z)B(y),\\
Q_{213} &= D(x,y,z)\big|_{|y|>|x|>|z|}B(y)A(x)C(z),\\
Q_{231} &= D(x,y,z)\big|_{|y|>|z|>|x|}B(y)C(z)A(x),\\
Q_{312} &= D(x,y,z)\big|_{|z|>|x|>|y|}C(z)A(x)B(y),\\
Q_{321} &= D(x,y,z)\big|_{|z|>|y|>|x|}C(z)B(y)A(x).
\end{aligned}$$

Then, again by cancellation, we have

$$\begin{aligned}
&(Q_{123}-Q_{213})-(Q_{312}-Q_{321})\\
&=\big((Q_{123}-Q_{132})-(Q_{231}-Q_{321})\big)-\big((Q_{213}-Q_{231})-(Q_{132}-Q_{312})\big).
\end{aligned}$$

The result follows similarly by taking $\mathrm{Res}_x\,\mathrm{Res}_y$ after carefully manipulating series under local commutativity. See [7] and [11] for details.

Alternatively, for $A(z), B(z), C(z)$ as above and $v \in \mathbf{M}$, note that there exists an element $w(x,y,z,\xi,\eta,\zeta) \in \mathbf{M}((x,y,z,\xi,\eta,\zeta))$ such that

$$\begin{aligned}
A(x)B(y)C(z)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|x|>|y|>|z|},\\
A(x)C(z)B(y)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|x|>|z|>|y|},\\
B(y)A(x)C(z)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|y|>|x|>|z|},\\
B(y)C(z)A(x)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|y|>|z|>|x|},\\
C(z)A(x)B(y)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|z|>|x|>|y|},\\
C(z)B(y)A(x)v &= w(x,y,z,x-y,x-z,y-z)\big|_{|z|>|y|>|x|}.
\end{aligned}$$

Set $W(x,y,z,\xi,\eta,\zeta) = \xi^r\eta^p\zeta^q w(x,y,z,\xi,\eta,\zeta)$. Then the Borcherds identity (2.19) follows by taking Res_ζ of the identity

$$\begin{aligned}
&\mathrm{Res}_\xi\ W(\xi+\zeta+z,\zeta+z,z,\xi,\xi+\zeta,\zeta)\big|_{|\xi|<|\zeta+z|,|\xi|<|\zeta|<|z|}\\
&=\mathrm{Res}_\eta\ W(\eta+z,\zeta+z,z,\eta-\zeta,\eta,\zeta)\big|_{|\zeta|<|\eta|<|z|}\\
&\quad-\mathrm{Res}_\eta\ W(\eta+z,\zeta+z,z,\eta-\zeta,\eta,\zeta)\big|_{|\eta|<|\zeta|<|z|},
\end{aligned}$$

which is in fact a variant of (2.6). Note that the region $|\xi| < |\zeta + z|$ attached in the left-hand side can be replaced by $|\xi + \zeta| < |z|$ without affecting the expansion.

1.2.3.2 Vertex Algebra of Series

We will say that a subset $\mathcal{S}$ of $\mathrm{Hom}(\mathbf{M}, \mathbf{M}((z)))$ is locally commutative if so are all the pairs of series belonging to $\mathcal{S}$, including the pairs of the same series.

By combining the properties of the residue products obtained so far, we arrive at the following result due to H. S. Li.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 2 Vertex algebras, abstract versus realization

Abstract vertex algebra		Vertex algebra of series	
$\mathbf{V}$	abstract vector space	$\mathcal{V}$	subspace of $\mathrm{Hom}(\mathbf{M}, \mathbf{M}((z)))$
$a_{(n)}b$	abstract products	$A(z)_{(n)}B(z)$	residue products
$\mathbf{1}$	the vacuum	$I(z) = I$	the identity series
$T^{(k)}$	translation operators	$\partial_z^{(k)}$	divided derivatives

Corollary 2.6 *Let $\mathcal{V}$ be a vector space consisting of series on a vector space satisfying the following conditions:*

(1) $\mathcal{V}$ is locally truncated and locally commutative.
(2) $\mathcal{V}$ is closed under the residue products.
(3) $\mathcal{V}$ contains the identity series.

Then $\mathcal{V}$ becomes a vertex algebra by the residue products.

We will call the vertex algebra thus obtained a *vertex algebra of series* for short. See Table 2 for comparison of abstract vertex algebras and vertex algebras of series.

We have shown that a locally commutative subspace of $\mathrm{Hom}(\mathbf{M}, \mathbf{M}((z)))$ automatically becomes a vertex algebra by the residue products if it is closed under the residue products and contains $I(z)$. Conversely, any vertex algebra $\mathbf{V}$ is realized in this way by letting $\mathbf{M}$ be the vertex algebra $\mathbf{V}$ itself and considering the image $\mathcal{V}$ of the generating series map $Y(-, z)$. (cf. Section 1.2.4.)

In this regard, the axioms for vertex algebras are seen to be modelled on the properties of locally truncated locally commutative series with respect to the residue products in the same way as those for groups are modelled on the properties of bijective transformations of a set with respect to composition of transformations.

1.2.3.3 Generation by a set of series

The following is called *Dong's Lemma*.

Lemma 2.7 *Let $A(z)$, $B(z)$, and $C(z)$ be locally truncated series on* $\mathbf{V}$. *If they are locally commutative with each other, then the residue products $A(z)_{(n)}B(z)$ and $C(z)$ are locally commutative for all $n \in \mathbb{Z}$.*

Let $\mathcal{S}$ be a locally commutative subset of $\mathrm{Hom}(\mathbf{M}, \mathbf{M}((z)))$. Then, by Lemma, we may construct a vertex algebra $\mathcal{V}$ by repeatedly applying residue products to the series belonging to $\mathcal{S}$:

$$I(z),\ A(z),\ A(z)_{(m)}B(z),\ (A(z)_{(m)}B(z))_{(n)}C(z),\ A(z)_{(m)}(B(z)_{(n)}C(z)),\ \text{etc.}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

We will denote the vertex algebra $\mathcal{V}$ thus defined by $\langle \mathcal{S} \rangle_{\mathrm{RP}}$ and call it the vertex algebra of series *generated by* $\mathcal{S}$ with respect to the residue products.

By the associativity formula (VA) for $\mathcal{V} = \langle \mathcal{S} \rangle_{\mathrm{RP}}$, the space is spanned by the elements of the following form:

$$A^1(z)_{(n_1)} \cdots A^k(z)_{(n_k)} I(z),$$

where $k \in \mathbb{N}$, $A^1(z), \ldots, A^k(z) \in \mathcal{S}$ and $n_1, \ldots, n_k \in \mathbb{Z}$ and the operations are taken from right to left.

1.2.4 Identification of Vertex Algebras

Let $\mathbf{V}$ be a vertex algebra and $\mathcal{V}$ the image of the generating series map:

$$Y(-,z)\colon \mathbf{V} \longrightarrow \mathrm{Hom}(\mathbf{V}, \mathbf{V}((z))).$$

Then $\mathcal{V}$ is a vertex algebra of series by Corollary 2.6, and the map $Y(-,z)$ is a homomorphism of vertex algebras by the associativity formula

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)} Y(b,z).$$

Moreover, $Y(-,z)$ is an isomorphism onto $\mathcal{V}$ with the inverse given by

$$\sigma_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbf{V}, \ \ Y(a,z) \mapsto Y(a,z)\mathbf{1}|_{z=0}.$$

Now, forget the vertex algebra structure on $\mathbf{V}$, but retain that Y is a linear isomorphism onto a vertex algebra $\mathcal{V}$ of series on $\mathbf{V}$ with the inverse given as above. Then the vertex algebra structure on $\mathbf{V}$ is reconstructed from $\mathcal{V}$ as

$$a_{(n)}b = \sigma_{\mathcal{V}}(Y(a,z)_{(n)} Y(b,z)).$$

In this section, we will construct a vertex algebra structure on a vector space $\mathbf{V}$ by identifying it with a vertex algebra of series along this line.

1.2.4.1 Creativity and the State Map

Let $\mathbf{V}$ be a vector space equipped with a candidate $\mathbf{1} \in \mathbf{V}$ of the vacuum. To relate a vertex algebra of series on $\mathbf{V}$ to a vertex algebra structure on $\mathbf{V}$, we introduce the concept of creativity for series.

A series $A(z)$ on $\mathbf{V}$ is said to be *creative* (with respect to $\mathbf{1} \in \mathbf{V}$) if

$$A_n \mathbf{1} = 0, \ \ n \geq 0,$$

that is, $A(z)\mathbf{1} \in \mathbf{V}[[z]]$. We will say that a subset $\mathcal{V} \subset (\mathrm{End}\,\mathbf{V})[[z, z^{-1}]]$ is creative if so is every series in it.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Consider the following map, which we will call the *state map*:

$$\sigma\colon (\operatorname{End}\mathbf{V})[[z,z^{-1}]] \longrightarrow \mathbf{V}, \quad A(z) \mapsto A_{-1}\mathbf{1}.$$

If $A(z)$ is creative, then $\sigma(A(z)) = A(z)\mathbf{1}\big|_{z=0}$.

Lemma 2.8 *Let $A(z)$ and $B(z)$ be locally truncated series on $\mathbf{V}$. If they are creative, then so are the residue products $A(z)_{(n)}B(z)$, for which*

$$\sigma(A(z)_{(n)}B(z)) = A_nB_{-1}\mathbf{1}$$

holds for all $n \in \mathbb{Z}$.

1.2.4.2 Identification by the State Map

Let $\mathcal{V}$ be a vertex algebra of series on $\mathbf{V}$. Assume that $\mathcal{V}$ is creative and the state map restricts to a bijection from $\mathcal{V}$ onto $\mathbf{V}$:

$$\sigma_{\mathcal{V}} = \sigma|_{\mathcal{V}} : \mathcal{V} \xrightarrow{\sim} \mathbf{V}, \quad A(z) \mapsto A_{-1}\mathbf{1}.$$

Then we may transfer the vertex algebra structure on $\mathcal{V}$ to $\mathbf{V}$ via $\sigma_{\mathcal{V}}$. That is, for any $a, b \in \mathbf{V}$, choose $A(z), B(z) \in \mathcal{V}$ so that $\sigma(A(z)) = a$ and $\sigma(B(z)) = b$, and define the product as

$$a_{(n)}b = \sigma(A(z)_{(n)}B(z)).$$

By Lemma 2.8, we actually have

$$a_{(n)}b = A_nB_{-1}\mathbf{1} = A_nb.$$

As this holds for all $b \in \mathbf{V}$, we have

$$Y(a,z) = A(z).$$

Thus the map Y agrees with the inverse of $\sigma_{\mathcal{V}}$. Moreover, since the vacuum of $\mathcal{V}$ is the identity series $I(z)$, the vacuum of $\mathbf{V}$ is given by $\sigma(I(z)) = I_{-1}\mathbf{1} = \mathbf{1}$.

In summary, we have the following theorem.

Theorem 2.9 *Let $\mathcal{V}$ be a vertex algebra of series on $\mathbf{V}$ and assume that it is creative with respect to $\mathbf{1} \in \mathbf{V}$. If $\sigma_{\mathcal{V}}\colon \mathcal{V} \to \mathbf{V}$ is bijective, then $\mathbf{V}$ carries a unique structure of a vertex algebra such that*

$$Y(A_{-1}\mathbf{1}, z) = A(z) \ \textit{for all } A(z) \in \mathcal{V},$$

with vacuum $\mathbf{1}$.

Consider now the case when $\mathcal{V}$ is a vertex algebra of series generated by a locally commutative subset of $\operatorname{Hom}(\mathbf{V}, \mathbf{V}((z)))$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Corollary 2.10 *Let* $\mathbf{V}$ *be a vector space,* $\mathbf{1}$ *an element of* $\mathbf{V}$*, and* $\mathcal{S}$ *a set of locally truncated series on* $\mathbf{V}$ *satisfying the following conditions:*

(1) $\mathcal{S}$ *is locally commutative.*
(2) $\mathcal{S}$ *is creative with respect to* $\mathbf{1}$.
(3) $\sigma_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbf{V}$ *is bijective for* $\mathcal{V} = \langle \mathcal{S} \rangle_{\mathrm{RP}}$.

Then $\mathbf{V}$ *carries a unique structure of a vertex algebra such that*

$$Y(S_{-1}\mathbf{1}, z) = S(z) \ \textit{for all} \ S(z) \in \mathcal{S}$$

with vacuum $\mathbf{1}$.

1.2.4.3 Bijectivity by Translation Covariance

Let $\mathcal{V}$ be a vertex algebra of series on a vector space $\mathbf{V}$ and assume that $\mathcal{V}$ is creative with respect to $\mathbf{1} \in \mathbf{V}$. To transfer the vertex algebra structure to $\mathbf{V}$ by applying Theorem 2.9, we need know that the restriction $\sigma_{\mathcal{V}} = \sigma|_{\mathcal{V}}$ of the state map is bijective onto $\mathbf{V}$.

For the convenience of readers, we will show that, under translation covariance, knowing surjectivity suffices, although we will not use this result in the rest of the sections.

Let $\mathbb{T} = (T^{(k)})$ be a sequence of operators on $\mathbf{V}$ indexed by positive integers:

$$T^{(k)} : \mathbf{V} \longrightarrow \mathbf{V} \ \ (k = 1, 2, \cdots).$$

We set $T^{(0)} = I$ and assume that the operators are iterative and annihilating $\mathbf{1}$ in the following senses, respectively:

$$T^{(i)}T^{(j)} = \binom{i+j}{i} T^{(i+j)} \ \text{ for } i, j \geq 0, \ \text{ and } \ T^{(k)}\mathbf{1} = 0 \ \text{ for } k \geq 1.$$

For an indeterminate x, define the formal exponentials $e^{\pm xT}$ by

$$e^{xT} = \sum_{k=0}^{\infty} x^k T^{(k)}, \ \ e^{-xT} = \sum_{k=0}^{\infty} (-x)^k T^{(k)}.$$

Then, by iterativity, they are inverse to each other:

$$e^{-xT} e^{xT} = 1 = e^{xT} e^{-xT}.$$

Now, let $A(z)$ be a series on $\mathbf{V}$. It is said to be *translation covariant* (with respect to $\mathbb{T}$) if

$$e^{xT} A(z) e^{-xT} = e^{x\partial_z} A(z),$$

or equivalently,

$$e^{xT} A(z) e^{-xT} = A(x + z)\big|_{|x|<|z|}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

If $A(z)$ is translation covariant, then the coefficients of $v(z) = A(z)\mathbf{1}$ satisfy

$$T^{(k)}v_n = (-1)^k \binom{n}{k} v_{n-k}.$$

Therefore, $A(z)$ is creative and $v_{-k-1} = T^{(k)}v_{-1}$.

We will say that a subset $\mathcal{V} \subset (\operatorname{End}\mathbf{V})[[z, z^{-1}]]$ is translation covariant if so is every series in it.

The following result is a variant of what is called *Goddard's uniqueness theorem.*

Proposition 2.11 *Let $\mathcal{V}$ be a locally commutative and translation covariant subspace of* $\operatorname{Hom}(\mathbf{V}, \mathbf{V}((z)))$. *If $\sigma_{\mathcal{V}}$ is surjective, then it is bijective.*

Proof. Assume that $\sigma_{\mathcal{V}}$ is surjective. The result follows by showing injectivity of p and i in the diagram below:

$$\begin{tikzcd}
\operatorname{Hom}(\mathbf{V}, \mathbf{V}((z))) \arrow[r] \arrow[rr, bend left, "\sigma"] & \mathbf{V}[[z]] \arrow[r] & \mathbf{V} \\
\mathcal{V} \arrow[u, phantom, "\cup"] \arrow[r, "p"] & \mathcal{V}\mathbf{1} \arrow[u, phantom, "\cup"] \arrow[r, "i"] & \mathbf{V} \arrow[u, phantom, "\|"] \\
A(z) \arrow[r, mapsto] & A(z)\mathbf{1} \arrow[r, mapsto] & A_{-1}\mathbf{1}.
\end{tikzcd}$$

Assume $A_{-1}\mathbf{1} = 0$. Then, by translation covariance, $A_{-k-1}\mathbf{1} = T^{(k)}A_{-1}\mathbf{1} = 0$ for all $k \geq 0$, thus $A(z)\mathbf{1} = 0$ since $A(z)$ is creative. Now, for any $b \in \mathbf{V}$, by surjectivity of $\sigma_{\mathcal{V}}$, there exists $B(z) \in \mathcal{V}$ such that $B_{-1}\mathbf{1} = b$. Since $A(z)$ and $B(z)$ are locally commutative, we have, for some $N \in \mathbb{N}$,

$$z^N A(z)b = (z-y)^N A(z)B(y)\mathbf{1}\big|_{y=0} = (z-y)^N B(y)A(z)\mathbf{1}\big|_{y=0} = 0.$$

Hence $A(z)b = 0$ holds for all $b \in \mathbf{V}$. Thus $A(z) = 0$. □

Therefore, by Theorem 2.9, we have the following result.

Theorem 2.12 *Let* $\mathbf{V}$ *be a vector space,* $\mathbb{T}$ *a sequence of iterative operators annihilating* $\mathbf{1} \in \mathbf{V}$*, and* $\mathcal{V}$ *a vertex algebra of series on* $\mathbf{V}$ *satisfying:*

(1) *$\mathcal{V}$ is translation covariant with respect to $\mathbb{T}$.*
(2) *The state map σ restricts to a surjective map from $\mathcal{V}$ onto* $\mathbf{V}$.

Then $\mathbf{V}$ *carries a unique structure of a vertex algebra such that*

$$Y(A_{-1}\mathbf{1}, z) = A(z) \ \textit{for all}\ \ A(z) \in \mathcal{V}$$

with vacuum $\mathbf{1}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

To deal with vertex algebras generated by series, we note the following.

Lemma 2.13 *Let $A(z)$ and $B(z)$ be locally truncated series on $\mathbf{V}$. If they are translation covariant, then so are the residue products $A(z)_{(n)}B(z)$ for all $n \in \mathbb{Z}$.*

Proof. For $n \geq 0$, by $e^{x\partial_y}e^{x\partial_z}(y-z)^n = (y-z)^n$ and the Leibniz rule, we have

$$\begin{aligned} &e^{x\partial_z}\operatorname{Res}_y(y-z)^n A(y)B(z) = \operatorname{Res}_y e^{x\partial_z}((y-z)^n A(y)B(z)) \\ &= \operatorname{Res}_y e^{x\partial_y}e^{x\partial_z}((y-z)^n A(y)B(z)) = \operatorname{Res}_y(y-z)^n(e^{x\partial_y}A(y))(e^{x\partial_z}B(z)), \end{aligned}$$

as well as the one with the positions of $A(y)$ and $B(z)$ switched. Therefore, we have $e^{x\partial_z}(A(z)_{(n)}B(z)) = (e^{x\partial_z}A(z))_{(n)}(e^{x\partial_z}B(z))$, and the result is now clear by $e^{xT}(A(z)_{(n)}B(z))e^{-xT} = (e^{xT}A(z)e^{-xT})_{(n)}(e^{xT}B(z)e^{-xT})$. The same proof works for $n < 0$ as well by replacing $(y-z)^n$ with its expansions. □

Therefore, we arrive at the following corollary, a variant of what is called the *existence theorem of Frenkel–Kac–Radul–Wang*.

Corollary 2.14 *Let $\mathbf{V}$ be a vector space, $\mathbb{T}$ a sequence of iterative operators annihilating $\mathbf{1} \in \mathbf{V}$, and $\mathcal{S}$ a set of locally truncated series on $\mathbf{V}$ satisfying:*

(1) *$\mathcal{S}$ is locally commutative.*
(2) *$\mathcal{S}$ is translation covariant with respect to $\mathbb{T}$.*
(3) *The state map σ restricts to a surjective map from $\langle\mathcal{S}\rangle_{\mathrm{RP}}$ onto $\mathbf{V}$.*

Then $\mathbf{V}$ carries a unique structure of a vertex algebra such that

$$Y(S_{-1}\mathbf{1}, z) = S(z) \quad \text{for all} \quad S(z) \in \mathcal{S}$$

with vacuum $\mathbf{1}$.

Note 2.15. The equality $e^{x\partial_z}(A(z)_{(n)}B(z)) = (e^{x\partial_z}A(z))_{(n)}(e^{x\partial_z}B(z))$ achieved in the proof of Lemma 2.13 also follows from the Borcherds identity if $A(z)$ and $B(z)$ are not only locally truncated but also locally commutative.

1.2.5 Representations and Modules

Recall that a representation of a commutative associative algebra $\mathbf{C}$ is a vector space $\mathbf{M}$ equipped with a homomorphism $\rho\colon \mathbf{C} \longrightarrow \operatorname{End}\mathbf{M}$ of algebras. However, since $\operatorname{End}\mathbf{M}$ is not commutative in general, let us alternatively define a representation of $\mathbf{C}$ to be a vector space $\mathbf{M}$ equipped with a map

$$\rho\colon \mathbf{C} \longrightarrow \operatorname{End}\mathbf{M}$$

satisfying the following properties:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

(1) The image of ρ is a commutative associative algebra of operators on **M**.
(2) The map ρ gives a homomorphism of commutative associative algebras onto the image.

Then, letting the action of **C** on **M** be given by

$$\mathbf{C} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (a, v) \mapsto av = \rho(a)v,$$

the vector space **M** becomes a module over **C**.

In this section, we will follow this line to define the equivalent concepts of representations of a vertex algebra and modules over a vertex algebra.

1.2.5.1 Representations of Vertex Algebras

Let **V** be a vertex algebra and **M** a vector space. Consider a sequence (ρ_n) of countably many maps from **V** to End **M** indexed by integers $n \in \mathbb{Z}$,

$$\rho_n : \mathbf{V} \longrightarrow \operatorname{End} \mathbf{M}, \quad n \in \mathbb{Z}.$$

For an indeterminate z, let us write the generating series as

$$\rho(a, z) = \sum_n \rho_n(a) z^{-n-1}, \quad a \in \mathbf{V},$$

which give rise to a map

$$\rho(-, z) : \mathbf{V} \longrightarrow \operatorname{Hom}(\mathbf{M}, \mathbf{M}[[z, z^{-1}]]), \quad a \mapsto \rho(a, z).$$

The sequence (ρ_n) is said to be a *representation* of the vertex algebra **V** on the vector space **M** if the following conditions are satisfied:

(1) The image of $\rho(-, z)$ is a vertex algebra of series on **M**.
(2) The map $\rho(-, z)$ gives a homomorphism of vertex algebras onto the image.

Note that (1) in particular says that $\rho(a, z)$ is locally truncated for any $a \in \mathbf{V}$.

Let $\mathcal{V}$ be a vertex algebra of series on a vector space **M**. Then the obvious maps

$$\rho_n : \mathcal{V} \longrightarrow \operatorname{End} \mathbf{M}, \quad A(z) \mapsto A_n$$

form a representation of $\mathcal{V}$ on **M**.

1.2.5.2 Modules over Vertex Algebras

Let $\rho = (\rho_n)$ be a representation of a vertex algebra **V** on a vector space **M**. Consider the corresponding actions of **V** on **M** given by

$$\mathbf{V} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (a, v) \mapsto a_n v = \rho_n(a)(v).$$

Then the conditions for representations imply the following properties:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

(M0) Local truncation. For any $a \in \mathbf{V}$ and $v \in \mathbf{M}$, there exists an $N \in \mathbb{N}$ such that

$$a_{N+i}v = 0 \text{ for all } i \geq 0.$$

(M1) Borcherds identity. For all $a, b \in \mathbf{V}$, $v \in \mathbf{M}$, and $p, q, r \in \mathbb{Z}$:

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{p+q-i}v = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a_{p+r-i}b_{q+i}v - \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} b_{q+r-i}a_{p+i}v.$$

(M2) Identity. For any $v \in \mathbf{M}$ and $n \in \mathbb{Z}$:

$$\mathbf{1}_n v = \begin{cases} 0 & (n \neq -1), \\ v & (n = -1). \end{cases}$$

A vector space $\mathbf{M}$ equipped with actions satisfying (M0)–(M2) is called a *module* over $\mathbf{V}$ or a $\mathbf{V}$*-module.* Conversely, any module $\mathbf{M}$ over $\mathbf{V}$ gives rise to a representation of $\mathbf{V}$ by letting $\rho_n(a) = a_n$ for $a \in \mathbf{V}$ and $n \in \mathbb{Z}$. Thus the concepts of modules and representations are essentially the same.

The generating series $\rho(a, z)$ for a $\mathbf{V}$-module $\mathbf{M}$ are usually denoted

$$Y_{\mathbf{M}}(a, z) = \sum_n a_n z^{-n-1}.$$

In particular, the vertex algebra $\mathbf{V}$ itself is thought of as a module over $\mathbf{V}$, called the *adjoint module*, for which $Y_{\mathbf{V}}(a, z) = Y(a, z)$ for all $a \in \mathbf{V}$.

1.2.5.3 Consequences of the Axioms

In the same way as in the case of the Borcherds identity for vertex algebras, we can derive various properties for modules. First note that (M0) means local truncation of the generating series $Y_{\mathbf{M}}(a, z)$:

$$Y_{\mathbf{M}}(a, z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))).$$

(MC) Commutator formula. For all $a, b \in \mathbf{V}$ and $m, n \in \mathbb{Z}$:

$$[a_m, b_n] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{m+n-i}.$$

(MA) Associativity formula. For all $a, b \in \mathbf{V}$, and $m, n \in \mathbb{Z}$:

$$(a_{(m)}b)_n v = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a_{m-i}b_{n+i}v - \sum_{i=0}^{\infty} (-1)^{m-i} \binom{m}{i} b_{m+n-i}a_i v.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

In terms of generating series, they are written respectively as follows:

$$[Y_{\mathbf{M}}(a,y), Y_{\mathbf{M}}(b,z)] = \sum_{i=0}^{\infty} Y_{\mathbf{M}}(a_{(i)}b,z)\delta^{(i)}(y,z),$$
$$Y_{\mathbf{M}}(a_{(m)}b,z) = Y_{\mathbf{M}}(a,z)_{(m)}Y_{\mathbf{M}}(b,z) \quad (m \in \mathbb{Z}).$$

(MLC) Local commutativity. For any $a, b \in \mathbf{V}$, there exists an $N \in \mathbb{N}$ such that:

$$(y-z)^N Y_{\mathbf{M}}(a,y)Y_{\mathbf{M}}(b,z) = (y-z)^N Y_{\mathbf{M}}(b,z)Y_{\mathbf{M}}(a,y).$$

(MLA) Local associativity. For any $a \in \mathbf{V}$ and $v \in \mathbf{M}$, there exists an $L \in \mathbb{N}$ such that for all $b \in \mathbf{V}$:

$$(x+z)^L Y_{\mathbf{M}}(Y_{\mathbf{V}}(a,x)b,z)v = (x+z)^L Y_{\mathbf{M}}(a,x+z)\big|_{|x|>|z|} Y_{\mathbf{M}}(b,z)v.$$

Recall the translation operators:

$$T^{(k)}\colon \mathbf{V} \longrightarrow \mathbf{V}, \;\; a \mapsto T^{(k)}a = a_{(-k-1)}\mathbf{1}.$$

The following property follows from (M1) and (M2):

(MT) Translation. For all $a \in \mathbf{V}$, $v \in \mathbf{M}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$:

$$(T^{(k)}a)_n v = (-1)^k \binom{n}{k} a_{n-k}v.$$

Note 2.16. 1. The commutator formula (MC) is equivalent to the OPE

$$Y_{\mathbf{M}}(a,y)Y_{\mathbf{M}}(b,z) \simeq Y_{\mathbf{M}}(b,z)Y_{\mathbf{M}}(a,y) \sim \sum_{i=0}^{\infty} \frac{Y_{\mathbf{M}}(a_{(i)}b,z)}{(y-z)^{i+1}},$$

while the Borcherds identity (M1) is equivalent to

$$Y_{\mathbf{M}}(a,y)Y_{\mathbf{M}}(b,z) \simeq Y_{\mathbf{M}}(b,z)Y_{\mathbf{M}}(a,y) \simeq Y_{\mathbf{M}}(Y_{\mathbf{V}}(a,y-z)b,z).$$

Thus the Borcherds identity (M1) is characterized by the following condition: There exists an element $w(x,y,z) \in \mathbf{M}((x,y,z))$ such that

$$Y_{\mathbf{M}}(a,y)Y_{\mathbf{M}}(b,z)v = w(y-z,y,z)\big|_{|y|>|z|},$$
$$Y_{\mathbf{M}}(b,z)Y_{\mathbf{M}}(a,y)v = w(y-z,y,z)\big|_{|y|<|z|},$$
$$Y_{\mathbf{M}}(Y(a,x)b,z)v = w(x,x+z,z)\big|_{|x|<|y|}.$$

The Borcherds identity (V1) for vertex algebras, which is seen to be a particular case of (M1), is also formulated in the same way. 2. In the definition of modules, the Borcherds identity (M1) can be replaced by local associativity (MLA) thanks to skew-symmetry (VS) for the vertex algebra (cf. [8]).

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Bibliographic Notes

General references for Section 1.2 are Kac [6], Matsuo and Nagatomo [7], Frenkel and Ben-Zvi [8], and Rosellen [11]. It is more or less straightforward to describe the materials covered in Section 1.2 over commutative rings. See Mason [78] for accounts over $\mathbb{Z}$.

For operator product expansions in various models in physics, consult [13] (cf. [19]). Formulation of local commutativity (locality) in terms of formal series is due to [4]. See [6] and [7] for formulation of operator product expansion in terms of formal series, and Tsuchiya and Kanie [96] for an earlier analytic formulation.

The statement of Theorem 2.5 is due to [7] (cf. [11]), although the result was implicitly given previously by Li in [70] in his proof of Corollary 2.6.

Construction of vertex algebras of series is due to [70], while properties of the residue products and their relation to vertex algebras are studied by Lian and Zuckerman [77]. See Goddard [20] for the original form of the uniqueness theorem and its applications, and Frenkel et al. [55] for the existence theorem.

1.3 Examples of Vertex Algebras

In Section 1.3, we will describe some standard examples of vertex algebras, the Heisenberg vertex algebra, the affine vertex algebras, and the Virasoro vertex algebras. Yet another class of standard examples, the lattice vertex algebras, will be described in Section 1.4.

For affine and Virasoro vertex algebras, we will first describe the *universal* ones as vector spaces and characterize the vertex algebra structure on them by conditions over a field of any characteristic not 2, and then describe some simple quotients over the field $\mathbb{C}$ of complex numbers.

1.3.1 Heisenberg Vertex Algebra

Let us start by constructing the Heisenberg vertex algebra of rank one briefly described in Subsection 1.1.4.2. Generalization to higher rank is straightforward (cf. Subsection 1.4.2.1).

Although they are particular cases of affine vertex algebras described in the next section, the Heisenberg vertex algebras have distinguished properties and play an important role in constructing the lattice vertex algebras in Section 1.4.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.3.1.1 Heisenberg Algebra

Let a_n with $n \in \mathbb{Z}$ and ζ be indeterminates and set

$$\hat{\mathfrak{h}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{F}a_n \oplus \mathbb{F}\zeta.$$

Then $\hat{\mathfrak{h}}$ becomes a Lie algebra by the bracket

$$[a_m, a_n] = m\delta_{m+n,0}\zeta, \quad [\zeta, a_n] = 0.$$

It is an infinite-dimensional *Heisenberg Lie algebra.*

Consider the quotient $\mathbf{U}(\hat{\mathfrak{h}}, 1)$ of the universal enveloping algebra $\mathbf{U}(\hat{\mathfrak{h}})$ by the two-sided ideal generated by $\zeta - 1$, where the scalar multiples of the unity of $\mathbf{U}(\hat{\mathfrak{h}})$ are identified with the scalars:

$$\mathbf{U}(\hat{\mathfrak{h}}, 1) = \mathbf{U}(\hat{\mathfrak{h}})/(\zeta - 1).$$

We will denote the images of the generators a_n by the same symbol. Then

$$[a_m, a_n] = m\delta_{m+n,0} \tag{3.1}$$

for $m, n \in \mathbb{Z}$, where the bracket denotes the commutator.

The algebra $\mathbf{U}(\hat{\mathfrak{h}}, 1)$ is the associative algebra generated by the symbols a_n, $n \in \mathbb{Z}$ subject to (3.1) as the fundamental relations. We will call it the *Heisenberg algebra.*

Let $\mathbf{M}$ be a $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module, regard the element a_n with $n \in \mathbb{Z}$ as operators acting on $\mathbf{M}$, and consider the generating series which we will call a *current* following physics terminology:

$$a(z) = \sum_n a_n z^{-n-1}.$$

As mentioned at the beginning of Section 1.2.2, we have

$$a(y)a(z) \simeq a(z)a(y) \sim \frac{1}{(y-z)^2}, \tag{3.2}$$

where the numerator 1 means the identity series $I(z)$. Note in particular that the current $a(z)$ is locally commutative with itself.

1.3.1.2 Fock Modules

Let $\hat{\mathfrak{h}}_{<0}$ and $\hat{\mathfrak{h}}_{\geq 0}$ be the Lie subalgebras of $\hat{\mathfrak{h}}$ spanned by a_n with $n < 0$ and $n \geq 0$, respectively. They are commutative, and generate subalgebras of $\mathbf{U}(\hat{\mathfrak{h}}, 1)$ isomorphic to the symmetric algebras $\mathbf{S}(\hat{\mathfrak{h}}_{<0})$ and $\mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})$, respectively. By PBW for $\mathbf{U}(\hat{\mathfrak{h}})$, as vector spaces,

$$\mathbf{U}(\hat{\mathfrak{h}}, 1) = \mathbf{S}(\hat{\mathfrak{h}}_{<0}) \otimes \mathbf{S}(\hat{\mathfrak{h}}_{\geq 0}). \tag{3.3}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The element a_0 is central and, when acting on a module, its eigenvalue is called the *charge*.

Now, for each scalar λ, define a one-dimensional $\mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})$-module $\mathbb{F}\mathbf{v}_\lambda$ by

$$a_n\mathbf{v}_\lambda = \begin{cases} 0 & (n \geq 1), \\ \lambda\mathbf{v}_\lambda & (n = 0), \end{cases}$$

and consider the universal $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module generated by $\mathbf{v}_\lambda$ given by

$$\mathbf{F}_\lambda = \mathbf{U}(\hat{\mathfrak{h}}, 1) \otimes_{\mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})} \mathbb{F}\mathbf{v}_\lambda$$

on which $\hat{\mathfrak{h}}$ acts by left multiplication on $\mathbf{U}(\hat{\mathfrak{h}}, 1)$. The $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module $\mathbf{F}_\lambda$ thus obtained is called the *Fock module* of *charge* λ, also called the *Fock space* of charge λ.

Let us denote the element $1 \otimes \mathbf{v}_\lambda$ simply by $\mathbf{v}_\lambda$. By (3.3), we have

$$\mathbf{F}_\lambda = \mathbf{S}(\hat{\mathfrak{h}}_{<0})\mathbf{v}_\lambda \simeq \mathbf{S}(\hat{\mathfrak{h}}_{<0}).$$

The Fock module $\mathbf{F}_\lambda$ is characterized by the following universal property:

For any $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module $\mathbf{M}$ and $\mathbf{w} \in \mathbf{M}$ satisfying

$$a_n\mathbf{w} = \begin{cases} 0 & (n \geq 1), \\ \lambda\mathbf{w} & (n = 0), \end{cases}$$

there exists a unique homomorphism $\psi : \mathbf{F}_\lambda \longrightarrow \mathbf{M}$ of $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-modules sending $\mathbf{v}_\lambda$ to $\mathbf{w}$.

Here the condition of ψ means that the following diagram commutes:

$$\begin{tikzcd} \mathbb{F}\mathbf{v}_\lambda \arrow[r] \arrow[d] & \mathbb{F}\mathbf{w} \arrow[d] \\ \mathbf{F}_\lambda \arrow[r, "\psi"'] & \mathbf{M}. \end{tikzcd}$$

Here the upper arrow sends $\mathbf{v}_\lambda$ to $\mathbf{w}$ and the vertical ones are inclusions.

We may further identify the Fock module with the polynomial ring via the linear isomorphism as in Table 3 defined by

$$\mathbb{F}[x_1, x_2, \cdots] \xrightarrow{\sim} \mathbf{F}_\lambda, \quad p(x_1, x_2, \cdots) \mapsto p(a_{-1}, a_{-2}, \cdots)\mathbf{v}_\lambda.$$

Then the actions of a_n for $n \neq 0$ turn out to be the same as (1.5) given by differential operators, while a_0 acts as multiplication by λ.

Among the Fock modules, the module $\mathbf{F}_0$ of charge 0 is called the *vacuum module* and the vector $\mathbf{v}_0$ the *vacuum*.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 3 Fock module of charge λ

0	1	2	3	4
1	x_1	x_2	x_3	x_4
		x_1^2	x_1x_2	x_1x_3
			x_1^3	x_2x_2
				$x_1^2x_2$
A basis of $\mathbb{F}[x_1, x_2, \cdots]$				x_1^4
$\mathbf{v}_\lambda$	$a_{-1}\mathbf{v}_\lambda$	$a_{-2}\mathbf{v}_\lambda$	$a_{-3}\mathbf{v}_\lambda$	$a_{-4}\mathbf{v}_\lambda$
		$a_{-1}a_{-1}\mathbf{v}_\lambda$	$a_{-1}a_{-2}\mathbf{v}_\lambda$	$a_{-1}a_{-3}\mathbf{v}_\lambda$
			$a_{-1}a_{-1}a_{-1}\mathbf{v}_\lambda$	$a_{-2}a_{-2}\mathbf{v}_\lambda$
				$a_{-1}a_{-1}a_{-2}\mathbf{v}_\lambda$
A basis of $\mathbf{F}_\lambda$				$a_{-1}a_{-1}a_{-1}a_{-1}\mathbf{v}_\lambda$

1.3.1.3 Vertex Algebra of Series on Fock Modules

Regard the current $a(z)$ as a series acting on $\mathbf{F}_\lambda$ for a $\lambda \in \mathfrak{h}^*$. Since it is locally truncated and locally commutative with itself, it generates a vertex algebra of series.

Recall that it is the span of the series obtained by repeatedly applying residue products of the current $a(z)$, which is actually spanned by the series of the form $a(z)_{(n_1)} \cdots a(z)_{(n_k)} I(z)$, where the residue products are taken from right to left. Let us denote it by

$$\mathcal{F}_0(\lambda) = \operatorname{Span}\left\{ a(z)_{(n_1)} \cdots a(z)_{(n_k)} I(z) \,\middle|\, k \in \mathbb{N}, n_1, \ldots, n_k \in \mathbb{Z} \right\},$$

where λ signifies that the space consists of series acting on $\mathbf{F}_\lambda$.

Since $\mathcal{F}_0(\lambda)$ is a vertex algebra, the commutator formula is available. To describe it, consider the residue products of the current $a(z)$ with itself, which can be read off by extracting coefficients in the OPE (3.2) as

$$a(z)_{(n)} a(z) = \begin{cases} 0 & (n \geq 2), \\ 1 & (n = 1), \\ 0 & (n = 0). \end{cases}$$

Therefore, the commutator formula reads

$$[a(z)_{(m)}, a(z)_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a(z)_{(i)} a(z))_{(m+n-i)} = m\delta_{m+n,0}.$$

Remarkably, this is the same as the Heisenberg commutation relation (3.1), and the space $\mathcal{F}_0(\lambda)$ becomes a $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module by

$$a_n : \mathcal{F}_0(\lambda) \longrightarrow \mathcal{F}_0(\lambda), \quad X(z) \mapsto a(z)_{(n)} X(z).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 4 Fock module of charge 0, abstract versus realization

0		2	3
$\mathbf{v}_\lambda$	$a_{-1}\mathbf{v}_\lambda$	$a_{-2}\mathbf{v}_\lambda$	$a_{-3}\mathbf{v}_\lambda$
		$a_{-1}a_{-1}\mathbf{v}_\lambda$	$a_{-1}a_{-2}\mathbf{v}_\lambda$
A basis of $\mathbf{F}_0$			$a_{-1}a_{-1}a_{-1}\mathbf{v}_\lambda$
$I(z)$	$a(z)_{(-1)}I(z)$	$a(z)_{(-2)}I(z)$	$a(z)_{(-3)}I(z)$
		$a(z)_{(-1)}a(z)_{(-1)}I(z)$	$a(z)_{(-1)}a(z)_{(-2)}I(z)$
A basis of $\mathcal{F}_0(\lambda)$			$a(z)_{(-1)}a(z)_{(-1)}a(z)_{(-1)}I(z)$

Moreover, since the identity series $I(z) = 1$ satisfies

$$a(z)_{(n)}I(z) = 0 \quad (n \geq 0),$$

the universal property of the Fock module implies that there exists a unique homomorphism of $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-modules sending the vector $\mathbf{v}_0$ to the identity series $I(z)$ on $\mathbf{F}_\lambda$,

$$\psi_\lambda : \mathbf{F}_0 \longrightarrow \mathcal{F}_0(\lambda), \quad \mathbf{v}_0 \mapsto I(z), \tag{3.4}$$

which is surjective since the $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module $\mathcal{F}_0(\lambda)$ is generated by $I(z)$ (cf. Table 4).

1.3.1.4 Identification of Heisenberg Vertex Algebra

For $\lambda = 0$, the current $a(z)$ is creative with respect to $\mathbf{v}_0$ and the map ψ_0 is inverse to the state map, hence the state map restricted to $\mathcal{F}_0(0)$ is bijective:

$$\sigma_{\mathcal{F}_0(0)} : \mathcal{F}_0(0) \xrightarrow{\sim} \mathbf{F}_0.$$

Therefore, we may transfer the vertex algebra structure on $\mathcal{F}_0(0)$ to $\mathbf{F}_0$ via the map $\sigma_{\mathcal{F}_0(0)}$, and general theory in Section 1.2 yields the following result, which restates Proposition 1.11.

Proposition 3.1 *The Fock module* $\mathbf{F}_0$ *of charge* 0 *carries a unique structure of a vertex algebra such that*

$$Y(a_{-1}\mathbf{v}_0, z) = a(z)$$

with vacuum $\mathbf{1} = \mathbf{v}_0$

For each $\lambda \in \mathfrak{h}^*$, the map (3.4) turns out to be a representation of $\mathbf{F}_0$, giving rise to a structure of a module over the Heisenberg vertex algebra $\mathbf{F}_0$ on the space $\mathbf{F}_\lambda$.

By considering the action of Heisenberg algebra, it is not difficult to show that, over a field of characteristic zero, the modules $\mathbf{F}_\lambda$ with $\lambda \in \mathfrak{h}^*$ are simple, as well as the Heisenberg vertex algebra $\mathbf{F}_0$ itself.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The construction of the Virasoro vector (1.8) is restated as

$$\omega = \frac{1}{2}a_{-1}a_{-1}\mathbf{1}. \tag{3.5}$$

The space $\mathbf{F}_0$ is given a grading by

$$\deg a_{-k_1-1}\cdots a_{-k_i-1}\mathbf{v}_0 = k_1 + \cdots + k_i,$$

which agrees with the eigenvalue for the action of L_0.

Let $\mathbf{F}_{0,d}$ denote the subspace of degree d. Then distribution of degrees is encoded in the *graded dimension*:

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{F}_{0,d} = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{\phi(q)} = \frac{q^{1/24}}{\eta(\tau)},$$

where $\phi(q)$ is the Euler function and $\eta(\tau)$ the Dedekind eta function with $q = e^{2\pi\sqrt{-1}\tau}$.

1.3.2 Affine Vertex Algebras

The Heisenberg Lie algebra as in the preceding section is actually a particular example of an *affine Lie algebra.*

In this section, we will describe affine Lie algebras and the associated vertex algebras.

1.3.2.1 Affine Lie Algebras

A bilinear form $(\ |\ )$ on a Lie algebra $\mathfrak{g}$ is said to be *invariant* (with respect to the adjoint action of $\mathfrak{g}$) if, for all $X, Y, Z \in \mathfrak{g}$,

$$([X,Y]|Z) = (X|[Y,Z]).$$

Let $\mathbb{F}[t,t^{-1}]$ denote the ring of Laurent polynomials in t and let K be an indeterminate.

For a Lie algebra $\mathfrak{g}$, set

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{F}[t,t^{-1}] \oplus \mathbb{F}K.$$

Denote the element $X \otimes t^n$ for $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$ as

$$X_n = X \otimes t^n.$$

Given a bilinear form $(\ |\ )$ on $\mathfrak{g}$, define a bracket operation by setting

$$[X_m, Y_n] = [X,Y]_{m+n} + m\delta_{m+n,0}(X|Y)K, \quad [K, X_n] = 0.$$

If the bilinear form is symmetric and invariant, then the bracket equips $\hat{\mathfrak{g}}$ with a structure of a Lie algebra. The Lie algebra $\hat{\mathfrak{g}}$ thus obtained is called the *affine Lie algebra* associated with $\mathfrak{g}$ and $(\ |\ )$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

A $\hat{\mathfrak{g}}$-module on which the central element K acts by a scalar k is said to be of *level* k, which is the same as a module over the quotient

$$\mathbf{U}(\hat{\mathfrak{g}}, k) = \mathbf{U}(\hat{\mathfrak{g}})/(K - k),$$

where the scalar k is identified with the multiple of the unity by k.

Let $\mathbf{M}$ be a $\hat{\mathfrak{g}}$-module of level k, regard the elements X_n for each $X \in \mathfrak{g}$ as operators on $\mathbf{M}$, and consider the generating series, again called a *current*:

$$X(z) = \sum_n X_n z^{-n-1}.$$

Then the commutation relation for $[X_m, Y_n]$ is equivalently described by the OPE:

$$X(y)Y(z) \simeq Y(z)X(y) \sim \frac{[X,Y](z)}{y-z} + \frac{k(X|Y)}{(y-z)^2}. \tag{3.6}$$

In particular, $X(z)$ and $Y(z)$ are locally commutative with each other.

1.3.2.2 Generalized Verma Modules

Consider the following Lie subalgebras of the affine Lie algebra $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes \mathbb{F}[t^{-1}]t^{-1}, \quad \hat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{F}[t].$$

They generate subalgebras of $\mathbf{U}(\hat{\mathfrak{g}}, k)$ isomorphic to $\mathbf{U}(\hat{\mathfrak{g}}_{<0})$ and $\mathbf{U}(\hat{\mathfrak{g}}_{\geq 0})$, respectively, and PBW for $\mathbf{U}(\hat{\mathfrak{g}})$ implies

$$\mathbf{U}(\hat{\mathfrak{g}}, k) = \mathbf{U}(\hat{\mathfrak{g}}_{<0}) \otimes \mathbf{U}(\hat{\mathfrak{g}}_{\geq 0}).$$

Let V be a $\mathfrak{g}$-module and regard it as a $\mathbf{U}(\hat{\mathfrak{g}}_{\geq 0})$-module in the following way: for $X \in \mathfrak{g}$ and $v \in V$,

$$X_n v = \begin{cases} 0 & (n \geq 1), \\ Xv & (n = 0). \end{cases}$$

Define a $\hat{\mathfrak{g}}$-module $\mathbf{M}_k(V)$ by

$$\mathbf{M}_k(V) = \mathbf{U}(\hat{\mathfrak{g}}, k) \otimes_{\mathbf{U}(\hat{\mathfrak{g}}_{\geq 0})} V,$$

where the action of $\hat{\mathfrak{g}}$ is by left multiplication on $\mathbf{U}(\hat{\mathfrak{g}}, k)$. This is a particular type of what is called the *generalized Verma module.*

When V is the one-dimensional trivial $\mathfrak{g}$-module $\mathbb{F}\mathbf{v}_0$, the module $\mathbf{M}_k(\mathbb{F}\mathbf{v}_0)$ is called the *universal vacuum module,* and the vector $\mathbf{v}_0$ the *vacuum vector.* We will often identify $\mathbb{F}\mathbf{v}_0$ with $\mathbb{F}$ and denote the universal vacuum module by $\mathbf{M}_k(\mathbb{F})$, which has an obvious universal property by construction.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.3.2.3 Affine Vertex Algebras

Let V be a $\mathfrak{g}$-module and consider the induced module $\mathbf{M}_k(V)$. Regard the currents $X(z)$ for $X \in \mathfrak{g}$ as series acting on $\mathbf{M}_k(V)$, which are locally truncated. Since they are locally commutative, they generate a vertex algebra of series:

$$\mathcal{M}_k(\mathbb{F})_V = \left\{ X^1(z)_{(n_1)} \cdots X^l(z)_{(n_l)} I(z) \,\middle|\, \begin{matrix} l \in \mathbb{N}, X^1, \ldots, X^l \in \mathfrak{g} \\ n_1, \ldots, n_l \in \mathbb{Z} \end{matrix} \right\}.$$

By the OPE (3.6),

$$X(z)_{(n)} Y(z) = \begin{cases} 0 & (n \geq 2), \\ k(X|Y) & (n = 1), \\ [X,Y](z) & (n = 0). \end{cases}$$

The commutator formula implies

$$[X(z)_{(m)}, Y(z)_{(n)}] = [X,Y](z)_{(m+n)} + k(X|Y) m \delta_{m+n,0}.$$

The identity series satisfies

$$X(z)_{(n)} I(z) = 0 \ \ (n \geq 0).$$

Therefore, by the universal property of $\mathbf{M}_k(\mathbb{F})$, there exists a unique homomorphism of $\mathbf{U}(\mathfrak{g}, k)$-modules sending the vector $\mathbf{v}_0$ to the identity series $I(z)$:

$$\psi_V : \mathbf{M}_k(\mathbb{F}) \longrightarrow \mathcal{M}_k(\mathbb{F})_V, \ \ \mathbf{v}_0 \mapsto I(z),$$

which is surjective since the $\mathbf{U}(\mathfrak{g}, k)$-module $\mathcal{M}_k(\mathbb{F})_V$ is generated by $I(z)$.

When V is the one-dimensional trivial $\mathfrak{g}$-module $\mathbb{F} = \mathbb{F}\mathbf{v}_0$, the currents are creative with respect to $\mathbf{v}_0$, and the map $\psi = \psi_{\mathbb{F}}$ is inverse to the state map.

Proposition 3.2 *The universal vacuum module $\mathbf{M}_k(\mathbb{F})$ of level k over the affine Lie algebra $\hat{\mathfrak{g}}$ carries a unique structure of a vertex algebra such that*

$$Y(X_{-1}\mathbf{v}_0, z) = X(z), \ \ X \in \mathfrak{g},$$

with vacuum $\mathbf{1} = \mathbf{v}_0$.

The vertex algebra thus obtained is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k. The module $\mathbf{M}_k(V)$ associated with a $\mathfrak{g}$-module V becomes a module over the vertex algebra $\mathbf{M}_k(\mathbb{F})$.

A quotient of $\mathbf{M}_k(\mathbb{F})$ is generally called an *affine vertex algebra* associated with $\mathfrak{g}$ at level k. The structure of such a quotient, including the simple one, heavily depends on the Lie algebra $\mathfrak{g}$ and the level k.

Note 3.3. Consider the subspace of $\mathbf{M}_k(\mathbb{F})$ spanned by $X_{(-1)}\mathbf{1}$ with$X \in \mathfrak{g}$:

$$\mathbf{M}_k(\mathbb{F})_1 = \mathrm{Span}\left\{ X_{(-1)}\mathbf{1} \,\middle|\, X \in \mathfrak{g} \right\}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Then the 0th product equips it with a structure of a Lie algebra and the coefficient to **1** of the 1st product gives a symmetric bilinear form on it. The map

$$i\colon \mathfrak{g} \longrightarrow \mathbf{M}_k(\mathbb{F})_1, \quad X \mapsto X_{(-1)}\mathbf{1}$$

is an isomorphism of Lie algebras, which is an isometry multiplied by k.

1.3.2.4 Integrable Highest Weight Modules over $\widehat{\mathfrak{sl}}_2$

Let us consider the case when $\mathfrak{g}$ is the three-dimensional simple Lie algebra $\mathfrak{sl}_2$. We assume that the base field is $\mathbb{C}$.

The Lie algebra $\mathfrak{sl}_2$ is spanned by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with the brackets

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

An invariant bilinear form on $\mathfrak{sl}_2$ is a scalar multiple of the Killing form. We normalize it as

$$\begin{aligned}(H|H) = 2, \quad (E|F) = (F|E) = 1,\\ (E|E) = (F|F) = (H|E) = (H|F) = 0.\end{aligned}$$

Let $\widehat{\mathfrak{sl}}_2$ denote the associated affine Lie algebra, which is (the derived algebra of) the affine Kac–Moody algebra of type $A_1^{(1)}$.

Finite-dimensional simple $\mathfrak{sl}_2$-modules are classified by their dimensions. We will denote the $(2j+1)$-dimensional simple module by V_j, where j is a nonnegative half integer. The representation V_j is said to be of *spin* j.

1. The module V_0 corresponds to the one-dimensional *trivial representation.*
2. The module $V_{1/2}$ corresponds to the two-dimensional *vector representation*, the representation defining $\mathfrak{sl}_2$ by 2×2 matrices.
3. The module V_1 corresponds to the three-dimensional *adjoint representation*, the representation by the adjoint action of $\mathfrak{sl}_2$ on itself.

Let $\mathbf{M}(k,j)$ denote the module $\mathbf{M}_k(V_j)$ and $\mathbf{L}(k,j)$ its simple quotient. The image of an element of $\mathbf{M}(k,j)$ in the quotient $\mathbf{L}(k,j)$ will be denoted by the same symbol by abuse of notation.

For a positive integer k, consider the $k+1$ simple quotients

$$\mathbf{L}(k,0), \mathbf{L}(k,1/2), \ldots, \mathbf{L}(k,k/2).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 5 Universal and simple $\widehat{\mathfrak{sl}}_2$-modules

0	1	2	0	1	2
		$E_{-1}E_{-1}\mathbf{v}_0$			
		$E_{-2}\mathbf{v}_0$			$E_{-2}\mathbf{v}_0$
	$E_{-1}\mathbf{v}_0$	$H_{-1}E_{-1}\mathbf{v}_0$		$E_{-1}\mathbf{v}_0$	
		$F_{-1}E_{-1}\mathbf{v}_0$			
$\mathbf{v}_0$	$H_{-1}\mathbf{v}_0$	$H_{-2}\mathbf{v}_0$	$\mathbf{v}_0$	$H_{-1}\mathbf{v}_0$	$H_{-2}\mathbf{v}_0$
		$H_{-1}H_{-1}\mathbf{v}_0$			$H_{-1}H_{-1}\mathbf{v}_0$
	$F_{-1}\mathbf{v}_0$	$F_{-1}H_{-1}\mathbf{v}_0$		$F_{-1}\mathbf{v}_0$	
		$F_{-2}\mathbf{v}_0$			$F_{-2}\mathbf{v}_0$
A basis of $\mathbf{M}(k,0)$		$F_{-1}F_{-1}\mathbf{v}_0$	A basis of $\mathbf{L}(1,0)$		
$\underline{\mathbf{1}}$	$\underline{\mathbf{3}}$	$\underline{\mathbf{1}} \oplus \underline{\mathbf{3}} \oplus \underline{\mathbf{5}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{3}}$	$\underline{\mathbf{1}} \oplus \underline{\mathbf{3}}$

Then they form an important class of representations of the affine Kac–Moody Lie algebra $\widehat{\mathfrak{sl}}_2$, called the *integrable highest weight representations* of level k. Among them, $\mathbf{L}(k,0)$ is a simple vertex algebra, and the rest as well as itself are simple modules over it.

For example, when $k = 1$, the vector $E_{-1}E_{-1}\mathbf{v}_0$ generates a maximal proper submodule of $\mathbf{M}(1,0)$ and the quotient $\mathbf{L}(1,0)$ is a simple vertex algebra (Table 5). The last rows exhibit the decomposition of each subspace under the action of $\mathfrak{sl}_2$ given by E_0, H_0, F_0, where $\underline{\mathbf{1}} = V_0$, $\underline{\mathbf{3}} = V_1$, and $\underline{\mathbf{5}} = V_2$.

For $k \neq -2$, we may consider the *Sugawara vector* defined by

$$\omega_k = \frac{1}{2(k+2)}\left(\frac{1}{2}H_{-1}H_{-1} + E_{-1}F_{-1} + F_{-1}E_{-1}\right)\mathbf{v}_0. \tag{3.7}$$

Then it becomes a Virasoro vector of central charge $c_k = 3k/(k+2)$. That is, the operators $L_n = \omega_{(n+1)}$ with $n \in \mathbb{Z}$ satisfy the commutation relation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} c_k \delta_{m+n,0}, \quad c_k = \frac{3k}{k+2}.$$

We take the Virasoro vector ω_k defined by (3.7) as the standard choice for the affine vertex algebra associated with $\mathfrak{sl}_2$.

Note 3.4. 1. In this construction, the Lie algebra $\mathfrak{sl}_2$ can be replaced by any finite-dimensional simple Lie algebra $\mathfrak{g}$. Among the integrable highest weight modules over the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ of level k, for which k is a positive integer, the vacuum module $\mathbf{L}(k,0)$ becomes a vertex algebra and the simple modules are classified as $\mathbf{L}(k,\lambda)$ where λ runs over the *dominant integral weights of level k*. For details, see [17]. 2. The construction (3.7) is called the *Sugawara construction*, which works for any finite-dimensional simple Lie algebra $\mathfrak{g}$ by means of the Casimir element of $\mathfrak{g}$ as long as $k + h^\vee \neq 0$, where $h^\vee$ the dual Coxeter number, that is, half the value of the Casimir action on the

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

adjoint module, and the central charge of the resulting Virasoro action is given by $c_k = k \dim \mathfrak{g}/(k + h^\vee)$. For example, if $\mathfrak{g} = \mathfrak{sl}_2$, then $h^\vee = 2$, so $c_1 = 1$, $c_2 = 3/2$, $c_3 = 9/5$, etc.

1.3.3 Virasoro Vertex Algebras

In this section, we will describe the universal Virasoro vertex algebras and their simple quotients. There is a particularly nice family of such quotients, called the *Virasoro minimal models.*

1.3.3.1 Virasoro Algebras

Let L_n, $n \in \mathbb{Z}$ and C be indeterminates and set

$$\mathrm{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{F} L_n \oplus \mathbb{F} C.$$

Then Vir becomes a Lie algebra, called the *Virasoro algebra*, by the bracket

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0} C, \quad [C, L_n] = 0.$$

A Vir-module on which the central element C acts by a scalar c is said to be of *central charge* c. It is equivalently described as a module over the quotient

$$\mathbf{U}(\mathrm{Vir}, c) = \mathbf{U}(\mathrm{Vir})/(C - c),$$

where the scalar c is identified with the multiple of the unity by c.

Let $\mathbf{M}$ be a Vir-module of central charge c, regard the element L_n for $n \in \mathbb{Z}$ as operators acting on $\mathbf{M}$, and consider the generating series which we denote, following physics, as follows:

$$T(z) = \sum_n L_n z^{-n-2}.$$

Then the Virasoro commutation relation is equivalently described by the OPE:

$$T(y)T(z) \simeq T(z)T(y) \sim \frac{\partial T(z)}{y - z} + \frac{2T(z)}{(y - z)^2} + \frac{c/2}{(y - z)^4}.$$

In particular, $T(z)$ is locally commutative with itself.

1.3.3.2 Verma Modules

Let $\mathrm{Vir}_{<0}$ and $\mathrm{Vir}_{\geq 0}$ be the subspaces of Vir spanned by L_n with $n < 0$ and $n \geq 0$, respectively, which form Lie subalgebras of Vir. They generate subalgebras of $\mathbf{U}(\mathrm{Vir}, c)$ isomorphic to $\mathbf{U}(\mathrm{Vir}_{<0})$ and $\mathbf{U}(\mathrm{Vir}_{\geq 0})$, respectively, and

$$\mathbf{U}(\mathrm{Vir}, c) = \mathbf{U}(\mathrm{Vir}_{<0}) \otimes \mathbf{U}(\mathrm{Vir}_{\geq 0}).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 6 Virasoro Verma module

0	1	2	3	4
$\mathbf{v}_h$	$L_{-1}\mathbf{v}_h$	$L_{-2}\mathbf{v}_h$	$L_{-3}\mathbf{v}_h$	$L_{-4}\mathbf{v}_h$
		$L_{-1}L_{-1}\mathbf{v}_h$	$L_{-2}L_{-1}\mathbf{v}_h$	$L_{-2}L_{-2}\mathbf{v}_h$
			$L_{-1}L_{-1}L_{-1}L_{-1}\mathbf{v}_h$	$L_{-3}L_{-1}\mathbf{v}_h$
				$L_{-2}L_{-1}L_{-1}\mathbf{v}_h$
A basis of $\mathbf{M}(c,h)$				$L_{-1}L_{-1}L_{-1}L_{-1}\mathbf{v}_h$

For each scalar h, define a one-dimensional $\mathbf{U}(\mathrm{Vir}_{\geq 0})$-module $\mathbb{F}\mathbf{v}_h$ by

$$L_n\mathbf{v}_h = \begin{cases} 0 & (n \geq 1), \\ h\mathbf{v}_h & (n = 0), \end{cases}$$

and a Vir-module $\mathbf{M}(c,h)$ by

$$\mathbf{M}(c,h) = \mathbf{U}(\mathrm{Vir}, c) \otimes_{\mathbf{U}(\mathrm{Vir}_{\geq 0})} \mathbb{F}\mathbf{v}_h,$$

where the action of Vir is by left multiplication on $\mathbf{U}(\mathrm{Vir}, c)$ (cf. Table 6). The resulting Vir-module is a highest weight module, called the *Verma module* of highest (conformal) weight h (although the value of the weight h is actually the lowest).

An element v of a Vir-module M is said to be a *singular vector* (for the Virasoro action) or a *primary vector* of M if the following condition holds:

$$L_n v = 0 \;\; (n \geq 1).$$

One often assumes that v is an eigenvector with respect to the action of L_0.

1.3.3.3 Virasoro Vertex Algebras

The module $\mathbf{M}(c,0)$ actually does not carry a natural structure of a vertex algebra. Indeed, if so with $\mathbf{1} = \mathbf{v}_0$, then, by the creation property, we must have

$$L_{-1}\mathbf{1} = \omega_{(0)}\mathbf{1} = 0.$$

We are thus led to consider a Virasoro module generated by a highest weight vector that is annihilated not only by L_n for $n \geq 0$ but also by L_{-1}.

To construct a universal one among such, let $\mathrm{Vir}_{\geq -1}$ be the subspace of Vir spanned by L_n with $n \geq -1$, which becomes a Lie subalgebra of Vir and generates a subalgebra of $\mathbf{U}(\mathrm{Vir}, c)$ isomorphic to $\mathbf{U}(\mathrm{Vir}_{\geq -1})$.

Consider the one-dimensional trivial $\mathbf{U}(\mathrm{Vir}_{\geq -1})$-module with $\mathbf{v}_0$ a basis and define a Vir-module by

$$\mathbf{V}(c) = \mathbf{U}(\mathrm{Vir}, c) \otimes_{\mathbf{U}(\mathrm{Vir}_{\geq -1})} \mathbb{F}\mathbf{v}_0.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 7 Universal vacuum module

0	1	2	3	4	5	6
$\mathbf{v}_0$		$L_{-2}\mathbf{v}_0$	$L_{-3}\mathbf{v}_0$	$L_{-4}\mathbf{v}_0$	$L_{-5}\mathbf{v}_0$	$L_{-6}\mathbf{v}_0$
				$L_{-2}L_{-2}\mathbf{v}_0$	$L_{-2}L_{-3}\mathbf{v}_0$	$L_{-2}L_{-4}\mathbf{v}_0$
						$L_{-3}L_{-3}\mathbf{v}_0$
A basis of $\mathbf{V}(c)$						$L_{-2}L_{-2}L_{-2}\mathbf{v}_0$

Then $\mathbf{V}(c)$ is a highest weight Vir-module, which we will call the *universal vacuum module.* (See Table 7.)

The module $\mathbf{V}(c)$ is also described as a quotient of $\mathbf{M}(c,0)$ as

$$\mathbf{V}(c) = \mathbf{M}(c,0)/\mathbf{U}(\mathrm{Vir},c)L_{-1}\mathbf{v}_0,$$

where $\mathbf{U}(\mathrm{Vir},c)L_{-1}\mathbf{v}_0$ is the submodule generated by $L_{-1}\mathbf{v}_0$. Note that the vector $L_{-1}\mathbf{v}_0$ is a singular vector of weight 1 in $\mathbf{M}(c,0)$.

Proposition 3.5 *The universal vacuum module $\mathbf{V}(c)$ of central charge c over the Virasoro algebra carries a unique structure of a vertex algebra such that*

$$Y(L_{-2}\mathbf{v}_0,z) = T(z)$$

with vacuum $\mathbf{1} = \mathbf{v}_0$.

The vertex algebra $\mathbf{V}(c)$ thus obtained is called the *universal Virasoro vertex algebra* of central charge c. It is generated by the Virasoro vector $\omega = L_{-2}\mathbf{1}$. The quotients of $\mathbf{M}(c,h)$ are modules over $\mathbf{V}(c)$ for any $h \in \mathbb{F}$.

The simple quotient of $\mathbf{V}(c)$ is called the *simple Virasoro vertex algebra* and denoted $\mathbf{L}(c,0)$. For example, the Virasoro vector (3.5) generates a vertex subalgebra in the Heisenberg vertex algebra of rank one, which is isomorphic to $\mathbf{L}(1,0)$ over $\mathbb{C}$.

Theory of simple Virasoro vertex algebras $\mathbf{L}(c,0)$ heavily relies on representation theory of the Virasoro algebra, for its structure and properties seriously change by presence of singular vectors in $\mathbf{V}(c)$ for special values of c.

1.3.3.4 Virasoro Minimal Models

In this subsection, we will work over the field $\mathbb{C}$ of complex numbers.

Let p,q be a pair of coprime positive integers and consider the rational number $c_{p,q}$ defined by

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

For positive integers r, s, consider the following numbers:

$$h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}.$$

We have the following list of simple Virasoro modules of central charge $c_{p,q}$:

$$\mathbf{L}(c_{p,q}, h_{r,s}) \ \ (1 \le r \le q - 1, \ 1 \le s \le p - 1), \tag{3.8}$$

which constitute *Virasoro minimal models* in physics. Among them, $\mathbf{L}(c_{p,q}, 0)$ is a simple vertex algebra and the rest as well as itself are the simple modules over it. By $h_{r,s} = h_{q-r,p-s}$, we have $(p - 1)(q - 1)/2$ simple modules.

When $(p, q) = (m + 3, m + 2)$ for some $m = 1, 2, \cdots$, set

$$c_m = c_{m+3,m+2} = 1 - \frac{6}{(m + 2)(m + 3)}.$$

The corresponding Virasoro modules in (3.8) are precisely the unitarizable ones among the modules in the minimal models.

Here are some examples.

1. For $m = 1$, we have $(p, q) = (4, 3)$ and $c = 1/2$. The simple vertex algebra $\mathbf{L}(1/2, 0)$ is related to the *Ising model* in physics. The list of simple modules is as follows:
$$\mathbf{L}(1/2, 0), \ \ \mathbf{L}(1/2, 1/2), \ \ \mathbf{L}(1/2, 1/16).$$
2. For $m = 2$, we have $(p, q) = (5, 4)$ and $c_{p,q} = 7/10$. The simple vertex algebra $\mathbf{L}(7/10, 0)$ is related to the *tricritical Ising model* in physics. There are six simple modules.
3. For $m = 3$, we have $(p, q) = (6, 5)$ and $c_{p,q} = 4/5$. The simple vertex algebra $\mathbf{L}(4/5, 0)$ is related to the *three-state Potts model* in physics. There are ten simple modules.

The corresponding lists of highest weights $h_{r,s}$ are given in Table 8, respectively. Such a table is called the *Kac table* or the *conformal grid.*

Here is an example of a nonunitary minimal model.

For $(p, q) = (5, 2)$, the simple vertex algebra $\mathbf{L}(-22/5, 0)$ is related to the *Lee–Yang model* in physics. The list of simple modules is as follows:

$$\mathbf{L}(-22/5, 0), \ \ \mathbf{L}(-22/5, -1/5).$$

The last model is interesting in its relation to the Rogers–Ramanujan identities.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 8 Virasoro minimal models

3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$
s/r	1	2

$c = 1/2$

4	$\frac{3}{2}$	$\frac{7}{16}$	0
3	$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
2	$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
1	0	$\frac{7}{16}$	$\frac{3}{2}$
s/r	1	2	3

$c = 7/10$

5	3	$\frac{7}{5}$	$\frac{2}{5}$	0
4	$\frac{13}{8}$	$\frac{21}{40}$	$\frac{1}{40}$	$\frac{1}{8}$
3	$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{3}$
2	$\frac{1}{8}$	$\frac{1}{40}$	$\frac{21}{40}$	$\frac{13}{8}$
1	0	$\frac{2}{5}$	$\frac{7}{5}$	3
s/r	1	2	3	4

$c = 4/5$

1.3.3.5 Ising Model and Majorana Fermions

Let us briefly describe an alternative construction of the simple Virasoro vertex algebra $\mathbf{L}(1/2, 0)$ related to the Ising model over $\mathbb{C}$.

Recall that the simple Virasoro vertex algebra $\mathbf{L}(1, 0)$ of central charge 1 is realized by the standard Virasoro vector in the Heisenberg vertex algebra $\mathbf{F}_0 = \mathbb{C}[x_1, x_2, \cdots]$, which is a mathematical formulation of free boson theory in physics.

In contrast, the simple Virasoro vertex algebra $\mathbf{L}(1/2, 0)$ of central charge $1/2$, is realized by replacing the polynomial ring $\mathbb{C}[x_1, x_2, \cdots]$ by the exterior algebra $\Lambda(x_1, x_2, \cdots)$, resulting in the theory of *free Majorana fermions*, where the structure is described not by a vertex algebra but a vertex *superalgebra,* which is, roughly speaking, obtained by replacing the commutator by the anti-commutator when the operators are both from the odd subspace.

Let us consider the associative algebra generated by

$$\cdots, \psi_{-3/2}, \psi_{-1/2}, \psi_{1/2}, \cdots$$

subject to the fundamental relations

$$\psi_{m+1/2}\psi_{n+1/2} + \psi_{n+1/2}\psi_{m+1/2} = \delta_{m+n+1,0}.$$

Let us denote it by $\mathbf{A}^{\psi}$, which is the counterpart of $\mathbf{U}(\hat{\mathfrak{h}}, 1)$, and consider the generating series

$$\psi(z) = \sum_n \psi_{n+1/2} z^{-n-1}.$$

Then, we have $\psi(y)\psi(z) + \psi(z)\psi(y) = \delta(y, z)$ and, by considering local anti-commutativity instead of local commutativity, we have

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\psi(y)\psi(z) \simeq -\psi(z)\psi(y) \sim \frac{1}{y-z}.$$

In particular, $\psi(z)$ is locally anticommutative with itself.

Let $\mathbf{A}^{\psi}_{>0}$ be the subalgebra generated by $\psi_{n+1/2}$ with $n \geq 0$ and consider the one-dimensional $\mathbf{A}^{\psi}_{>0}$-module $\mathbb{C}\mathbf{v}_0$ characterized by

$$\psi_{n+1/2}\mathbf{v}_0 = 0, \;\; n \geq 0.$$

Define the fermionic Fock space $\mathbf{F}_\psi$ by setting

$$\mathbf{F}_\psi = \mathbf{A}^\psi \otimes_{\mathbf{A}^\psi_{>0}} \mathbb{C}\mathbf{v}_0 \simeq \Lambda(x_1, x_2, \cdots).$$

The space $\mathbf{F}_\psi$ carries a unique structure of a vertex superalgebra such that $Y(\psi_{-1/2}\mathbf{v}_0, z) = \psi(z)$ with vacuum $\mathbf{1} = \mathbf{v}_0$.

By analogy with the construction of the standard Virasoro vector (3.5) for the Heisenberg vertex algebra, consider the vector

$$\omega = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}\mathbf{v}_0.$$

Then it generates a vertex subalgebra isomorphic to $\mathbf{L}(1/2, 0)$, and

$$\mathbf{F}_\psi \simeq \underbrace{\mathbf{L}(1/2,0)}_{\mathbf{F}^+_\psi} \oplus \underbrace{\mathbf{L}(1/2,1/2)}_{\mathbf{F}^-_\psi},$$

where $\mathbf{F}^{\pm}_\psi$ are the eigenspaces of the involution θ induced by the action $\psi \mapsto -\psi$ on the generator $\psi = \psi_{-1/2}\mathbf{v}_0$. We can then readily read off their graded dimensions as

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{F}^{\pm}_{\psi,d} = \frac{1}{2}\Big(\prod_{k=0}^{\infty}(1+q^{k+1/2}) \pm \prod_{k=0}^{\infty}(1-q^{k+1/2})\Big),$$

where the degree is given by setting $\deg \psi_{-n-1/2} = n + 1/2$.

Note 3.6. 1. The fermionic construction as described is useful in constructing VOAs associated with binary even codes introduced by Miyamoto [84] (cf. [81]) and consequently in studying framed VOAs including the moonshine module (cf. [43]). See Subsection 1.4.4.3 for a related construction. 2. The module $\mathbf{L}(1/2, 1/16)$ can be constructed by considering the *twisted module* over vertex superalgebra $\mathbf{F}_\psi$. See Section 1.5 for the concept and examples of twisted modules over *vertex algebras.*

Bibliographic Notes

General references are Kac [6], Frenkel and Ben-Zvi [8], and Lepowsky and Li [10], for Section 1.3. For descriptions of algebras appearing in various models in physics, consult Di Francesco et al. [13] (cf. Ginsparg [19]).

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Construction of affine and Virasoro vertex algebras are due to Frenkel and Zhu [60]. Our exposition follows Li [70]. See also Primc [88] for a related work.

See Kac [17] and Lepowsky and Li [10] (cf. Tsuchiya, Ueno, and Yamada [97]) for generalities on affine Kac–Moody algebras, and Kac, Raina, and Rozhkovskaya [18] and Iohara and Koga [15] for the Virasoro algebra.

For the construction of the Ising model $\mathbf{L}(1/2, 0)$ by Majorana fermions, see [54], [13], and [18].

1.4 Lattice Vertex Algebras

From here on, we will work over the field $\mathbb{C}$ of complex numbers, although most of the results hold over a field of characteristic zero.

Recall that a (nondegenerate integral) *lattice* is a free $\mathbb{Z}$-module L of finite rank equipped with a nondegenerate symmetric bilinear form valued in $\mathbb{Z}$:

$$(\mid) : L \times L \longrightarrow \mathbb{Z}.$$

Theory of lattices is important in many areas of mathematics.

For a lattice L, consider the Heisenberg Lie algebra $\hat{\mathfrak{h}}$; that is, the affine Lie algebra associated with the commutative Lie algebra $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and the bilinear form extending that of the lattice, and the direct sum of the Fock modules of charge belonging to the lattice:

$$\mathbf{V}_L = \bigoplus_{\lambda \in L} \mathbf{F}_\lambda.$$

Then the vertex algebra structure on the Heisenberg vertex algebra $\mathbf{F}_0$ can be extended to $\mathbf{V}_L$ by using the *vertex operators* in a natural but subtle way.

In Section 1.4, we will describe properties of vertex operators and the way how to construct a vertex algebra structure on $\mathbf{V}_L$.

Note that the Fock modules are written as $\mathbf{F}_\lambda = \mathbf{S}(\hat{\mathfrak{h}}_{<0})\mathbf{v}_\lambda$, where $\mathbf{S}(\hat{\mathfrak{h}}_{<0})$ denotes the symmetric algebra over $\hat{\mathfrak{h}}_{<0}$. Thus the lattice vertex algebra is also written as

$$\mathbf{V}_L = \mathbf{S}(\hat{\mathfrak{h}}_{<0}) \otimes \mathbb{C}[L]$$

by identifying the vector $\mathbf{v}_\lambda$ with the basis vector e^λ of the group algebra.

1.4.1 Series with Homomorphism Coefficients

In the previous sections, we have considered series with operator coefficients, where the operators are endomorphisms of a vector space. In this section,

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

we will consider slightly more general types of series, whose coefficients are homomorphisms (linear maps) between vector spaces.

1.4.1.1 Local Truncation and Residue Products

Let $\mathbf{M}$ and $\mathbf{N}$ be vector spaces and let $\Phi(z)$ be a series with coefficients in $\mathrm{Hom}(\mathbf{M}, \mathbf{N})$:

$$\Phi(z) = \sum_n \Phi_n z^{-n-1}, \quad \Phi_n : \mathbf{M} \longrightarrow \mathbf{N}.$$

Let $A(z)$ be a series on $\mathbf{M}$ and $\mathbf{N}$, that is, a series with coefficients in the set

$$\mathrm{Hom}((\mathbf{M}, \mathbf{N}), (\mathbf{M}, \mathbf{N})) = \mathrm{Hom}(\mathbf{M}, \mathbf{M}) \oplus \mathrm{Hom}(\mathbf{N}, \mathbf{N}).$$

Denote the actions of the coefficients on $\mathbf{M}$ and $\mathbf{N}$ by the same symbol:

$$A(z) = \sum_n A_n z^{-n-1}, \quad A_n : \mathbf{M} \longrightarrow \mathbf{M}, \ \mathbf{N} \longrightarrow \mathbf{N}.$$

We may then consider the compositions of the coefficients as in the diagram:

$$\begin{tikzcd}
\mathbf{M} \arrow[r, "\Phi_n"] \arrow[d, "A_m"'] & \mathbf{N} \arrow[d, "A_m"] \\
\mathbf{M} \arrow[r, "\Phi_n"'] & \mathbf{N}.
\end{tikzcd}$$

We say that $\Phi(z)$ is locally truncated if it belongs to $\mathrm{Hom}(\mathbf{M}, \mathbf{N}((z)))$. The concept of local truncation for $A(z)$ is defined in an obvious way, and denote the set of such series as

$$\mathrm{Hom}\left(((\mathbf{M}, \mathbf{N}), ((\mathbf{M}, \mathbf{N})((z)))\right) = \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))) \oplus \mathrm{Hom}(\mathbf{N}, \mathbf{N}((z))).$$

If $A(z)$ and $\Phi(z)$ are locally truncated, then the residue products make sense for all $m \in \mathbb{Z}$:

$$\begin{aligned} A(z)_{(m)}\Phi(z) = {} & \mathrm{Res}_y (y-z)^m\big|_{|y|>|z|} A(y)\Phi(z) \\ & - \mathrm{Res}_y (y-z)^m\big|_{|y|>|z|} \Phi(z)A(y). \end{aligned}$$

Explicitly, the coefficients of $A(z)_{(m)}\Phi(z) = \sum_n (A_{(m)}\Phi)_n z^{-n-1}$ are given by

$$(A_{(m)}\Phi)_n = \sum_{i=0}^{\infty} (-1)^m \binom{m}{i} A_{m-i}\Phi_{n+i} - \sum_{i=0}^{\infty} (-1)^{m-i} \binom{m}{i} \Phi_{m+n-i} A_i,$$

as in the case of series acting on a vector space.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.4.1.2 Local Commutativity and Borcherds Identity for Series

The series $A(z)$ and $\Phi(z)$ are *locally commutative* if the following holds for some $N \in \mathbb{N}$:

$$(y-z)^N A(y)\Phi(z) = (y-z)^N \Phi(z)A(y).$$

For such series, we have the OPE

$$A(y)\Phi(z) \simeq \Phi(z)A(y) \sim \sum_{k=0}^{N-1} \frac{\Psi_k(z)}{(y-z)^{k+1}},$$

by which the residue products are found as

$$A(z)_{(m)}\Phi(z) = \begin{cases} 0 & (N \leq m), \\ \Psi_m(z) & (0 \leq m < N). \end{cases}$$

Let $A(z), B(z)$ be locally truncated series acting on $\mathbf{M}$ and $\mathbf{N}$ and $\Phi(z)$ a locally truncated series with coefficients in $\mathrm{Hom}(\mathbf{M}, \mathbf{N})$:

$$A(z), B(z) \in \mathrm{Hom}((\mathbf{M}, \mathbf{N}), (\mathbf{M}, \mathbf{N})((z))), \quad \Phi(z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{N}((z))).$$

If they are locally commutative with each other, then the Borcherds identity

$$\begin{aligned} &\sum_{i=0}^{\infty} \binom{p}{i} (A(z)_{(r+i)}B(z))_{(p+q-i)}\Phi(z) \\ &\quad = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} A(z)_{(p+r-i)}(B(z)_{(q+i)}\Phi(z)) \\ &\qquad - \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} B(z)_{(q+r-i)}(A(z)_{(p+i)}\Phi(z)) \end{aligned}$$

holds for all $p, q, r \in \mathbb{Z}$ with respect to the residue products.

1.4.1.3 OPEs in General Settings

Let $\mathbf{L}, \mathbf{M}_1, \mathbf{M}_2, \mathbf{N}$ be vector spaces and consider locally truncated series

$$\Psi(z) \in \mathrm{Hom}((\mathbf{L}, \mathbf{M}_1), (\mathbf{M}_2, \mathbf{N})((z))), \quad \Phi(z) \in \mathrm{Hom}((\mathbf{L}, \mathbf{M}_2), (\mathbf{M}_1, \mathbf{N})((z)))$$

and their compositions

$$\Phi(y)\Psi(z) \in \mathrm{Hom}(\mathbf{L}, \mathbf{N}((y))((z))), \quad \Psi(z)\Phi(y) \in \mathrm{Hom}(\mathbf{L}, \mathbf{N}((z))((y))),$$

where

$$\begin{tikzcd} & \mathbf{M}_1 \arrow[dr, "\Psi_m"] & \\ \mathbf{L} \arrow[ur, "\Phi_n"] \arrow[dr, "\Psi_m"'] & & \mathbf{N}. \\ & \mathbf{M}_2 \arrow[ur, "\Phi_n"'] & \end{tikzcd}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Consider the residue products

$$\Phi(z)_{(m)}\Psi(z) = \mathrm{Res}_y (y-z)^m\big|_{|y|>|z|}\Phi(y)\Psi(z) \\ - \mathrm{Res}_y (y-z)^m\big|_{|y|>|z|}\Psi(z)\Phi(y).$$

We will say that $\Phi(z)$ and $\Psi(z)$ are locally commutative if, for some $N \in \mathbb{N}$,

$$(y-z)^N\Phi(y)\Psi(z) = (y-z)^N\Psi(z)\Phi(y).$$

Then, for some series $\Gamma_0(z), \ldots, \Gamma_{N-1}(z) \in \mathrm{Hom}(\mathbf{L},\mathbf{N})((z))$, the OPE

$$\Phi(y)\Psi(z) \simeq \Psi(z)\Phi(y) \sim \sum_{k=0}^{N-1} \frac{\Gamma_k(z)}{(y-z)^{k+1}}$$

holds in the obvious sense, and the residue products are given by

$$\Phi(z)_{(m)}\Psi(z) = \begin{cases} 0 & (m \geq N), \\ \Gamma_m(z) & (0 \leq m < N) \end{cases}$$

for $m \in \mathbb{N}$.

1.4.1.4 Formal Taylor Expansion

For series $\Phi(z)$ and $\Psi(z)$ as in Subsection 1.4.1.3, assume that the composite $\Phi(y)$
$\Psi(z)$ is written in the following form with some $m_0 \in \mathbb{Z}$:

$$\Phi(y)\Psi(z) = (y-z)^{-m_0-1}\big|_{|y|>|z|}\Gamma(y,z), \quad \Gamma(y,z) \in \mathrm{Hom}(\mathbf{L},\mathbf{N}((y,z))).$$

Then Taylor expansion of $\Gamma(y,z)$ yields

$$\Gamma(x+z,z)\big|_{|x|<|z|} = \sum_{i=0}^{\infty} x^i \partial_y^{(i)}\Gamma(y,z)\big|_{y=z}.$$

Therefore, by substitution $x = y - z$,

$$\Phi(y)\Psi(z) = x^{-m_0-1}\big|_{|y|>|z|}\Gamma(y,z) \\ = \sum_{i=0}^{\infty} x^{-m_0+i-1}\big|_{|y|>|z|}\partial_y^{(i)}\Gamma(y,z)\big|_{y=z}.$$

Hence the residue products are determined as

$$\Phi(z)_{(m)}\Psi(z) = \begin{cases} 0 & (m > m_0), \\ \partial_y^{(i)}\Gamma(y,z)\big|_{y=z} & (m = m_0 - i \leq m_0). \end{cases}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.4.2 Vertex Operators

Let us now describe *vertex operators*, which are the main ingredients in constructing lattice vertex algebras.

A *vertex operator* is a series of the form

$$V_\lambda(z) = \exp\Big(-\sum_{n<0} \lambda_n \frac{z^{-n}}{n}\Big) \exp\Big(-\sum_{n>0} \lambda_n \frac{z^{-n}}{n}\Big) e^\lambda z^{\lambda_0},$$

where λ_n with $n \in \mathbb{Z}$ are actions of the Heisenberg algebra (of higher rank in general). Thus, contrary to its name, a vertex operator is not a single operator, but a series with operator coefficients.

1.4.2.1 Heisenberg Vertex Algebras of Higher Rank

Let $\mathfrak{h}$ be a finite-dimensional vector space, and let $(\ |\ )$ be a symmetric bilinear form on $\mathfrak{h}$, which we assume to be nondegenerate.

Regard $\mathfrak{h}$ as a commutative Lie algebra and consider the affine Lie algebra associated with $\mathfrak{h}$ and $(\ |\ )$:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K.$$

Define the *Heisenberg algebra* associated with $\mathfrak{h}$ and $(\ |\ )$ by

$$\mathbf{U}(\hat{\mathfrak{h}}, 1) = \mathbf{U}(\hat{\mathfrak{h}})/(K-1).$$

We will sometimes call a $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module a *Heisenberg module.*

Let $\hat{\mathfrak{h}}_{<0}$ and $\hat{\mathfrak{h}}_{\geq 0}$ be the commutative Lie subalgebra of $\hat{\mathfrak{h}}$ spanned by $\mathfrak{h} \otimes t^n$ with $n < 0$ and $n \geq 0$, respectively, for which

$$\mathbf{U}(\hat{\mathfrak{h}}, 1) = \mathbf{S}(\hat{\mathfrak{h}}_{<0}) \otimes \mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})$$

as a vector space.

Let $\lambda \in \mathfrak{h}^*$ be a linear form on $\mathfrak{h}$, where $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. Consider the one-dimensional $\mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})$-module $\mathbb{C}\mathbf{v}_\lambda$ given as follows for all $h \in \mathfrak{h}$ and $n \geq 0$:

$$h_n \mathbf{v}_\lambda = \begin{cases} 0 & (n \geq 1), \\ \lambda(h)\mathbf{v}_\lambda & (n = 0). \end{cases}$$

The *Fock module* of *charge* λ is the $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-module:

$$\mathbf{F}_\lambda = \mathbf{U}(\hat{\mathfrak{h}}, 1) \otimes_{\mathbf{S}(\hat{\mathfrak{h}}_{\geq 0})} \mathbb{C}\mathbf{v}_\lambda.$$

It is isomorphic to $\mathbf{S}(\hat{\mathfrak{h}}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}\mathbf{v}_\lambda$ as a vector space by PBW. Having this in mind, we often write

$$\mathbf{F}_\lambda = \mathbf{S}(\hat{\mathfrak{h}}_{<0})\mathbf{v}_\lambda.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The Fock module $\mathbf{F}_0$ carries a natural structure of a vertex algebra, and $\mathbf{F}_\lambda$ are simple modules over the vertex algebra $\mathbf{F}_0$ for all $\lambda \in \mathfrak{h}^*$.

The construction of the standard Virasoro vector (3.5) generalizes to higher rank cases by

$$\omega = \frac{1}{2} \sum_{i=1}^{d} a^i_{-1} a_{i,-1} \mathbf{v}_0,$$

where $(a^1, \dots, a^d)$ and $(a_1, \dots, a_d)$ are dual bases of $\mathfrak{h}$ with respect to the nondegenerate bilinear form $(-|-)$.

1.4.2.2 The Operators e^λ and z^{λ_0}

From here on, we identify the vector space $\mathfrak{h}$ with its dual $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ by the symmetric bilinear form $(\,|\,)$, which we have assumed to be nondegenerate, so that $\lambda(h) = (\lambda|h)$ for $\lambda, h \in \mathfrak{h}$.

For $\lambda \in \mathfrak{h}$, there exists a unique homomorphism of $\mathbf{S}(\hat{\mathfrak{h}}_{<0})$-modules sending $\mathbf{v}_\mu$ to $\mathbf{v}_{\lambda+\mu}$, which we denote by

$$e^\lambda : \mathbf{F}_\mu \longrightarrow \mathbf{F}_{\lambda+\mu}, \quad \mathbf{v}_\mu \mapsto \mathbf{v}_{\lambda+\mu}.$$

Next, for $\lambda, \mu \in \mathfrak{h}$ satisfying $(\lambda|\mu) \in \mathbb{Z}$, define

$$z^{\lambda_0} : \mathbf{F}_\mu \longrightarrow \mathbf{F}_\mu z^{(\lambda|\mu)}, \quad v \mapsto z^{\lambda_0} v = z^{(\lambda|\mu)} v.$$

Then, for $\lambda, h \in \mathfrak{h}$ and $n \in \mathbb{Z}$,

$$[h_n, e^\lambda] = (\lambda|h)\delta_{n,0} e^\lambda, \quad [h_n, z^{\lambda_0}] = 0.$$

The operators z^{λ_0} and e^μ do not commute, but satisfy

$$z^{\lambda_0} e^\mu = z^{(\lambda|\mu)} e^\mu z^{\lambda_0} \tag{4.1}$$

for $\lambda, \mu \in \mathfrak{h}$.

1.4.2.3 Vertex Operators

For $\lambda \in \mathfrak{h}$, consider the following expression:

$$V_\lambda(z) = \exp\left(\sum_{n<0} \lambda_n \frac{z^{-n}}{-n}\right) \exp\left(\sum_{n>0} \lambda_n \frac{z^{-n}}{-n}\right) e^\lambda z^{\lambda_0},$$

where the sums are over negative and positive integers, respectively, and the exponential of a series is defined as

$$\exp x(z) = \sum_{k=0}^{\infty} \frac{x(z)^k}{k!}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

To see the meaning and well-definedness of $V_\lambda(z)$, note the following structure:

$$\underbrace{\exp\left(\sum_{n<0}\lambda_n\frac{z^{-n}}{-n}\right)}_{\substack{\text{positive powers}\\ \text{coefficients in } \hat{\mathfrak{h}}_{<0}}}\ \underbrace{\exp\left(\sum_{n>0}\lambda_n\frac{z^{-n}}{-n}\right)}_{\substack{\text{negative powers}\\ \text{coefficients in } \hat{\mathfrak{h}}_{>0}}}\ \underbrace{e^{\lambda}}_{\substack{\text{shifting}\\ \text{charge}}}\ \underbrace{z^{\lambda_0}}_{\substack{\text{shifting}\\ \text{exponent}}}.$$

Let $\mu \in \mathfrak{h}$ satisfy $(\lambda|\mu) \in \mathbb{Z}$. Then, for any element $P \in \mathbf{S}(\hat{\mathfrak{h}}_{<0})$,

$$z^{\lambda_0}P\mathbf{v}_\mu = P\mathbf{v}_\mu z^{(\lambda|\mu)},\ e^{\lambda}z^{\lambda_0}P\mathbf{v}_\mu = P\mathbf{v}_{\lambda+\mu}z^{(\lambda|\mu)}.$$

Since if $n_1 + \cdots + n_k$ is sufficiently large, then $\lambda_{n_1}\cdots\lambda_{n_k}P\mathbf{v}_{\lambda+\mu} = 0$, we have

$$\underbrace{\exp\left(\sum_{n>0}\lambda_n\frac{z^{-n}}{-n}\right)}_{\substack{\text{negative powers}\\ \text{coefficients in } \hat{\mathfrak{h}}_{>0}}} P\mathbf{v}_{\lambda+\mu}z^{(\lambda|\mu)} \in \mathbf{F}_{\lambda+\mu}[z^{-1}]z^{(\lambda|\mu)},$$

thus

$$\underbrace{\exp\left(\sum_{n<0}\lambda_n\frac{z^{-n}}{-n}\right)}_{\text{nonnegative powers}}\ \underbrace{\exp\left(\sum_{n>0}\lambda_n\frac{z^{-n}}{-n}\right)P\mathbf{v}_{\lambda+\mu}z^{(\lambda|\mu)}}_{\substack{\text{finitely many terms with}\\ \text{negative powers}}} \in \mathbf{F}_{\lambda+\mu}((z))z^{(\lambda|\mu)}.$$

Therefore, for $\mu \in \mathfrak{h}^*$ with $(\lambda|\mu) \in \mathbb{Z}$, the expression $V_\lambda(z)$ gives rise to a locally truncated series with coefficients being maps from $\mathbf{F}_\mu$ to $\mathbf{F}_{\lambda+\mu}$:

$$(\lambda|\mu) \in \mathbb{Z} \implies V_\lambda(z) \in \mathrm{Hom}(\mathbf{F}_\mu, \mathbf{F}_{\lambda+\mu}((z))).$$

The series $V_\lambda(z)$ thus constructed is called the *vertex operator.*

Note 4.1. 1. Following the physics literatures, formally write

$$\phi_\lambda(z) = \phi_\lambda(z)_{<0} + \phi_\lambda(z)_{>0} + \lambda_0 \log z + \lambda,$$

where

$$\phi_\lambda(z)_{<0} = \sum_{n<0}\lambda_n\frac{z^{-n}}{-n},\ \ \phi_\lambda(z)_{>0} = \sum_{n>0}\lambda_n\frac{z^{-n}}{-n}.$$

Then we have $\partial\phi_\lambda(z) = \lambda(z)$ so that the expression $\phi_\lambda(z)$ is thought of as the "indefinite integral" of the series $\lambda(z)$. 2. The vertex operator $V_\lambda(z)$ as defined here is thought of as a regularization of the divergent expression $e^{\phi_\lambda(z)}$ by "normal ordering" and often denoted as

$$:e^{\phi_\lambda(z)}: = \exp\left(\phi_\lambda(z)_{<0}\right)\exp\left(\phi_\lambda(z)_{>0}\right)e^{\lambda}z^{\lambda_0}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.4.2.4 Commutation with Currents

Let X, Y be operators or series on a vector space such that $\exp Y = \sum_{k=0}^{\infty} Y^k/k!$ makes sense in an appropriate way. Then

$$[X,Y] \text{ commutes with } Y \implies [X, \exp Y] = [X,Y] \exp Y,$$

where the bracket refers to the commutator.

By using this, we have

$$[h(y), V_\lambda(z)] = \lambda(h) V_\lambda(z) \delta(y,z).$$

In particular, the current $h(z)$ and the vertex operator $V_\lambda(z)$ are locally commutative and their OPE is given by

$$h(y)V_\lambda(z) \simeq V_\lambda(z)h(y) \sim \frac{\lambda(h)}{y-z} V_\lambda(z).$$

We thus have

$$h(z)_{(n)} V_\lambda(z) = \begin{cases} 0 & (n \geq 1), \\ \lambda(h) V_\lambda(z) & (n = 0). \end{cases}$$

For each $\lambda \in \mathfrak{h}^*$, repeatedly apply the residue products by the currents to the vertex operator $V_\lambda(z)$, and let $\mathcal{F}_\lambda$ denote the span of such series:

$$\mathcal{F}_\lambda = \operatorname{Span} \left\{ h^1(z)_{(n_1)} \cdots h^k(z)_{(n_k)} V_\lambda(z) \,\middle|\, \begin{matrix} k \in \mathbb{N}, h^1, \ldots, h^k \in \mathfrak{h} \\ n_1, \ldots, n_k \in \mathbb{Z} \end{matrix} \right\}. \tag{4.2}$$

Then it becomes an $\mathbf{F}_0$-module by the residue products. By the OPE, it is isomorphic to the Fock module $\mathbf{F}_\lambda$ of charge λ as a Heisenberg module.

1.4.3 Residue Products of Vertex Operators

Assume that $\lambda, \mu, \nu \in L$ satisfy $(\lambda|\mu), (\mu|\nu), (\lambda|\nu) \in \mathbb{Z}$ and consider the vertex operators

$$\begin{aligned} V_\lambda(z) &\in \operatorname{Hom}((\mathbf{F}_\nu, \mathbf{F}_{\mu+\nu}), (\mathbf{F}_{\lambda+\nu}, \mathbf{F}_{\lambda+\mu+\nu})((z))), \\ V_\mu(z) &\in \operatorname{Hom}((\mathbf{F}_\nu, \mathbf{F}_{\lambda+\nu}), (\mathbf{F}_{\mu+\nu}, \mathbf{F}_{\lambda+\mu+\nu})((z))). \end{aligned}$$

Their coefficients fit in

$$\begin{tikzcd} & \mathbf{F}_{\mu+\nu} \arrow[dr, "V_{\lambda,m}"] & \\ \mathbf{F}_\nu \arrow[ur, "V_{\mu,n}"] \arrow[dr, "V_{\lambda,m}"'] & & \mathbf{F}_{\lambda+\mu+\nu} \\ & \mathbf{F}_{\lambda+\nu}. \arrow[ur, "V_{\mu,n}"'] & \end{tikzcd}$$

We are interested in commutation of $V_\lambda(y)$ and $V_\mu(z)$ for $\lambda, \mu \in L$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.4.3.1 Commutation of Vertex Operators

Let X, Y be operators or series on a vector space such that $\exp X$, $\exp Y$, and $\exp[X,Y]$ make sense in an appropriate way. Then

$$[X,Y] \text{ commutes with } X \text{ and } Y \Longrightarrow \exp X \exp Y = \exp[X,Y] \exp Y \exp X.$$

Apply this to the following partial product of $V_\lambda(z)V_\mu(y)$:

$$\exp\Big(\sum_{n<0} \lambda_n \frac{y^{-n}}{-n}\Big) \exp\underbrace{\Big(\sum_{n>0} \lambda_n \frac{y^{-n}}{-n}\Big)}_{X} \exp\underbrace{\Big(\sum_{n<0} \mu_n \frac{z^{-n}}{-n}\Big)}_{Y} \exp\Big(\sum_{n>0} \mu_n \frac{z^{-n}}{-n}\Big).$$

Then, since

$$\begin{aligned}\Big[\underbrace{\sum_{n>0} \lambda_n \frac{y^{-n}}{-n}}_{X}, \underbrace{\sum_{n<0} \mu_n \frac{z^{-n}}{-n}}_{Y}\Big] &= \sum_{m>0}\sum_{n<0} [\lambda_m, \mu_n] \frac{y^{-m}z^{-n}}{mn} \\ &= (\lambda|\mu) \sum_{m>0} \frac{y^{-m}z^m}{-m} = (\lambda|\mu) \log\Big(1 - \frac{z}{y}\Big),\end{aligned}$$

we have

$$\begin{aligned}&\overbrace{\exp\Big(\sum_{n>0} \lambda_n \frac{y^{-n}}{-n}\Big)}^{\exp X} \overbrace{\exp\Big(\sum_{n<0} \mu_n \frac{z^{-n}}{-n}\Big)}^{\exp Y} \\ &\quad = \underbrace{\Big(1 - \frac{z}{y}\Big)^{(\lambda|\mu)}\Big|}_{\exp[X,Y]}{}_{|y|>|z|} \underbrace{\exp\Big(\sum_{n<0} \mu_n \frac{z^{-n}}{-n}\Big)}_{\exp Y} \underbrace{\exp\Big(\sum_{n>0} \lambda_n \frac{y^{-n}}{-n}\Big)}_{\exp X}.\end{aligned}$$

On the other hand, by (4.1),

$$e^\lambda \underline{y^{\lambda_0} e^\mu} z^{\mu_0} = y^{(\lambda|\mu)} e^\lambda \underline{e^\mu y^{\lambda_0}} z^{\mu_0} = y^{(\lambda|\mu)} e^{\lambda+\mu} y^{\lambda_0} z^{\mu_0}.$$

Combining them together, we arrive at

$$\begin{aligned} V_\lambda(y)V_\mu(z) &= (y-z)^{(\lambda|\mu)}\big|_{|y|>|z|} V_{\lambda,\mu}(y,z), \\ V_\mu(z)V_\lambda(y) &= (z-y)^{(\mu|\lambda)}\big|_{|y|<|z|} V_{\lambda,\mu}(y,z), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} V_{\lambda,\mu}(y,z) = \exp&\Big(\sum_{n<0} \frac{\lambda_n y^{-n} + \mu_n z^{-n}}{-n}\Big) \\ &\exp\Big(\sum_{n>0} \frac{\lambda_n y^{-n} + \mu_n z^{-n}}{-n}\Big) e^{\lambda+\mu} y^{\lambda_0} z^{\mu_0}. \end{aligned}$$

Therefore, if $(\lambda|\mu), (\mu|\nu), (\lambda|\nu) \in \mathbb{Z}$, then we have the following equalities as series with coefficients in $\mathrm{Hom}(\mathbf{F}_\nu, \mathbf{F}_{\lambda+\mu+\nu})$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1. If $(\lambda|\mu) \geq 0$, then
$$V_\lambda(y)V_\mu(z) = (-1)^{(\lambda|\mu)}V_\mu(z)V_\lambda(y).$$
2. If $(\lambda|\mu) < 0$, then, for $N = -(\lambda|\mu) \geq 0$,
$$(y-z)^N V_\lambda(y)V_\mu(z) = (-1)^{(\lambda|\mu)}(y-z)^N V_\mu(z)V_\lambda(y).$$

In particular, if $(\lambda|\mu) \in 2\mathbb{Z}$, then $V_\lambda(z)$ and $V_\mu(z)$ are locally commutative.

1.4.3.2 OPE of Vertex Operators

Consider the case with $(\lambda|\mu) \in \mathbb{Z}$. Then, by the last result, we have
$$V_\lambda(y)V_\mu(z) = (y-z)^{(\lambda|\mu)}\big|_{|y|>|z|}V_{\lambda,\mu}(y,z).$$

Applying Taylor expansion to $V_{\lambda,\mu}(y,z)$ in the first equality of (4.3), we have
$$V_{\lambda,\mu}(x+z,z) = e^\lambda \sum_{i=0}^{\infty} x^i \partial_y^{(i)} \Bigg(\exp\Big(\sum_{n<0} \frac{\lambda_n y^{-n}}{-n}\Big) V_\mu(z) \\ \exp\Big(\sum_{n>0} \frac{\lambda_n y^{-n}}{-n}\Big) y^{\lambda_0} \Bigg)\Bigg|_{y=z}.$$

The result can be written in a compact form as
$$V_{\lambda,\mu}(x+z,z)\big|_{|x|<|z|} = \exp\Big(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}\lambda(z)_{(-k-1)}\Big) V_{\lambda+\mu}(z).$$

When $(\lambda|\mu) \in 2\mathbb{Z}$, the residue products make sense and read
$$V_\lambda(z)_{(n)}V_\mu(z) = \begin{cases} 0 & (n \geq -(\lambda|\mu)), \\ V_{\lambda+\mu}(z) & (n = -(\lambda|\mu)-1), \\ {}^{\circ}_{\circ}\lambda(z)V_{\lambda+\mu}(z){}^{\circ}_{\circ} & (n = -(\lambda|\mu)-2), \\ \qquad \cdots\cdots\cdots. & \end{cases}$$

In particular, $V_\lambda(z)_{(n)}V_\mu(z)$ belongs to $\mathcal{F}_{\lambda+\mu}$ for all $n \in \mathbb{Z}$.

1.4.4 Lattice Vertex Algebras for Rank One Even Lattices

Let L be an even lattice and set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Consider the direct sum of vector spaces given by
$$\mathbf{V}_L = \bigoplus_{\lambda \in L} \mathbf{F}_\lambda,$$

where $\mathbf{F}_\lambda = \mathbf{S}(\hat{\mathfrak{h}}_{<0})\mathbf{v}_\lambda$ is the Fock module of charge λ.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

If L is an even lattice of rank one, then the bilinear form takes values in $2\mathbb{Z}$. In such a case, the vertex operators $V_\lambda(z)$ with $\lambda \in L$ are locally commutative, and general theory in Section 1.2 is available.

In this section, we will describe the vertex algebra structure on $\mathbf{V}_L$.

1.4.4.1 Lattice Vertex Algebras of Rank One

Let L be an even lattice of rank one. Recall the currents $h(z)$ with $h \in \mathfrak{h}$ and the vertex operators $V_\lambda(z)$ with $\lambda \in L$, and regard them as series acting on $\mathbf{V}_L$. Since L is a lattice, the series $V_\lambda(z)$ has integral exponents by $z^{\lambda_0}\mathbf{v}_\mu = z^{(\lambda|\mu)}\mathbf{v}_\mu$ for $\lambda, \mu \in L$. Thus:

$$h(z), V_\lambda(z) \in \mathrm{Hom}(\mathbf{V}_L, \mathbf{V}_L((z))).$$

As we have already seen, the currents and the vertex operators are locally truncated and locally commutative with themselves.

Consider the vertex algebra of series generated by the currents and vertex operators:

$$\mathcal{V}_L = \langle h(z), V_\lambda(z) | h \in \mathfrak{h}, \lambda \in L\rangle_{\mathrm{RP}}.$$

Since the vertex operators, as well as the currents, are creative with respect to the vacuum vector $\mathbf{v}_0$, the state map σ restricts to a map

$$\sigma_{\mathcal{V}_L} : \mathcal{V}_L \longrightarrow \mathbf{V}_L.$$

The OPEs of vertex operators show

$$\mathcal{V}_L = \bigoplus_{\lambda\in L} \mathcal{F}_\lambda,$$

where $\mathcal{F}_\lambda$ is defined by (4.2) with the operators replaced by those acting on $\mathbf{V}_L$, which is isomorphic to the Fock module $\mathbf{F}_\lambda$ via the state map $\sigma_{\mathcal{V}_L}$. Therefore, we have the following result.

Proposition 4.2 *Let L be an even lattice of rank one. Then the vector space $\mathbf{V}_L$ carries a unique structure of a vertex algebra with vacuum $\mathbf{1} = \mathbf{v}_0$ such that*

$$Y(h_{-1}\mathbf{v}_0, z) = h(z) \ \textit{and}\ Y(\mathbf{v}_\lambda, z) = V_\lambda(z)$$

for all $h \in \mathfrak{h}$ and $\lambda \in L$.

The vertex algebra $\mathbf{V}_L$ thus obtained is called the *lattice vertex algebra* associated with L. It is not so difficult to show that it is a simple vertex algebra.

Consider the group algebra $\mathbb{C}[L]$ of the lattice L spanned by e^λ, $\lambda \in L$. We often identify the vector $\mathbf{v}_\lambda$ with $1 \otimes e^\lambda$ or e^λ as in the following diagram:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\begin{array}{ccccccc} \mathbf{V}_L & = & \bigoplus_{\lambda\in L}\mathbf{F}_\lambda & = & \mathbf{S}(\hat{\mathfrak{h}}_{<0})\otimes\mathbb{C}[L] & \longleftarrow & \mathbb{C}[L] \\ & & \mathbf{v}_\lambda & \longleftrightarrow & 1\otimes e^\lambda & \longleftrightarrow & e^\lambda. \end{array}$$

Then the operator e^λ described in Subsection 1.4.2.2 is regarded as the multiplication by e^λ in $\mathbb{C}[L]$ and the conditions in Proposition 4.2 become

$$Y(h_{-1}\otimes e^0, z) = h(z) \ \text{ and } \ Y(1\otimes e^\lambda, z) = V_\lambda(z),$$

where e^0 is the unity of $\mathbb{C}[L]$.

The space $\mathbf{V}_L$ is given a grading by

$$\deg\, h_{-i_1}\cdots h_{-i_k}\mathbf{v}_\lambda = i_1+\cdots+i_k+\frac{(\lambda|\lambda)}{2},$$

which agrees with the eigenvalue for the action of L_0 with respect to the Virasoro vector ω of the Heisenberg vertex subalgebra $\mathbf{F}_0$.

If L is positive-definite, then the degree takes values in nonnegative integers and the subspace of degree 0 is spanned by $\mathbf{v}_0$, and the graded dimension is given by

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{V}_{L,d} = \sum_{\lambda\in L}\prod_{k=1}^{\infty}\frac{q^{(\lambda|\lambda)/2}}{1-q^k} = \frac{\Theta_L(\tau)}{\phi(q)} = q^{1/24}\frac{\Theta_L(\tau)}{\eta(\tau)},$$

where $\Theta_L(\tau)$ is the theta constant associated with the lattice L.

1.4.4.2 The Lattice of Type A_1

Let us take L to be the root lattice $\sqrt{2}\mathbb{Z}$ of type A_1, a unique lattice generated by a root α; that is, an element of squared norm 2:

$$A_1 = \mathbb{Z}\alpha, \ \ (\alpha|\alpha) = 2.$$

The structure of $\mathbf{V}_{A_1}$ looks as in Table 9. The graded dimension is given by

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{V}_{A_1,d} = \frac{\sum_n q^{n^2}}{\prod_{k=1}^{\infty}(1-q^k)} = \frac{\theta_3(\tau)}{\phi(q)} = q^{1/24}\frac{\theta_3(\tau)}{\eta(\tau)},$$

where $\theta_3(\tau)$ is the Jacobi theta constant.

Pick up the following elements of degree 1:

$$E = \mathbf{v}_\alpha, \ \ H = \alpha_{-1}\mathbf{v}_\lambda, \ \ F = \mathbf{v}_{-\alpha}.$$

Then the subspace of degree 1 becomes a Lie algebra isomorphic to $\mathfrak{sl}_2$ with respect to the bracket defined by $[X,Y] = X_{(0)}Y$:

$$\begin{aligned} &[H,E] = 2E, \ \ [H,F] = -2F, \ \ [E,F] = H, \\ &[H,H] = [F,F] = [E,E] = 0. \end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 9 Lattice vertex algebra of type A_1

0	1	2	3	4
				$\mathbf{v}_{2\alpha}$
	$\mathbf{v}_\alpha$	$\alpha_{-1}\mathbf{v}_\alpha$	$\alpha_{-2}\mathbf{v}_\alpha$	$\alpha_{-3}\mathbf{v}_\alpha$
			$\alpha_{-1}\alpha_{-1}\mathbf{v}_\alpha$	$\alpha_{-1}\alpha_{-2}\mathbf{v}_\alpha$
				$\alpha_{-1}\alpha_{-1}\alpha_{-1}\mathbf{v}_\alpha$
$\mathbf{v}_0$	$\alpha_{-1}\mathbf{v}_0$	$\alpha_{-2}\mathbf{v}_0$	$\alpha_{-3}\mathbf{v}_0$	$\alpha_{-4}\mathbf{v}_0$
		$\alpha_{-1}\alpha_{-1}\mathbf{v}_0$	$\alpha_{-1}\alpha_{-2}\mathbf{v}_0$	$\alpha_{-1}\alpha_{-3}\mathbf{v}_0$
			$\alpha_{-1}\alpha_{-1}\alpha_{-1}\mathbf{v}_0$	$\alpha_{-2}\alpha_{-2}\mathbf{v}_0$
				$\alpha_{-1}\alpha_{-1}\alpha_{-2}\mathbf{v}_0$
				$\alpha_{-1}\alpha_{-1}\alpha_{-1}\alpha_{-1}\mathbf{v}_0$
	$\mathbf{v}_{-\alpha}$	$\alpha_{-1}\mathbf{v}_{-\alpha}$	$\alpha_{-2}\mathbf{v}_{-\alpha}$	$\alpha_{-3}\mathbf{v}_{-\alpha}$
			$\alpha_{-1}\alpha_{-1}\mathbf{v}_{-\alpha}$	$\alpha_{-1}\alpha_{-2}\mathbf{v}_{-\alpha}$
				$\alpha_{-1}\alpha_{-1}\alpha_{-1}\mathbf{v}_{-\alpha}$
A basis of $V_{\sqrt{2}\mathbb{Z}}$				$\mathbf{v}_{-2\alpha}$

Moreover, the bilinear form defined by $(X|Y)\mathbf{1} = X_{(1)}Y$ becomes

$$(H|H) = 2, \ \ (E|F) = (F|E) = 1,$$
$$(E|E) = (F|F) = (H|E) = (H|F) = 0,$$

which is invariant with respect to the Lie bracket.

Therefore, there exists a homomorphism of vertex algebras from the universal affine vertex algebra $\mathbf{M}(1,0)$ associated with $\mathfrak{sl}_2$ at level $k = 1$. Since $\mathbf{V}_{A_1}$ is a simple vertex algebra generated by the degree 1 subspace, the map π induces an isomorphism of vertex algebras from $\mathbf{L}(1,0)$ onto $\mathbf{V}_{A_1}$:

$$\mathbf{L}(1,0) \xrightarrow{\sim} \mathbf{V}_{A_1}.$$

The $\mathbf{L}(1,0)$-module $\mathbf{L}(1,1/2)$ can be constructed as the space $\mathbf{V}_{A_1+1/\sqrt{2}}$ (cf. Subsection 1.4.5.3).

Note 4.3. 1. The Sugawara vector $\omega_k \in \mathbf{M}(k,0)$ given by (3.7) coincides for $k = 1$ with the Virasoro vector $\omega \in \mathbf{F}_0$ given by (3.5). 2. The construction of $\widehat{\mathfrak{sl}}_2$-modules as described above is a particular case of the famous *Frenkel–Kac construction* mentioned in the Introduction, which works for the root lattices of ADE type and realizes integrable highest weight representations of the corresponding affine Kac–Moody algebras of level 1.

1.4.4.3 The Lattice of Type D_1

Let L be the even lattice $2\mathbb{Z}$ generated by a norm 2 element as

$$2\mathbb{Z} = \sqrt{2}A_1 = \mathbb{Z}\beta, \ \ (\beta|\beta) = 4,$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 10 Lattice vertex algebra of type D_1

0	1	2	3	4
		$\mathbf{v}_\beta$	$\beta_{-1}\mathbf{v}_\beta$	$\beta_{-2}\mathbf{v}_\beta$
				$\beta_{-1}\beta_{-1}\mathbf{v}_\beta$
$\mathbf{v}_0$	$\beta_{-1}\mathbf{v}_0$	$\beta_{-2}\mathbf{v}_0$	$\beta_{-3}\mathbf{v}_0$	$\beta_{-4}\mathbf{v}_0$
		$\beta_{-1}\beta_{-1}\mathbf{v}_0$	$\beta_{-1}\beta_{-2}\mathbf{v}_0$	$\beta_{-1}\beta_{-3}\mathbf{v}_0$
			$\beta_{-1}\beta_{-1}\beta_{-1}\mathbf{v}_0$	$\beta_{-2}\beta_{-2}\mathbf{v}_0$
				$\beta_{-1}\beta_{-1}\beta_{-2}\mathbf{v}_0$
				$\beta_{-1}\beta_{-1}\beta_{-1}\beta_{-1}\mathbf{v}_0$
		$\mathbf{v}_{-\beta}$	$\beta_{-1}\mathbf{v}_{-\beta}$	$\beta_{-2}\mathbf{v}_{-\beta}$
A basis of $\mathbf{V}_{2\mathbb{Z}}$				$\beta_{-1}\beta_{-1}\mathbf{v}_{-\beta}$

which can be thought of as the root lattice of type D_n with n formally set to 1.

The structure of $\mathbf{V}_{2\mathbb{Z}}$ looks as in Table 10. Consider the elements

$$e^+ = \frac{\omega}{2} + \frac{\mathbf{v}_\beta + \mathbf{v}_{-\beta}}{4}, \quad e^- = \frac{\omega}{2} - \frac{\mathbf{v}_\beta + \mathbf{v}_{-\beta}}{4},$$

where ω is the Virasoro vector (3.5) with $a_n = (1/2)\beta_n$, $n \in \mathbb{Z}$. Then $e^\pm$ are Virasoro vectors of central charge $c = 1/2$ commuting with each other:

$$e^+_{(n)}e^- = 0 \ \ (n \geq 0), \quad e^\pm_{(n)}e^\pm = \begin{cases} 0 & (n \geq 4), \\ 1/4 & (n = 3), \\ 2e^\pm & (n = 1). \end{cases}$$

In fact, they generate vertex subalgebras isomorphic to the simple Virasoro vertex algebra of central charge $1/2$,

$$\langle e^+ \rangle_{\mathrm{VA}} \simeq \mathbf{L}(1/2, 0) \simeq \langle e^- \rangle_{\mathrm{VA}},$$

and we have a decomposition of the form

$$\mathbf{V}_{2\mathbb{Z}} = \underbrace{\mathbf{L}(1/2,0) \otimes \mathbf{L}(1/2,0)}_{\mathbf{V}^+_{2\mathbb{Z}}} \oplus \underbrace{\mathbf{L}(1/2,1/2) \otimes \mathbf{L}(1/2,1/2)}_{\mathbf{V}^-_{2\mathbb{Z}}},$$

where $\mathbf{V}^\pm_{2\mathbb{Z}}$ are the eigenspaces of an involution characterized by

$$\theta \colon \mathbf{V}_{2\mathbb{Z}} \longrightarrow \mathbf{V}_{2\mathbb{Z}}, \quad h \mapsto -h, \quad \mathbf{v}_\beta \mapsto \mathbf{v}_{-\beta}.$$

Note 4.4. 1. For more information on $\mathbf{V}^+_{2\mathbb{Z}}$ and its applications, see [43]. 2. Many interesting and useful examples of vertex algebras are found as subalgebras of the vertex algebras (or vertex superalgebras) associated with a rank one lattice. For example, the vertex algebra $\mathbf{V}^+_{\sqrt{6}\mathbb{Z}}$ is identified with what is called the

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

minimal W_4 algebra of central charge 1, which can be used to describe the actions of 4A elements of the Monster on the moonshine module $\mathbf{V}^\natural$ (cf. [79] and [93]).

1.4.5 Lattice Vertex Algebras for General Even Lattices

Recall the commutation of vertex operators described by (4.3), which implies, for sufficiently large N,

$$(y-z)^N V_\lambda(y)V_\mu(z) = (-1)^{(\lambda|\mu)}(y-z)^N V_\mu(z)V_\lambda(y).$$

For an even lattice L of higher rank, the value $(\lambda|\mu)$ for $\lambda, \mu \in L$ can be odd and, for such a case, the series $V_\lambda(z)$ and $V_\mu(z)$ are not locally commutative.

By this reason, we wish to modify the vertex operators by multiplying it by a sign factor in such a way that the resulting series become locally commutative and creative. We will then describe the lattice vertex algebras $\mathbf{V}_L$ in the same way as in the rank one case, and give a brief account on the simple modules over $\mathbf{V}_L$.

1.4.5.1 Cocycle Factors and Central Extensions

For $\lambda, \mu \in L$, the vertex operator $V_\lambda(z)$ restricts to

$$V_\lambda(z)\big|_{\mathbf{F}_\mu} : \mathbf{F}_\mu \longrightarrow \mathbf{F}_{\lambda+\mu}((z)).$$

We will multiply it by a sign factor so that the resulting series become locally commutative. To be more precise, consider a function

$$\varepsilon\colon L \times L \longrightarrow \{\pm 1\},$$

that is to be called a *cocycle factor*, and set

$$V_{\lambda,\varepsilon}(z)\big|_{\mathbf{F}_\mu} = \varepsilon(\lambda,\mu)V_\lambda(z)\big|_{\mathbf{F}_\mu}. \tag{4.4}$$

Then the condition on the function ε so that $V_{\lambda,\varepsilon}(z)$ become locally commutative and creative with respect to $\mathbf{v}_0$ are stated, respectively, as follows:

(1) For all $\lambda, \mu, \nu \in L$: $\varepsilon(\lambda, \mu+\nu)\varepsilon(\mu,\nu) = (-1)^{(\lambda|\mu)}\varepsilon(\mu, \lambda+\nu)\varepsilon(\lambda,\nu)$.
(2) For all $\lambda \in L$: $\varepsilon(0,\lambda) = 1 = \varepsilon(\lambda, 0)$.

These conditions imply:

(3) For all $\lambda, \mu, \nu \in L$: $\varepsilon(\lambda,\mu)\varepsilon(\lambda+\mu,\nu) = \varepsilon(\lambda,\mu+\nu)\varepsilon(\mu,\nu)$.
(4) For all $\lambda, \mu \in L$: $\varepsilon(\lambda,\mu)\varepsilon(\mu,\lambda) = (-1)^{(\lambda|\mu)}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

We actually have the equivalence

$$(1) + (2) \iff (2) + (3) + (4).$$

To see the meaning of the latter, consider a central extension of groups of the form

$$0 \longrightarrow \{\pm 1\} \longrightarrow \hat{\boldsymbol{L}} \xrightarrow{\ \pi\ } L \longrightarrow 0. \tag{4.5}$$

Choose a set-theoretical section $\iota\colon L \longrightarrow \hat{\boldsymbol{L}}$ and denote its value at λ by e_λ. Assume that the group structure on $\hat{\boldsymbol{L}}$ and the function $\varepsilon\colon L \times L \longrightarrow \{\pm 1\}$ are related by, for all $\lambda, \mu \in L$,

$$e_\lambda e_\mu = \varepsilon(\lambda, \mu) e_{\lambda+\mu}.$$

Then condition (2) says that e_0 is the identity element and (3) that the product is associative. In other words, the function $\varepsilon\colon L \times L \longrightarrow \{\pm 1\}$ satisfying (2) and (3) is a normalized 2-cocycle associated with the group extension (4.5).

Now the condition (4) means that the extension (4.5) is subject to

$$e_\lambda e_\mu = (-1)^{(\lambda|\mu)} e_\mu e_\lambda,$$

for which the correspondence $(e_\lambda, e_\mu) \mapsto e_\lambda e_\mu e_\lambda^{-1} e_\mu^{-1} = (-1)^{(\lambda|\mu)}$ is called the *commutator map.*

Note 4.5. As the vertex operators generate *intertwining operators* among the Fock modules (cf. Section 1.6.4), the problem of constructing a vertex algebra structure on $\mathbf{V}_L$ is a particular case of that for a direct sum of modules over a vertex algebra by intertwining operators. Such a problem is studied in detail for a good vertex algebra under certain conditions (cf. [37] and references therein), although $\mathbf{V}_L$ as a sum of Fock modules does not fulfill such conditions.

1.4.5.2 Lattice Vertex Algebras

Let us explicitly construct a cocycle factor as a bimultiplicative map:

$$\varepsilon(\lambda, \mu + \nu) = \varepsilon(\lambda, \mu)\varepsilon(\lambda, \nu), \ \ \varepsilon(\lambda + \mu, \nu) = \varepsilon(\lambda, \nu)\varepsilon(\mu, \nu).$$

Let $\alpha_1, \ldots, \alpha_n$ be a basis of L, set the values of $\varepsilon(\alpha_i, \alpha_j)$ so that

$$\varepsilon(\alpha_i, \alpha_j) = (-1)^{(\alpha_i|\alpha_j)} \varepsilon(\alpha_j, \alpha_i),$$

and extend it bimultiplicatively to the whole L. Then it indeed satisfies the conditions for a cocycle factor and the following proposition holds for the series $V_{\lambda,\varepsilon}(z)$ defined by (4.4).

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Proposition 4.6 *For an even lattice L and a cocycle factor ε, there exists a unique structure of a vertex algebra on the vector space $\mathbf{V}_L$ with vacuum $\mathbf{1} = \mathbf{v}_0$ such that*

$$Y(h_{-1}\mathbf{v}_0, z) = h(z) \ \textit{and} \ Y(\mathbf{v}_\lambda, z) = V_{\lambda,\varepsilon}(z)$$

for $h \in \mathfrak{h}$ and $\lambda, \mu \in L$.

The vertex algebra $\mathbf{V}_L$ is called the *lattice vertex algebra* associated with the even lattice L, which is a simple vertex algebra. The isomorphism class of the resulting vertex algebra does not depend on the choice of the cocycle factor.

The space $\mathbf{V}_L$ is given a grading as in the rank one case. If L is positive-definite of rank n, the graded dimension is given by

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{V}_{L,d} = \sum_{\lambda \in L} \prod_{k=1}^{\infty} \frac{q^{(\lambda|\lambda)/2}}{(1-q^k)^n} = \frac{\Theta_L(\tau)}{\phi(q)^n} = q^{n/24} \frac{\Theta_L(\tau)}{\eta(\tau)^n}.$$

Note 4.7. 1. The cocycle factor $\varepsilon \colon L \times L \longrightarrow \{\pm 1\}$ here can be chosen so that it is bimultiplicative and satisfies $\varepsilon(\lambda, \lambda) = (-1)^{(\lambda|\lambda)/2}$ for all $\lambda \in L$. 2. Let $\mathbf{F}_\lambda(d)$ be the subspace of $\mathbf{F}_\lambda$ of degree $(\lambda|\lambda)/2 + d$ for each $\lambda \in L$ and $d \in \mathbb{N}$. In particular, we have $\mathbf{F}_\lambda(0) = \mathbb{C}\mathbf{v}_\lambda$. Then the following properties hold:

(1) For all $h \in \mathfrak{h}$, $\lambda \in L$ and $k, m \in \mathbb{Z}$: $[h_k, \mathbf{v}_{\lambda(m)}] = \lambda(h)\mathbf{v}_{\lambda(k+m)}$.
(2) For all $\lambda, \mu \in L$ and $m \in \mathbb{Z}$: $\mathbf{v}_{\lambda(m)}\mathbf{v}_\mu \in \mathbf{F}_{\lambda+\mu}(-(\lambda|\mu) - m - 1)$.
(3) For all $\lambda, \mu \in L$: $\mathbf{v}_{\lambda(-(\lambda|\mu)-1)}\mathbf{v}_\mu = \pm \mathbf{v}_{\lambda+\mu}$.

These properties in fact characterize the vertex algebra structure of the lattice vertex algebra $\mathbf{V}_L$. (See the arguments in [76].)

1.4.5.3 Modules over Lattice Vertex Algebras

Let us briefly describe modules over the lattice vertex algebras. (See [40] for details.)

For a lattice L, the *dual lattice* is the set

$$L^\circ = \{\mu \in \mathfrak{h} \,|\, (\lambda|\mu) \in \mathbb{Z}\},$$

which is an additive subgroup of $\mathfrak{h}$ containing L. The values of the bilinear form on L° are rational numbers, and L° need not be a lattice in the sense we followed so far.

The isomorphism classes of simple $\mathbf{V}_L$-modules are in one-to-one correspondence with the cosets in L°/L. For each coset $M \in L^\circ/L$, it has the following shape:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\mathbf{V}_M = \bigoplus_{\lambda \in M} \mathbf{F}_\lambda.$$

In particular, the adjoint module $\mathbf{V}_L$ is the case with $M = L$.

Recall that a lattice is said to be *unimodular* if $L^\circ = L$. For a unimodular lattice L, the adjoint module is the only simple module over $\mathbf{V}_L$.

Here are examples of even unimodular positive-definite lattices. Note that the rank of such a lattice is a multiple of 8.

1. Rank 8. There is only one such lattice: the Gosset lattice E_8.
2. Rank 16. There are two such lattices: $E_8 \oplus E_8$ and D_{16}^+.
3. Rank 24. There are 24 such lattices, called *Niemeier lattices.* Among them, there is a distinguished one without roots called the *Leech lattice.*

The graded dimensions of the vertex algebras associated with the Gosset lattice E_8 and the Leech lattice Λ are given by

$$\sum_{d=0}^{\infty} q^d \dim \mathbf{V}_{E_8,d} = \frac{\Theta_{E_8}(\tau)}{\phi(q)^8} = q^{1/3}\frac{E_4(\tau)}{\eta(\tau)^8} \quad \text{and}$$
$$\sum_{d=0}^{\infty} q^d \dim \mathbf{V}_{\Lambda,d} = \frac{\Theta_{\Lambda}(\tau)}{\phi(q)^{24}} = q(j(\tau) - 720) = 1 + 24q + 196884q^2 + \cdots,$$

respectively, where $E_4(\tau)$ is the Eisenstein series of weight 4,

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q}{1 - q^n},$$

and $j(\tau)$ the elliptic modular function. The graded dimensions for the Niemeier lattices are all the same except for the coefficients to q.

Note 4.8. To construct the $\mathbf{V}_L$-module structures on $\mathbf{V}_M$ for cosets M in L°/L, we need to extend the factor ε to an appropriate one defined on $L \times L^\circ$ valued in a cyclic group containing $\{\pm 1\}$. See [40] for details.

Bibliographic Notes

Main references for Section 1.4 are Frenkel, Lepowsky, and Meurman [1], Dong and Lepowsky [4], and Dong [40]. See also Kac [6] and Lepowsky and Li [10]. For descriptions in models in physics, consult Di Francesco et al. [13]. For generalities on lattices, see Ebeling [14] and Conway and Sloan [12].

See Frenkel and Kac [57] and Kac [17] for Frenkel–Kac Construction (cf. (cf. Segal [92]), and Lepowsky and Wilson [69] for a slightly earlier construction of $\widehat{\mathfrak{sl}}_2$ by twisted vertex operators and Frenkel, Lepowsky, and Meurman [58] for generalizations.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Construction of the lattice vertex algebras, as well as introduction of the concept of vertex algebras, is due to Borcherds [32]. See Frenkel, Lepowsky, and Meurman [59] for an earlier attempt to construct and investigate the moonshine module $\mathbf{V}^\natural$ by means of vertex operators.

There are a huge number of applications of vertex operators in various fields of mathematics and physics.

1.5 Twisted Modules

Let $\mathbf{V}$ be a vertex algebra and θ an involution of $\mathbf{V}$, that is, an automorphism of order 2, and consider the subspace of fixed-points,

$$\mathbf{V}^+ = \{a \in \mathbf{V} \mid \theta a = a\},$$

which is a vertex subalgebra of $\mathbf{V}$. Let $i\colon \mathbf{V}^+ \longrightarrow \mathbf{V}$ denote the inclusion.

For any representation $\rho(-,z)\colon \mathbf{V} \longrightarrow \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z)))$, we obtain a representation of $\mathbf{V}^+$ by restriction as

$$\mathbf{V}^+ \xrightarrow{i} \mathbf{V} \xrightarrow{\rho(-,z)} \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z))).$$

Representation of $\mathbf{V}^+$ may also be obtained by restricting "generalized" representations of $\mathbf{V}$ in such a way that series in half-integral powers are involved but the restriction becomes valued in series with integral powers:

$$\begin{tikzcd} \mathbf{V}^+ \arrow[r, "i"] \arrow[drr] & \mathbf{V} \arrow[r, "\rho(-,z)"] & \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z^{1/2}))) \\ & & \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z))). \arrow[u] \end{tikzcd}$$

The concept of θ-twisted modules over $\mathbf{V}$ corresponds to such a generalization of representations of $\mathbf{V}$, achieved by appropriately generalizing the residue products to series with half-integral powers in z.

More generally, for an automorphism g of finite order N, the concept of g-twisted modules is defined by replacing $z^{1/2}$ with $z^{1/N}$:

$$\mathbf{V} \xrightarrow{\rho(-,z)} \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z^{1/N}))).$$

Although twisted modules are considered under presence of the action of an automorphism, generalization of the residue products works well for series with complex powers in z without actions of automorphisms.

In Section 1.5, we will first describe the general theory of twisted modules and then proceed to classical examples, the θ-twisted modules over the Heisenberg vertex algebras and the lattice vertex algebras, where θ is lift of the (-1)

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

-involution, the automorphism of order 2 induced by negation -1 of the generators of the Heisenberg algebra or the lattice.

We will work over the field $\mathbb{C}$ of complex numbers.

1.5.1 OPE of Shifted Series

In this section, we will generalize the residue products to those of the series whose exponents are integers shifted by a complex number for each. We will then appropriately adjust the concept of operator product expansion so that it fits such shifted series.

1.5.1.1 Preliminaries on Shifted Series

For a complex number α, consider the vector space $V[[z,z^{-1}]]z^{-\alpha}$ consisting of the formal expressions of the form

$$\sum_n v_{n+\alpha} z^{-n-\alpha-1},$$

where the summation is over the integers n. In this section, we will call such a series a series *shifted* by $\alpha \in \mathbb{C}$.

We will also consider subspaces such as $V((z))z^{-\alpha}$ or

$$V((y))((z))y^{-\alpha}z^{-\beta},\ \ V((z))((y))y^{-\alpha}z^{-\beta},\ \ V((y,z))y^{-\alpha}z^{-\beta}$$

for indeterminates y,z and complex numbers α,β.

Let x,y,z be indeterminates. We write

$$(x+z)^{n-\alpha}\big|_{|x|>|z|} = \sum_{i=0}^{\infty} \binom{n-\alpha}{i} x^{n-\alpha-i} z^i,$$
$$(x+z)^{n-\alpha}\big|_{|x|<|z|} = \sum_{i=0}^{\infty} \binom{n-\alpha}{i} x^i z^{n-\alpha-i}.$$

Note that

$$(x+z)^{n-\alpha}\big|_{|x|>|z|} \in \mathbb{C}[x,x^{-1}][[z]]x^{-\alpha} \subset \mathbb{C}((x))((z))x^{-\alpha},$$
$$(x+z)^{n-\alpha}\big|_{|x|<|z|} \in \mathbb{C}[z,z^{-1}][[x]]z^{-\alpha} \subset \mathbb{C}((z))((x))z^{-\alpha}.$$

Similarly, we write

$$(y-z)^{n-\alpha}\big|_{|y|>|z|} = \sum_{i=0}^{\infty} (-1)^i \binom{n-\alpha}{i} y^{n-\alpha-i} z^i \qquad \in \mathbb{C}((y))((z))y^{-\alpha},$$
$$(y-z)^{n-\alpha}\big|_{|y|<|z|} = \sum_{i=0}^{\infty} (-1)^{n-\alpha-i} \binom{n-\alpha}{i} y^i z^{n-\alpha-i} \in \mathbb{C}((z))((y))z^{-\alpha}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.1.2 Expansions of Shifted Series

Let x, y, z be indeterminates and α, β complex numbers. Consider the space

$$V((x, y, z))y^{-\alpha} z^{-\beta},$$

whose elements are written in the following form with some $L, M, N \in \mathbb{N}$:

$$w(x, y, z) = \frac{w_0(x, y, z)}{x^N y^{L+\alpha} z^{M+\beta}}, \quad w_0(x, y, z) \in V[[x, y, z]].$$

Since $x^N w(x, y, z) \in V[[x]]((y, z))y^{-\alpha} z^{-\beta}$,

$$(y - z)^N w(y - z, y, z)\big|_{|y|>|z|} = (y - z)^N w(y - z, y, z)\big|_{|y|<|z|}. \tag{5.1}$$

Similarly, $y^{L+\alpha} w(x, y, z) \in V[[y]]((x, z))z^{-\beta}$ implies

$$(x + z)^{L+\alpha} w(x, x + z, z)\big|_{|x|>|z|} = (x + z)^{L+\alpha} w(x, x + z, z)\big|_{|x|<|z|}.$$

Let $s(y, z)$ and $t(y, z)$ be series belonging to the spaces $V((y))((z))y^{-\alpha} z^{-\beta}$ and $V((z))((y))y^{-\alpha} z^{-\beta}$, respectively, and $w(x, y, z) \in V((x, y, z))y^{-\alpha} z^{-\beta}$ satisfy

$$\begin{aligned} s(y, z) &= w(y - z, y, z)\big|_{|y|>|z|} \in V((y))((z))y^{-\alpha} z^{-\beta}, \\ t(y, z) &= w(y - z, y, z)\big|_{|y|<|z|} \in V((z))((y))y^{-\alpha} z^{-\beta}. \end{aligned}$$

Consider the series $u(x, z) = w(x, x + z, z)\big|_{|x|<|z|}$. Then the coefficients $u_m(z)$ in $u(x, z) = \sum_m u_m(z)x^{-m-1}$ are determined by

$$\begin{aligned} u_m(z) = \sum_{i=0}^{\infty} \binom{-\alpha}{i} z^{-\alpha-i} \operatorname{Res}_y \Big(&(y - z)^{m+i}\big|_{|y|>|z|} y^{\alpha} s(y, z) \\ &- (y - z)^{m+i}\big|_{|y|<|z|} y^{\alpha} t(y, z) \Big), \end{aligned}$$

where the sum is a finite sum by (5.1). Note that $u_m(z)$ does not depend on the choice of α such that $y^{\alpha} w(x, y, z)$ is of integral powers in y.

Note 5.1. The right-hand side is obtained by formally calculating the expression

$$(y - z)^m s(y, z)\big|_{|y|>|z|} - (y - z)^m t(y, z)\big|_{|y|<|z|}.$$

Indeed, inserting $y^{-\alpha} y^{\alpha}$, we formally write it as

$$y^{-\alpha}(y - z)^m\big|_{|y|>|z|}\, y^{\alpha} s(y, z) - y^{-\alpha}(y - z)^m\big|_{|y|<|z|}\, y^{\alpha} t(y, z). \tag{5.2}$$

Replace the factor $y^{-\alpha}(y - z)^m$ by the expansion

$$(x + z)^{-\alpha}\big|_{|x|<|z|} x^m = \sum_{i=0}^{\infty} \binom{-\alpha}{i} z^{-\alpha-i} x^{m+i},$$

and then expand it by substitution $x = y - z$ in the respective regions. Although the result does not make sense as a series in y and z, the expression (5.2) turns out to give a well-defined expression by (5.1).

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.1.3 Residue Products of Shifted Series

Let $\mathbf{M}$ be a vector space and α a complex number. We will call such an element of $(\operatorname{End}\mathbf{M})[[z,z^{-1}]]z^{-\alpha}$ a series on $\mathbf{M}$ *shifted* by α. For such a series $A(z)$, set

$$A(z) = \sum_n A_{n+\alpha} z^{-n-\alpha-1},$$

where the sum is over the integers n and $A_{n+\alpha}$ are operators acting on $\mathbf{M}$.

We will say that $A(z)$ is *locally truncated* if $A(z)v \in \mathbf{M}((z))z^{-\alpha}$ for all $v \in \mathbf{M}$. The set of such series is identified with $\operatorname{Hom}(\mathbf{M}, \mathbf{M}((z))z^{-\alpha})$.

Let $A(z)$ and $B(z)$ be series on a vector space $\mathbf{M}$ shifted by complex numbers. They are said to be *locally commutative* if the following holds for some $N \in \mathbb{N}$:

$$(y-z)^N A(y)B(z) = (y-z)^N B(z)A(y).$$

This is the same as the case of series with integral powers.

Let $A(z)$ and $B(z)$ be locally truncated and *locally commutative* series on a vector space $\mathbf{M}$ shifted by complex numbers α and β, respectively. Define the nth residue product of $A(z)$ and $B(z)$ by

$$\begin{aligned} A(z)_{(n)}B(z) &= \sum_{i=0}^{\infty} \binom{-\alpha}{i} z^{-\alpha-i} \big(\operatorname{Res}_y (y-z)^{n+i}\big|_{|y|>|z|} y^{\alpha} A(y)B(z) \\ &\qquad\qquad - \operatorname{Res}_y (y-z)^{n+i}\big|_{|y|<|z|} y^{\alpha} B(z)A(y)\big) \\ &= \sum_{i=0}^{\infty} \binom{-\alpha}{i} (z^{\alpha}A(z))_{(n+i)} B(z)\, z^{-\alpha-i} \end{aligned}$$

for each $n \in \mathbb{Z}$.

As the identity series $I(z)$ is unshifted, the identity property holds:

$$I(z)_{(n)}A(z) = \begin{cases} 0 & (n \neq -1), \\ A(z) & (n = -1). \end{cases}$$

The relation $A(z)_{(n)}I(z) = 0$ for $n \geq 0$ is clear and

$$\begin{aligned} A(z)_{(-1)}I(z) &= \sum_{i=0}^{\infty} \binom{-\alpha}{i} z^{-\alpha-i} (z^{\alpha}A(z))_{(-1+i)} I(z) \\ &= z^{-\alpha}(z^{\alpha}A(z))_{(-1)}I(z) = z^{-\alpha}z^{\alpha}A(z) = A(z). \end{aligned}$$

Therefore, the creation property also holds:

$$A(z)_{(n)}I(z) = \begin{cases} 0 & (n \geq 0), \\ A(z) & (n = -1). \end{cases}$$

It is not difficult to show the relation

$$A(z)_{(-k-1)}I(z) = \partial^{(k)}A(z)$$

for $n = -k-1 < 0$ by $(z^{\alpha}A(z))_{(-k-1)}I(z) = \partial^{(k)}(z^{\alpha}A(z))$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.1.4 Modified OPE of Shifted Series

Let $A(z)$ and $B(z)$ be locally truncated locally commutative series on a vector space $\mathbf{M}$ shifted by α and β, respectively:

$$A(z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))z^{-\alpha}), \quad B(z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))z^{-\beta}).$$

Since $z^{\alpha}A(z)$ and $z^{\beta}B(z)$ are of integral powers, we have an OPE of the following form for some $N \in \mathbb{N}$ and series $C_0(z), \dots, C_{N-1}(z)$ with integral powers:

$$y^{\alpha}A(y)z^{\beta}B(z) \simeq z^{\beta}B(z)y^{\alpha}A(y) \sim \sum_{k=0}^{N-1} \frac{C_k(z)}{(y-z)^{k+1}}.$$

Therefore,

$$A(y)B(z) \simeq B(z)A(y) \sim y^{-\alpha}z^{-\beta} \sum_{k=0}^{N-1} \frac{C_k(z)}{(y-z)^{k+1}}. \tag{5.3}$$

Let us further expand $y^{-\alpha}$ by substitution $y = x + z$ in $|x| < |z|$, connect by $\approx$ if the two sides are related by this process, and neglect the regular part. Then the right-hand side of (5.3) results in

$$y^{-\alpha}z^{-\beta} \sum_{k=0}^{N-1} \frac{C_k(z)}{(y-z)^{k+1}} \approx \sum_{k=0}^{N-1} \frac{D_k(z)}{(y-z)^{k+1}},$$

where

$$D_k(z) = \sum_{i=0}^{N-k} \binom{-\alpha}{i} C_{k+i}(z) z^{-\alpha-\beta-i} \quad (0 \leq k < N).$$

We thus arrive at the following expression:

$$A(y)B(z) \simeq B(z)A(y) \approx \sum_{k=0}^{N-1} \frac{D_k(z)}{(y-z)^{k+1}}. \tag{5.4}$$

Let us call it (the singular part of) the modified OPE of $A(z)$ and $B(z)$.

The modified OPE (5.4) allows us to find the residue products for $m \in \mathbb{N}$ as

$$A(z)_{(m)}B(z) = \begin{cases} 0 & (N \leq m), \\ D_m(z) & (0 \leq m < N), \end{cases}$$

by the formula (5.4).

1.5.2 Shifted and Twisted Modules

In this section, we will generalize the concept of modules over a vertex algebra by replacing vertex algebra of series with vertex algebra of shifted series. Under

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

presence of the action of an automorphism g of finite order, those fitting g give rise to the concept of g-twisted modules.

1.5.2.1 Vertex Algebras of Shifted Series

Let us now consider a finite sum of series shifted by complex numbers. Such a sum $v(z)$ is written in the following form with some $k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$:

$$v(z) = \sum_{i=1}^{k} \sum_{n} v_{n+\alpha_i} z^{-n-\alpha_i-1}.$$

We will simply call such a series a *shifted series.* The concepts of local truncation, local commutativity, and residue products are generalized to shifted series in obvious ways.

The result shown Theorem 5.2 is a shifted analogue of Theorem 5.2 and can be proved in the same spirit.

Theorem 5.2 *Let $A(z), B(z), C(z)$ be locally truncated shifted series on a vector space. If they are locally commutative with each other, then the Borcherds identity*

$$\begin{aligned}
&\sum_{i=0}^{\infty} \binom{p}{i} (A(z)_{(r+i)} B(z))_{(p+q-i)} C(z) \\
&\qquad = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} A(z)_{(p+r-i)} (B(z)_{(q+i)} C(z) \\
&\qquad\quad - \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} B(z)_{(q+r-i)} (A(z)_{(p+i)} C(z)
\end{aligned}$$

holds for all $p, q, r \in \mathbb{Z}$.

As in the unshifted case, we readily obtain the following result due to Li and Roitman.

Corollary 5.3 *Let $\mathcal{V}$ be a vector space consisting of shifted series on a vector space* $\mathbf{M}$ *satisfying the following conditions.*

(1) *The space $\mathcal{V}$ is locally truncated and locally commutative.*
(2) *The space $\mathcal{V}$ is closed under the residue products.*
(3) *The space $\mathcal{V}$ contains the identity series.*

Then the space $\mathcal{V}$ becomes a vertex algebra by the residue products.

We will call the vertex algebra thus obtained a *vertex algebra of shifted series.*

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.2.2 Shifted Representations

Let us denote the sets of shifted series and shifted Laurent series with coefficients in a vector space V respectively by

$$V[[z,z^{-1}]]z^{\mathbb{C}} = \sum_{\alpha\in\mathbb{C}} V[[z,z^{-1}]]z^{-\alpha},$$
$$V((z))z^{\mathbb{C}} = \sum_{\alpha\in\mathbb{C}} V((z))z^{-\alpha} = \sum_{\alpha\in\mathbb{C}} V[[z]]z^{-\alpha}.$$

Let $\mathbf{V}$ be a vertex algebra and $\mathbf{M}$ a vector space, and consider a map of the following form:

$$\rho(-,z) : \mathbf{V} \longrightarrow \mathrm{Hom}(\mathbf{M}, \mathbf{M}[[z,z^{-1}]]z^{\mathbb{C}}), \quad a \mapsto \rho(a,z).$$

Such a map is said to be a *shifted representation* of $\mathbf{V}$ if the following conditions are satisfied:

(1) The image of $\rho(-,z)$ is a vertex algebra of shifted series on $\mathbf{M}$.
(2) The map $\rho(-,z)$ induces a homomorphism of vertex algebras onto its image.

Note that (1) in particular says that $\rho(a,z)$ is locally truncated for any $a \in \mathbf{V}$.

1.5.2.3 Twisted Modules

By abuse of notation, we will denote the image of a complex number $\alpha \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$ by the same symbol α.

Let g be an automorphism of a vertex algebra $\mathbf{V}$ of finite order:

$$g : \mathbf{V} \longrightarrow \mathbf{V}.$$

Let $N \in \mathbb{N}$ be the order of g and set

$$\Gamma = ((1/N)\mathbb{Z})/\mathbb{Z} = \{0, 1/N, \ldots, (N-1)/N\}.$$

Then the eigenspace decomposition of $\mathbf{V}$ with respect to the action of g is written as follows:

$$\mathbf{V} = \bigoplus_{\alpha\in\Gamma} \mathbf{V}^{\alpha}, \quad \mathbf{V}^{\alpha} = \left\{ a \in \mathbf{V} \,\middle|\, ga = e^{2\pi\sqrt{-1}\alpha} a \right\}.$$

Since g is an automorphism of $\mathbf{V}$, we have

$$\mathbf{V}^{\alpha}{}_{(m)}\mathbf{V}^{\beta} \subset \mathbf{V}^{\alpha+\beta}$$

for all $\alpha, \beta \in \Gamma$, and $m \in \mathbb{Z}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

In such a situation, a shifted representation given by a direct sum of maps

$$\rho_\alpha(-,z)\colon \mathbf{V}^\alpha \longrightarrow \operatorname{Hom}(\mathbf{M}, \mathbf{M}((z))z^{-\alpha})$$

is called a *g-twisted representation* of $\mathbf{V}$.

For each $\alpha \in \Gamma$ and $a \in \mathbf{V}_\alpha$, let us write

$$\rho_\alpha(a,z) = \sum_n a_{n+\alpha} z^{-n-\alpha-1},$$

so that the coefficients give rise to maps

$$\rho_{\alpha,n} : \mathbf{V}^\alpha \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (a,v) \mapsto a_{n+\alpha} v. \tag{5.5}$$

Then the direct sum of $\rho_\alpha(-,z)$ becomes a g-twisted representation of $\mathbf{V}$ if and only if the following properties holds:

(T0) Local truncation. For any $\alpha \in \Gamma$, $a \in \mathbf{V}^\alpha$ and $v \in \mathbf{M}$, there exists an $N \in \mathbb{N}$ such that

$$a_{N+\alpha+i} v = 0 \quad \text{for all } i \geq 0.$$

(T1) Borcherds identity. For all $\alpha, \beta \in \Gamma$, $a \in \mathbf{V}^\alpha$, $b \in \mathbf{V}^\beta$, $v \in \mathbf{M}$, and $p \in \mathbb{Z} + \alpha$, $q \in \mathbb{Z} + \beta$ and $r \in \mathbb{Z}$:

$$\begin{aligned}\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)} b)_{p+q-i} v = &\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a_{p+r-i}(b_{q+i} v)\\ &- \sum_{i=0}^{\infty} (-1)^{r-i} \binom{r}{i} b_{q+r-i}(a_{p+i} v).\end{aligned}$$

(T2) Identity. For any $v \in \mathbf{M}$ and $n \in \mathbb{Z}$:

$$\mathbf{1}_n v = \begin{cases} 0 & (n \neq -1), \\ v & (n = -1). \end{cases}$$

A sequence of maps as in (5.5) satisfying the properties (T0)–(T2) is called a *g-twisted module* over $\mathbf{V}$ or a *g-twisted* $\mathbf{V}$*-module*.

The generating series $\rho_\alpha(a,z)$ for $a \in \mathbf{V}^\alpha$ is usually written as

$$Y_{\mathbf{M}}(a,z) = \sum_n a_{n+\alpha} z^{-n-\alpha-1}.$$

A module over a vertex algebra in the ordinary sense is sometimes called an *untwisted module*. A g-twisted $\mathbf{V}$-module $\mathbf{M}$ is an untwisted module over the subalgebra $\mathbf{V}^0$.

In later sections, we will be concerned with the case $N = 2$, where the eigenspaces $\mathbf{V}^0$ and $\mathbf{V}^{1/2}$ are denoted by $\mathbf{V}^+$ and $\mathbf{V}^-$, respectively.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Note 5.3. In the physics literatures, the untwisted and twisted modules are often called the untwisted and twisted *sectors* (of the theory or the model under consideration), respectively.

1.5.3 Twisted Heisenberg Modules

In this section, we illustrate an example of a twisted module in the case of the Heisenberg vertex algebra with a particular involution θ; the one induced by negation of the standard generator, which is the most simple and classical example of twisted modules.

For simplicity, we will work with the Heisenberg vertex algebra of rank one following the description in Section 1.3.1. The higher rank cases are treated in the same way (cf. Subsection 1.5.4.1).

1.5.3.1 OPE of Twisted Current

Let $a_{n+1/2}$ with $n \in \mathbb{Z}$ and ζ be indeterminates and set

$$\hat{\mathfrak{h}}^{\mathrm{tw}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_{n+1/2} \oplus \mathbb{C}\zeta.$$

Then $\hat{\mathfrak{h}}^{\mathrm{tw}}$ becomes a Lie algebra by the bracket

$$[a_{m+1/2}, a_{n+1/2}] = (m+1/2)\delta_{m+n+1,0}\zeta, \quad [\zeta, a_{n+1/2}] = 0.$$

Consider the twisted Heisenberg algebra defined by

$$\mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1) = \mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}})/(\zeta - 1).$$

We will denote the images of the generators $a_{n+1/2}$ by the same symbol. Then

$$[a_{m+1/2}, a_{n+1/2}] = (m+1/2)\delta_{m+n+1,0}$$

for $m, n \in \mathbb{Z}$, where the bracket denotes the commutator.

Consider the *twisted current*

$$a^{\mathrm{tw}}(z) = \sum_n a_{n+1/2} z^{-(n+1/2)-1} = \sum_n a_{n+1/2} z^{-n-3/2}.$$

Then the commutation relation turns out to be expressed as

$$\begin{aligned}
[a^{\mathrm{tw}}(y), a^{\mathrm{tw}}(z)] &= \sum_m (m+1/2) y^{-m-1/2-1} z^{m-1/2} \\
&= \partial_z \big(y^{-1/2} z^{1/2} \delta(y,z) \big) \\
&= \frac{1}{2} y^{-1/2} z^{-1/2} \delta(y,z) + y^{-1/2} z^{1/2} \delta^{(1)}(y,z).
\end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 11 Twisted Fock module

	0	1/2	1	3/2	2
$\mathbf{F}^{\mathrm{tw}}$:	$\mathbf{v}^{\mathrm{tw}}$	$a_{-1/2}\mathbf{v}^{\mathrm{tw}}$	$(a_{-1/2})^2\mathbf{v}^{\mathrm{tw}}$	$a_{-3/2}\mathbf{v}^{\mathrm{tw}}$ $(a_{-1/2})^3\mathbf{v}^{\mathrm{tw}}$	$a_{-1/2}a_{-3/2}\mathbf{v}^{\mathrm{tw}}$ $(a_{-1/2})^4\mathbf{v}^{\mathrm{tw}}$

Therefore, we have the following OPE:

$$a^{\mathrm{tw}}(y)a^{\mathrm{tw}}(z) \simeq a^{\mathrm{tw}}(z)a^{\mathrm{tw}}(y) \sim \frac{1}{2}\frac{y^{-1/2}z^{-1/2}}{y-z} + \frac{y^{-1/2}z^{1/2}}{(y-z)^2}.$$

In particular, $a^{\mathrm{tw}}(z)$ is locally commutative with itself.

Expansion of the factor $y^{-1/2}$ by $y = x + z$ in $|x| < |z|$ yields

$$(x+z)^{-1/2}\Big|_{|x|<|z|} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k z^{-1/2-k}.$$

After some algebra, we find that the modified OPE is given by

$$a^{\mathrm{tw}}(y)a^{\mathrm{tw}}(z) \simeq a^{\mathrm{tw}}(z)a^{\mathrm{tw}}(y) \approx \frac{1}{(y-z)^2}.$$

Therefore, the shifted residue products for $n \in \mathbb{N}$ is determined as

$$a^{\mathrm{tw}}(z)_{(n)}a^{\mathrm{tw}}(z) = \begin{cases} 0 & (n \geq 2), \\ 1 & (n = 1), \\ 0 & (n = 0), \end{cases} \tag{5.6}$$

which is exactly of the same form as the untwisted case.

1.5.3.2 Twisted Fock Module

Consider the following subspaces of the Lie algebra $\hat{\mathfrak{h}}^{\mathrm{tw}}$:

$$\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0} = \mathrm{Span}\left\{a_{n+1/2}\middle| n < 0\right\}, \ \hat{\mathfrak{h}}^{\mathrm{tw}}_{>0} = \mathrm{Span}\left\{a_{n+1/2}\middle| n \geq 0\right\}.$$

They are commutative Lie subalgebras, and generate subalgebras of $\mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1)$ isomorphic to the symmetric algebras $\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0})$ and $\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{>0})$, respectively.

Let $\mathbb{C}\mathbf{v}^{\mathrm{tw}}$ be the one-dimensional trivial module over $\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{>0})$, for which

$$a_{n+1/2}\mathbf{v}^{\mathrm{tw}} = 0 \ \ (n \geq 0).$$

Define the twisted Fock module by

$$\mathbf{F}^{\mathrm{tw}} = \mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1) \otimes_{\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{>0})} \mathbb{C}\mathbf{v}^{\mathrm{tw}} \simeq \mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbb{C}\mathbf{v}^{\mathrm{tw}}.$$

See Table 11.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Unlike the untwisted case, there is no freedom of *charge,* for it is the eigenvalue of the action of the central element a_0, which is present in $\mathbf{U}(\hat{\mathfrak{h}}, 1)$ but absent in $\mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1)$.

1.5.3.3 Twisted $\mathbf{F}_0$-Module

Since the twisted current on $\mathbf{F}^{\mathrm{tw}}$ is locally truncated and locally commutative with itself, the twisted currents generate a vertex algebra, which we denote as

$$\mathcal{F}_0^{\mathrm{tw}} = \langle a^{\mathrm{tw}}(z) \rangle_{\mathrm{RP}}.$$

Now recall the (untwisted) Fock module $\mathbf{F}_0 = \mathbf{S}(\hat{\mathfrak{h}}_{<0})\mathbf{v}_0 = \mathbb{C}[x_1, x_2, \cdots]$ of charge 0. By the shifted OPE (5.6) and the universal property of the Fock module $\mathbf{F}_0$, there exists a unique homomorphism of $\mathbf{U}(\hat{\mathfrak{h}}, 1)$-modules sending the vacuum vector $\mathbf{v}_0$ to the identity series $I(z)$ on $\mathbf{F}^{\mathrm{tw}}$,

$$\psi^{\mathrm{tw}} : \mathbf{F}_0 \longrightarrow \mathcal{F}_0^{\mathrm{tw}}, \quad \mathbf{v}_0 \mapsto I(z),$$

giving rise to a shifted representation of the Heisenberg vertex algebra $\mathbf{F}_0$ on the twisted Fock module $\mathbf{F}^{\mathrm{tw}}$.

The corresponding shifted $\mathbf{F}_0$-module becomes a twisted module. To see it, consider the involution θ of the polynomial ring $\mathbf{F}_0 = \mathbb{C}[x_1, x_2, \cdots]$ which negates the indeterminates:

$$\theta \colon \mathbf{F}_0 \longrightarrow \mathbf{F}_0, \quad x_k \mapsto -x_k \quad (k = 1, 2, \cdots).$$

Then θ turns out to be an automorphism of a vertex algebra, actually determined by its action on the generator $x_1 = a_{-1}\mathbf{v}_0$, for which the eigenspace decomposition is of the form

$$\mathbf{F}_0 = \mathbf{F}_0^+ \oplus \mathbf{F}_0^-,$$

where

$$\mathbf{F}_0^+ = \{a \in \mathbf{F}_0 \,|\, \theta a = a\}, \quad \mathbf{F}_0^- = \{a \in \mathbf{F}_0 \,|\, \theta a = -a\}.$$

On the other hand, the space $\mathcal{F}_0^{\mathrm{tw}}$ decomposes according to the shifts of the series as

$$\mathcal{F}_0^{\mathrm{tw}} = \mathcal{F}_0^{\mathrm{tw},+} \oplus \mathcal{F}_0^{\mathrm{tw},-},$$

where

$$\begin{aligned} \mathcal{F}_0^{\mathrm{tw},+} &= \mathcal{F}_0^{\mathrm{tw}} \cap \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))), \\ \mathcal{F}_0^{\mathrm{tw},-} &= \mathcal{F}_0^{\mathrm{tw}} \cap \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))z^{-1/2}). \end{aligned}$$

For example,

$$I(z) \in \mathcal{F}_0^{\mathrm{tw},+}, \quad a^{\mathrm{tw}}(z) \in \mathcal{F}_0^{\mathrm{tw},-}, \quad a^{\mathrm{tw}}(z)_{(n)} a^{\mathrm{tw}}(z) \in \mathcal{F}_0^{\mathrm{tw},+} \quad (n \in \mathbb{Z}).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

In general, the residue products of even numbers of $a^{\mathrm{tw}}(z)$ belong to $\mathcal{F}_0^{\mathrm{tw},+}$, whereas odd numbers to $\mathcal{F}_0^{\mathrm{tw},-}$.

The shifted representation of $\mathbf{F}_0$ now decomposes into the direct sum of

$$\psi^{\mathrm{tw},+} : \mathbf{F}_0^+ \longrightarrow \mathcal{F}_0^{\mathrm{tw},+}, \ \ \psi^{\mathrm{tw},-} : \mathbf{F}_0^- \longrightarrow \mathcal{F}_0^{\mathrm{tw},-},$$

giving rise to a structure of a θ-twisted $\mathbf{F}_0$-module on the twisted Fock module $\mathbf{F}^{\mathrm{tw}}$.

As the space $\mathcal{F}_0^{\mathrm{tw},+}$ consists of series with integral powers and the fixed-point subspace $\mathbf{F}_0^+$ is a vertex subalgebra of $\mathbf{F}_0$, the map

$$\psi^{\mathrm{tw},+} : \mathbf{F}_0^+ \longrightarrow \mathcal{F}_0^{\mathrm{tw},+}$$

gives rise to an untwisted representation of $\mathbf{F}_0^+$ on $\mathbf{F}^{\mathrm{tw}}$.

1.5.3.4 Twisted Virasoro Actions

Recall the standard Virasoro vector for the Heisenberg vertex algebra given by (1.8) and (3.5):

$$\omega = \frac{1}{2}x_1^2 = \frac{1}{2}a_{-1}a_{-1}\mathbf{v}_0.$$

The corresponding twisted series on $\mathbf{F}^{\mathrm{tw}}$ turns out to be

$$T^{\mathrm{tw}}(z) = \frac{1}{2}a^{\mathrm{tw}}(z)_{(-1)}a^{\mathrm{tw}}(z)_{(-1)}I(z) = \frac{1}{2}a^{\mathrm{tw}}(z)_{(-1)}a^{\mathrm{tw}}(z).$$

After some algebra following the definition of the residue products,

$$T^{\mathrm{tw}}(z) = \frac{1}{2}z^{-1/2}\,{\circ\atop\circ}(z^{1/2}a^{\mathrm{tw}}(z))a^{\mathrm{tw}}(z){\circ\atop\circ} + \frac{1}{16}z^{-2}.$$

The Fourier modes of $T^{\mathrm{tw}}(z)$ generate a representation of Virasoro algebra of central charge 1. For example,

$$L_0^{\mathrm{tw}} = \frac{1}{16} + a_{-1/2}a_{1/2} + a_{-3/2}a_{3/2} + a_{-5/2}a_{5/2} + \cdots .$$

Note that the Virasoro vector ω belongs to the subspace $\mathbf{F}_0^+$.

1.5.4 Twisted Vertex Operators

In this section, we will generalize and continue the consideration of the preceding section for higher-rank Heisenberg vertex algebras, and introduce the twisted version of the vertex operators. The twisted vertex operators will be used to construct twisted modules over lattice vertex algebras in the next section.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.4.1 Twisted Currents of Higher Rank

Let $\mathfrak{h}$ be an n-dimensional vector space regarded as an abelian Lie algebra and $(\,|\,)$ a nondegenerate symmetric invariant bilinear form on $\mathfrak{h}$. Recall the Heisenberg algebra $\mathbf{U}(\hat{\mathfrak{h}}, 1)$ and related notations from Subsection 1.4.2.1.

Let us now consider the space

$$\hat{\mathfrak{h}}^{\mathrm{tw}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]t^{1/2} \oplus \mathbb{C}K.$$

Denote $h \otimes t^s$ by h_s for $h \in \mathfrak{h}$ and $s \in \mathbb{Z} + 1/2$ and equip $\hat{\mathfrak{h}}^{\mathrm{tw}}$ with a structure of a Lie algebra by setting, for $g, h \in \mathfrak{h}$,

$$[g_r, h_s] = r(g|h)\delta_{r+s,0}K.$$

Define the *twisted Heisenberg algebra* associated with $\mathfrak{h}$ and $(\,|\,)$ by

$$\mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1) = \mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}})/(K - 1).$$

Define the spaces such as $\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}$ and the twisted module space $\mathbf{F}^{\mathrm{tw}}$ as before. By PBW,

$$\mathbf{F}^{\mathrm{tw}} = \mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbb{C}\mathbf{v}^{\mathrm{tw}}$$

as vector spaces.

For $h \in \mathfrak{h}$ and $s \in \mathbb{Z} + 1/2$, denote the image of h_s in $\mathbf{U}(\hat{\mathfrak{h}}^{\mathrm{tw}}, 1)$ by the same symbol, and consider the *twisted current*

$$h^{\mathrm{tw}}(z) = \sum_s h_s z^{-s-1},$$

where the sum is over $s \in \mathbb{Z} + 1/2$. Then the twisted currents are locally truncated shifted series on the twisted Fock module, they are locally commutative, and their modified OPE is given by

$$g^{\mathrm{tw}}(z)_{(n)} h^{\mathrm{tw}}(z) = \begin{cases} 0 & (n \geq 2), \\ (g|h) & (n = 1), \\ 0 & (n = 0), \end{cases}$$

which is again of the same form as the untwisted case.

1.5.4.2 Twisted Vertex Operators

Regard $\lambda \in \mathfrak{h}^*$ as an element of $\mathfrak{h}$ via the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by $(\,|\,)$ and consider the actions of λ_s with $s \in \mathbb{Z} + 1/2$ on $\mathbf{F}^{\mathrm{tw}}$. The following expression is called the *twisted vertex operator*:

$$U^{\mathrm{tw}}_\lambda(z) = \exp\Big(-\sum_{s<0} \lambda_s \frac{z^{-s}}{s}\Big) \exp\Big(-\sum_{s>0} \lambda_s \frac{z^{-s}}{s}\Big).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Here s runs over the elements of $\mathbb{N} + 1/2$ and $-\mathbb{N} - 1/2$, respectively, and

$$U_\lambda^{\mathrm{tw}}(z) \in \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z^{1/2}))).$$

Note that

$$\begin{aligned} U_\lambda^{\mathrm{tw}}(z) + U_{-\lambda}^{\mathrm{tw}}(z) &\in \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))), \\ U_\lambda^{\mathrm{tw}}(z) - U_{-\lambda}^{\mathrm{tw}}(z) &\in \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))z^{1/2}). \end{aligned}$$

We will later modify $U_\lambda^{\mathrm{tw}}(z)$ in a suitable way and call the resulting expression by the same term.

For $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, we have

$$\left[h^{\mathrm{tw}}(y), U_\lambda^{\mathrm{tw}}(z)\right] = y^{-1/2} z^{1/2} \lambda(h) \delta(y, z) U_\lambda^{\mathrm{tw}}(z).$$

In particular, the twisted currents are locally commutative with twisted vertex operators, and the modified OPE reads

$$h^{\mathrm{tw}}(y) U_\lambda^{\mathrm{tw}}(z) \simeq U_\lambda^{\mathrm{tw}}(z) h^{\mathrm{tw}}(y) \approx \frac{\lambda(h)}{y - z} U_\lambda^{\mathrm{tw}}(z).$$

Hence the residue products for nonnegative n become

$$h^{\mathrm{tw}}(z)_{(n)} U_\lambda^{\mathrm{tw}}(z) = \begin{cases} 0 & (n \geq 1), \\ \lambda(h) U_\lambda^{\mathrm{tw}}(z) & (n = 0), \end{cases}$$

which is of the same form as the untwisted case.

1.5.4.3 Commutation Relations

For $\lambda, \mu \in \mathfrak{h}^*$, consider the composite $U_\lambda^{\mathrm{tw}}(y) U_\mu^{\mathrm{tw}}(z)$ of twisted vertex operators,

$$\exp\Big(\sum_{r<0} \lambda_r \frac{y^{-r}}{-r}\Big) \underline{\exp\Big(\sum_{r>0} \lambda_r \frac{y^{-r}}{-r}\Big)}\ \underline{\exp\Big(\sum_{s<0} \mu_s \frac{z^{-s}}{-s}\Big)} \exp\Big(\sum_{s>0} \mu_s \frac{z^{-s}}{-s}\Big),$$

and denote the expression obtained by switching the underlined factors by $U_{\lambda,\mu}^{\mathrm{tw}}(y, z)$, which is written as

$$U_{\lambda,\mu}^{\mathrm{tw}}(y, z) = \exp\Big(\sum_{r<0} \lambda_r \frac{y^{-r}}{-r} + \sum_{s<0} \mu_s \frac{z^{-s}}{-s}\Big) \exp\Big(\sum_{r>0} \lambda_r \frac{y^{-r}}{-r} + \sum_{s>0} \mu_s \frac{z^{-s}}{-s}\Big).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

By the commutation relations of twisted Heisenberg algebra,

$$\begin{aligned}\Big[\sum_{r>0}\lambda_r\frac{y^{-r}}{-r},\sum_{s<0}\mu_s\frac{z^{-s}}{-s}\Big] &= \sum_{r>0}\sum_{s<0}[\lambda_r,\mu_s]\frac{y^{-r}}{-r}\frac{z^{-s}}{-s}\\ &= \sum_{r>0}\sum_{s<0} r(\lambda|\mu)\delta_{r+s,0}\frac{y^{-r}}{-r}\frac{z^{-s}}{-s}\\ &= -(\lambda|\mu)\sum_{r>0}\frac{y^{-r}z^{r}}{r}.\end{aligned}$$

We have

$$-\sum_{r>0}\frac{y^{-r}z^{r}}{r} = \log\left(\frac{y^{1/2}-z^{1/2}}{y^{1/2}+z^{1/2}}\right)\Bigg|_{|y|>|z|},$$

where the region $|y| > |z|$ indicates that the expression is to be expanded in $|y^{1/2}| > |z^{1/2}|$. Therefore, we have the following equalities, with the latter obtained by switching the roles of $U_\lambda^{\mathrm{tw}}(y)$ and $U_\mu^{\mathrm{tw}}(z)$:

$$\begin{aligned}U_\lambda^{\mathrm{tw}}(y)U_\mu^{\mathrm{tw}}(z) &= \left(\frac{y^{1/2}-z^{1/2}}{y^{1/2}+z^{1/2}}\right)^{(\lambda|\mu)}\Bigg|_{|y|>|z|} U_{\lambda,\mu}^{\mathrm{tw}}(y,z),\\ U_\mu^{\mathrm{tw}}(z)U_\lambda^{\mathrm{tw}}(y) &= \left(\frac{z^{1/2}-y^{1/2}}{z^{1/2}+y^{1/2}}\right)^{(\lambda|\mu)}\Bigg|_{|z|>|y|} U_{\lambda,\mu}^{\mathrm{tw}}(y,z).\end{aligned} \tag{5.7}$$

Therefore, it follows that, for sufficiently large N,

$$(y-z)^N U_\lambda^{\mathrm{tw}}(y)U_\mu^{\mathrm{tw}}(z) = (-1)^{(\lambda|\mu)}(y-z)^N U_\mu^{\mathrm{tw}}(z)U_\lambda^{\mathrm{tw}}(y),$$

since the numerators of the right-hand sides of

$$\frac{y^{1/2}-z^{1/2}}{y^{1/2}+z^{1/2}} = \frac{\left(y^{1/2}-z^{1/2}\right)^2}{y-z},\quad \frac{y^{1/2}+z^{1/2}}{y^{1/2}-z^{1/2}} = \frac{\left(y^{1/2}+z^{1/2}\right)^2}{y-z}$$

are symmetric polynomials in $y^{1/2}$ and $z^{1/2}$.

1.5.4.4 Correction of Operators

Consider the following expansion:

$$\left(y^{1/2}+z^{1/2}\right)^{-2(\lambda|\mu)} \approx 2^{-2(\lambda|\mu)}z^{-2(\lambda|\mu)/2}+\cdots.$$

Then, by the first equality in (5.7),

$$U_\lambda^{\mathrm{tw}}(y)U_\mu^{\mathrm{tw}}(z) \approx \underbrace{2^{-2(\lambda|\mu)}z^{-(\lambda|\mu)}}(y-z)^{(\lambda|\mu)}\big|_{|y|>|z|}\left(U_{\lambda+\mu}^{\mathrm{tw}}(z)+\cdots\right).$$

To remove the unpleasant factors, define, for $\lambda \in \mathfrak{h}^*$:

$$\tilde{U}_\lambda^{\mathrm{tw}}(z) = 2^{-(\lambda|\lambda)}z^{-(\lambda|\lambda)/2}U_\lambda^{\mathrm{tw}}(z). \tag{5.8}$$

Then we have

$$\tilde{U}_\lambda^{\mathrm{tw}}(y)\tilde{U}_\mu^{\mathrm{tw}}(z) \approx (y-z)^{(\lambda|\mu)}\big|_{|y|>|z|}\left(\tilde{U}_{\lambda+\mu}^{\mathrm{tw}}(z)+\cdots\right).$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Note that the equalities in (5.7) become

$$\tilde{U}^{\text{tw}}_{\lambda}(y)\tilde{U}^{\text{tw}}_{\mu}(z) = \left(\frac{y^{1/2}-z^{1/2}}{y^{1/2}+z^{1/2}}\right)^{(\lambda|\mu)}\Bigg|_{|y|>|z|}\tilde{U}^{\text{tw}}_{\lambda,\mu}(y,z),$$
$$\tilde{U}^{\text{tw}}_{\mu}(z)\tilde{U}^{\text{tw}}_{\lambda}(y) = \left(\frac{z^{1/2}-y^{1/2}}{z^{1/2}+y^{1/2}}\right)^{(\lambda|\mu)}\Bigg|_{|z|>|y|}\tilde{U}^{\text{tw}}_{\lambda,\mu}(y,z),$$

where, for $\lambda,\mu\in\mathfrak{h}^*$,

$$\tilde{U}^{\text{tw}}_{\lambda,\mu}(y,z) = 2^{-(\lambda|\lambda)-(\mu|\mu)}y^{-(\lambda|\lambda)/2}z^{-(\mu|\mu)/2}U^{\text{tw}}_{\lambda,\mu}(y,z).$$

Therefore, by the same argument as in the preceding subsection, we have, for sufficiently large N,

$$(y-z)^N\tilde{U}^{\text{tw}}_{\lambda}(y)\tilde{U}^{\text{tw}}_{\mu}(z) = (-1)^{(\lambda|\mu)}(y-z)^N\tilde{U}^{\text{tw}}_{\mu}(z)\tilde{U}^{\text{tw}}_{\lambda}(y).$$

In particular, if $(\lambda|\mu)$ is even, then the twisted vertex operators $\tilde{U}^{\text{tw}}_{\lambda}(z)$ and $\tilde{U}^{\text{tw}}_{\mu}(z)$ are locally commutative.

1.5.5 Twisted Modules for Rank One Even Lattices

In this section, we will describe θ-twisted modules over the lattice vertex algebra $\mathbf{V}_L$ with respect to a lift θ of the (-1)-involution induced from the (-1)-involution of the lattice:

$$\theta : L \longrightarrow L,\quad \lambda\mapsto -\lambda.$$

We will first describe them for the rank 1 cases in this section, and then proceed to higher-rank cases in the next section.

1.5.5.1 Shifted $\mathbf{V}_L$-Module

Let L be an even lattice of rank one, set $\mathfrak{h} = L\otimes_{\mathbb{Z}}\mathbb{C}$, and extend the bilinear form on L to $\mathfrak{h}$. Recall the lattice vertex algebra:

$$\mathbf{V}_L = \bigoplus_{\lambda\in L}\mathbf{F}_\lambda$$

Since L is of rank 1, we have $(\lambda|\mu)\in 2\mathbb{Z}$ for all $\lambda,\mu\in L$.

Consider the twisted Fock module $\mathbf{F}^{\text{tw}}$ for the twisted Heisenberg algebra and the twisted currents $h^{\text{tw}}(z)$ for $h\in\mathfrak{h}$ acting on it. Let $V^{\text{tw}}_{\lambda}(z)$ denote the twisted vertex operator $\tilde{U}^{\text{tw}}_{\lambda}(z)$ with correction factors given by (5.8), and simply call it the twisted vertex operator from now on:

$$V^{\text{tw}}_{\lambda}(z) = \tilde{U}^{\text{tw}}_{\lambda}(z).$$

Then the twisted currents and the twisted vertex operators become locally commutative, and they generate a vertex algebra by the residue products, which we denote by $\mathcal{V}^{\text{tw}}_L$:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\mathcal{V}_L^{\mathrm{tw}} = \langle h^{\mathrm{tw}}(z), V_\lambda^{\mathrm{tw}}(z) \,|\, h \in \mathfrak{h}, \lambda \in L\rangle_{\mathrm{RP}}.$$

Moreover, for all $n \in \mathbb{Z}$,

$$V_\lambda^{\mathrm{tw}}(z)_{(n)} V_\mu^{\mathrm{tw}}(z) \in \mathcal{F}_\lambda^{\mathrm{tw}}(\lambda + \mu).$$

Therefore, we have

$$\mathcal{V}_L^{\mathrm{tw}} = \bigoplus_{\lambda \in L} \mathcal{F}_\lambda^{\mathrm{tw}},$$

where $\mathcal{F}_\lambda^{\mathrm{tw}}$ is the $\mathbf{F}_0$-submodule generated by $V_\lambda^{\mathrm{tw}}(z)$, which is isomorphic to $\mathbf{F}_\lambda$ as an $\mathbf{F}_0$-module.

It is therefore likely that there exists a homomorphism of vertex algebras satisfying

$$\rho^{\mathrm{tw}} \colon \mathbf{V}_L \longrightarrow \mathcal{V}_L^{\mathrm{tw}}, \quad \mathbf{v}_\lambda \mapsto V_\lambda^{\mathrm{tw}}(z).$$

This is indeed the case, and the map ρ^{tw} is actually an isomorphism of vertex algebras, since properties enough to characterize the lattice vertex algebra have already been verified. We thus have a shifted representation

$$\rho^{\mathrm{tw}} \colon \mathbf{V}_L \longrightarrow \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z^{1/2})))$$

of the lattice vertex algebra $\mathbf{V}_L$ on the twisted Fock module $\mathbf{F}^{\mathrm{tw}}$.

1.5.5.2 Twisted $\mathbf{V}_L$-Module

The shifted $\mathbf{V}_L$-module $\mathbf{F}^{\mathrm{tw}}$ constructed in the preceding subsection can be given a structure of a twisted module with respect to a lift θ of the (-1)-involution of the lattice L.

Let θ denote the involution of $\mathfrak{h}$ induced from the (-1)-involution of L; that is,

$$\theta \colon \mathfrak{h} \longrightarrow \mathfrak{h}, \quad h \mapsto -h.$$

Then θ induces an involution of the Heisenberg algebra $\mathbf{U}(\hat{\mathfrak{h}}, 1)$, hence of $\mathbf{S}(\hat{\mathfrak{h}}_{<0})$, and of the lattice vertex algebra as

$$\theta \colon \mathbf{V}_L \longrightarrow \mathbf{V}_L, \quad P\mathbf{v}_\lambda \mapsto (\theta P)(\hat{\theta}\mathbf{v}_\lambda),$$

where $P \in \mathbf{S}(\hat{\mathfrak{h}}_{<0})$ and $\hat{\theta}\mathbf{v}_\lambda = \mathbf{v}_{-\lambda}$. The eigenspace decomposition with respect to the action of θ looks thus:

$$\mathbf{V}_L = \mathbf{V}_L^+ \oplus \mathbf{V}_L^-, \tag{5.9}$$

where $\mathbf{V}_L^\pm = \{a \in \mathbf{V}_L \mid \theta a = \pm a\}$ are the eigenspaces.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Recall the vertex algebra $\mathcal{V}_L^{\mathrm{tw}}$ of shifted series on the twisted Fock module $\mathbf{F}^{\mathrm{tw}}$ generated by the twisted currents and the twisted vertex operators. Then it decomposes as

$$\mathcal{V}_L^{\mathrm{tw}} = \mathcal{V}_L^{\mathrm{tw},+} \oplus \mathcal{V}_L^{\mathrm{tw},-},$$

where

$$\begin{aligned}\mathcal{V}_L^{\mathrm{tw},+} &= \mathcal{V}_L^{\mathrm{tw}} \cap \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))),\\ \mathcal{V}_L^{\mathrm{tw},-} &= \mathcal{V}_L^{\mathrm{tw}} \cap \mathrm{Hom}(\mathbf{F}^{\mathrm{tw}}, \mathbf{F}^{\mathrm{tw}}((z))z^{1/2}).\end{aligned}$$

By the very definition of the twisted vertex operators, we have

$$V_\lambda^{\mathrm{tw}}(z) + V_{-\lambda}^{\mathrm{tw}}(z) \in \mathcal{V}_L^{\mathrm{tw},+} \text{ and } V_\lambda^{\mathrm{tw}}(z) - V_{-\lambda}^{\mathrm{tw}}(z) \in \mathcal{V}_L^{\mathrm{tw},-}.$$

Therefore, it turns out that the shifted representation $\mathbf{F}^{\mathrm{tw}}$ is actually a θ-twisted $\mathbf{V}_L$-module by the decomposition (5.9).

Proposition 5.4 *Let L be an even lattice of rank 1. Then the twisted Fock module $\mathbf{F}^{\mathrm{tw}}$ carries a unique structure of a θ-twisted $\mathbf{V}_L$-module such that*

$$Y_{\mathbf{F}^{\mathrm{tw}}}(h_{-1}\mathbf{1}, z) = h^{\mathrm{tw}}(z) \text{ and } Y_{\mathbf{F}^{\mathrm{tw}}}(\mathbf{v}_\lambda, z) = V_\lambda^{\mathrm{tw}}(z)$$

for $h \in \mathfrak{h}$ and $\lambda \in L$, respectively.

Note 5.5. 1. There is another θ-twisted $\mathbf{V}_L$-module structure on the same space $\mathbf{F}^{\mathrm{tw}}$ given by $V_\beta^{\mathrm{tw}}(z) = -\tilde{U}_\beta^{\mathrm{tw}}(z)$ for a generator β of L. In other words, for $\lambda = n\beta$,

$$V_{n\beta}^{\mathrm{tw}}(z) = (-1)^n \tilde{U}_{n\beta}^{\mathrm{tw}}(z).$$

The two θ-twisted modules correspond to the two 1-dimensional representations of L as a group, the trivial representation and the sign representation.
2. The θ-twisted $\mathbf{V}_L$-modules for a rank one even lattice L are actually classified by representations of $L/2L \simeq \mathbb{Z}/2\mathbb{Z}$. See Subsection 1.5.6.3 for more details.

1.5.6 Twisted Modules for General Even Lattices

Let us now describe θ-twisted modules over lattice vertex algebras associated with general even lattices, where the cocycle factor has to be taken into account. We will first describe a lift θ of the (-1)-involution of the lattice L in terms of the central extension $\hat{L}$ of the lattice by $\{\pm 1\}$, and then classify $\hat{L}$-modules $\mathbf{T}$ such that the tensor products $\mathbf{F}^{\mathrm{tw}} \otimes \mathbf{T}$ carry structures of θ-twisted modules.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.5.6.1 Relation to Central Extensions

Let L be an even lattice, choose a cocycle factor $\varepsilon\colon L \times L \longrightarrow \{\pm 1\}$, and consider the corresponding central extension as in Section 1.4.5:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \hat{L} \xrightarrow{\pi} L \longrightarrow 0.$$

Let $\lambda \mapsto e_\lambda$ be a set-theoretical section of π such that $e_\lambda e_\mu = \varepsilon(\lambda,\mu)e_{\lambda+\mu}$.

Recall that the lattice vertex algebra can be written as

$$\mathbf{V}_L = \mathbf{S}(\hat{\mathfrak{h}}_{<0}) \otimes \mathbb{C}[L],$$

on which the generating series $Y(\mathbf{v}_\lambda, z)$ given by the series $V_{\lambda,\varepsilon}(z)$ defined as in (4.4) factors as

$$Y(\mathbf{v}_\lambda, z) = U_\lambda(z) \otimes e_\lambda z^{\lambda_0},$$

where $U_\lambda(z)$ is the part acting on $\mathbf{S}(\hat{\mathfrak{h}}_{<0})$ and e_λ sends e^μ to $\varepsilon(\lambda,\mu)e^{\lambda+\mu}$ so that $\mathbb{C}[L]$ becomes an $\hat{L}$-module on which the image of -1 in $\hat{L}$ acts by -1.

We are to construct a shifted $\mathbf{V}_L$-module on the tensor product of the form

$$\mathbf{M} = \mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbf{T}$$

so that the generating series for $\mathbf{v}_\lambda$ acts in the form

$$Y_{\mathbf{M}}(\mathbf{v}_\lambda, z) = \tilde{U}^{\mathrm{tw}}_\lambda(z) \otimes e^{\mathrm{tw}}_\lambda,$$

where $\tilde{U}^{\mathrm{tw}}_\lambda(z)$ refers to the series defined by (5.8) acting on $\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0})$ and e^{tw}_λ is an operator acting on $\mathbf{T}$.

Indeed, if $\mathbf{T}$ is an $\hat{L}$-module on which e^{tw}_λ operates by the action of e_λ on $\mathbf{T}$ in such a way that the image of -1 in $\hat{L}$ acts by -1, then

$$e^{\mathrm{tw}}_\lambda e^{\mathrm{tw}}_\mu = (-1)^{(\lambda|\mu)} e^{\mathrm{tw}}_\mu e^{\mathrm{tw}}_\lambda.$$

Hence the series $Y_{\mathbf{M}}(\mathbf{v}_\lambda, z)$ become locally commutative with each other, thus giving rise to a structure of a shifted $\mathbf{V}_L$-module on the space $\mathbf{M} = \mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbf{T}$ by the same argument as in the rank one case.

1.5.6.2 Lifts of (−1)-Involutions

To generalize the construction of *θ-twisted* modules for rank one lattices to the higher rank cases, let us lift the (-1)-involution of the lattice L to an automorphism of the lattice vertex algebra $\mathbf{V}_L$.

To this end, first choose a lift $\hat{\theta}$ of $\theta = -1$ on L to an automorphism of $\hat{\boldsymbol{L}}$.

$$\begin{tikzcd}
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, equal] & \hat{L} \arrow[r] \arrow[d, "\hat{\theta}"] & L \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \{\pm 1\} \arrow[r] & \hat{L} \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Note that, by definition,

$$\hat{\theta} e_\lambda \in \{\pm e_{-\lambda}\} = \{\pm (e_\lambda)^{-1}\}.$$

Such an automorphism $\hat{\theta}$ automatically becomes an involution. There is actually a canonical choice defined by $\hat{\theta} e_\lambda = (-1)^{(\lambda|\lambda)/2}(e_\lambda)^{-1}$ for each $\lambda \in L$.

Now define the action of θ on $\mathbf{V}_L$ by

$$P\mathbf{v}_\lambda \mapsto (\theta P)(\theta \mathbf{v}_\lambda),$$

where $\theta\mathbf{v}_\lambda$ is given by $1 \otimes \hat{\theta} e_\lambda$ under the identification of $\mathbf{v}_\lambda$ with $1 \otimes e_\lambda$. Then the lattice vertex algebra decomposes as

$$\mathbf{V}_L = \mathbf{V}_L^+ \oplus \mathbf{V}_L^-, \quad \text{where} \quad \mathbf{V}_L^\pm = \{a \in \mathbf{V}_L \mid \theta a = \pm a\}$$

as in the rank one case. The isomorphism class of the vertex algebra $\mathbb{V}_L^+$ does not depend on the choice of the lift $\hat{\theta}$ (cf. [43]).

1.5.6.3 Construction of Twisted $\mathbf{V}_L$-Modules

In order for the shifted representation $\mathbf{M} = \mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbf{T}$ to become a θ-twisted $\mathbf{V}_L$-module, the generating series $Y_{\mathbf{M}}(\mathbf{v}_\lambda, z) = \tilde{U}^{\mathrm{tw}}_\lambda(z) \otimes e^{\mathrm{tw}}_\lambda$ must satisfy

$$\begin{aligned} &Y_{\mathbf{M}}(\mathbf{v}_\lambda, z) + Y_{\mathbf{M}}(\theta\mathbf{v}_\lambda, z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))), \\ &Y_{\mathbf{M}}(\mathbf{v}_\lambda, z) - Y_{\mathbf{M}}(\theta\mathbf{v}_\lambda, z) \in \mathrm{Hom}(\mathbf{M}, \mathbf{M}((z))z^{1/2}). \end{aligned}$$

Since $\tilde{U}^{\mathrm{tw}}_\lambda(z)$ already satisfy this property, it remains to impose the following condition on the $\hat{L}$-module $\mathbf{T}$: for all $\lambda \in L$,

$$(\hat{\theta} e_\lambda)\big|_{\mathbf{T}} = e_\lambda\big|_{\mathbf{T}}.$$

To describe it, consider the following set:

$$K = \left\{(\theta g)g^{-1} \,\middle|\, g \in \hat{\boldsymbol{L}}\right\} \subset \hat{\boldsymbol{L}}.$$

Then the condition holds if and only if K acts on $\mathbf{T}$ by 1.

It is readily checked that K is a central subgroup of $\hat{L}$, which fits in the following commutative diagram of exact sequences:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, "\sim"] \arrow[d] & 2L \arrow[d] & \\
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, equal] & \hat{L} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{\pm 1\} \arrow[r] & \hat{L}/K \arrow[r] \arrow[d] & L/2L \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Therefore, the space $\mathbf{S}(\hat{\mathfrak{h}}^{\mathrm{tw}}_{<0}) \otimes \mathbf{T}$ carries a structure of a θ-twisted $\mathbf{V}_L$-module if and only if $\mathbf{T}$ is an $\hat{L}/K$-module on which $(-1)K$ acts by -1.

To classify such $\hat{L}/K$-modules, let R be the radical of the commutator map modulo 2:

$$R = \{\lambda \in L \,|\, (\lambda|L) \subset 2\mathbb{Z}\}\,.$$

Then the inverse image of R in $\hat{L}$ agrees with the center of $\hat{L}$, and the number of central characters of $\hat{L}/K$ of which $(-1)K$ acts by -1 agrees with $|R/2L|$. For each such character χ, we have a unique $\hat{L}$-module $\mathbf{T}_\chi$ satisfying the desired properties, whose dimension is given by

$$\dim_{\mathbb{C}} \mathbf{T}_\chi = |L/R|^{1/2}.$$

See [1] for details.

Let us finally mention the case when L is the Leech lattice Λ, the unique even unimodular positive-definite lattice of rank 24 without roots. Being unimodular implies that the commutator map induces a nondegenerate bilinear form on $\Lambda/2\Lambda$, hence the radical R agrees with 2Λ. Therefore, there exists a unique $\hat{\Lambda}$-module $\mathbf{T}$ of which $(-1)K$ acts by -1, and its dimension is given by

$$\dim_{\mathbb{C}} \mathbf{T} = |\Lambda/2\Lambda|^{1/2} = \sqrt{2^{24}} = 2^{12}.$$

The central extension $\hat{\Lambda}/K$ is actually the extraspecial 2-group 2^{1+24}_+ for the canonical choice of the lift θ.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Bibliographic Notes

Main references for Section 1.5 are Frenkel et al. [1], Dong [41], and Li [71]. For descriptions of twisted modules in models in physics, consult Di Francesco et al. [13] (cf. Ginsparg [19]).

Our treatments by introducing the concepts of shifted series and shifted representation are more or less straightforward generalizations of those described in [71] along the line of [7]. See Roitman [89], Frenkel and Szczensny [56], and Doyon et al. [51] for related works.

Properties of twisted vertex operators are described in detail in [1] by explicit calculations. The idea of using characterization of lattice vertex algebras is stated in [89]. For earlier works on twisted vertex operators, see Lepowsky and Wilson [69], Frenkel, Lepowsky, and Meurman [58], and Lepowsky [68].

For twisted module for general finite-order automorphisms of lattice vertex algebras, see Dong and Lepowsky [45] and Bakalov and Kac [30].

Often useful in constructing twisted modules is an operator $\Delta(z)$ introduced by H. S. Li [71], called the *Delta operator*, which transforms a twisted module to another.

Under certain circumstances, existence of some twisted modules is guaranteed by general theory of modular invariance by Dong, Li, and Mason [46].

1.6 Vertex Operator Algebras

Recall that standard examples of vertex algebras, such as the lattice vertex algebra associated with a positive-definite even lattice, often admit an action of the Virasoro algebra that is *internal* in the sense that it is given by the actions of an element of the vertex algebra. Such a vector is called a *conformal vector* if the Virasoro action includes the translation operator. The presence of a conformal vector leads to additional features such as gradings and various transformation formulas.

A vertex algebra may have many conformal vectors, and it is natural to specify one and to impose appropriate conditions on it. The resulting concept is that of a *vertex operator algebra* (VOA). Thus a VOA is a pair $(\mathbf{V}, \omega)$ of a vertex algebra $\mathbf{V}$ and a conformal vector ω satisfying a number of conditions.

Section 1.6 is a brief introduction to theory of vertex operator algebras. We will start by describing consequences of the presence of a conformal vector and then give terminologies specific to VOAs. As a VOA is graded, we may talk about the category of $\mathbb{N}$-graded modules for which the simple objects are controlled by the top subspaces by means of *Zhu's algebra*, an associative algebra universally constructed by the VOA structure alone without addressing

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

modules. Two more important topics, fusion rules among modules and modular invariance of conformal characters, are also included.

We will work over the field $\mathbb{C}$ of complex numbers.

1.6.1 Conformal Vectors

In this section, we will give the precise definition of conformal vectors of a vertex algebra and describe various consequences of the presence of a conformal vector. We will show that a grading is given to a vertex algebra by L_0 action for the choice of the conformal vector and that various transformation formulas hold by a part of the Virasoro action.

1.6.1.1 Virasoro Vectors in Vertex Algebras

Let $\mathbf{V}$ be a vertex algebra. Recall that an element $e \in \mathbf{V}$ is called a *Virasoro vector* if there exists a scalar c_e, called the *central charge*, such that

$$e_{(n)}e = \begin{cases} 0 & (n \geq 4), \\ (c_e/2)\mathbf{1} & (n = 3), \\ 2e & (n = 1), \end{cases}$$

from which the properties $e_{(0)}e = Te$ and $e_{(2)}e = 0$ follow, where $T = T^{(1)}$ is the translation operator.

Set $L^e_n = e_{(n+1)}$ so that

$$Y(e,z) = \sum_n L^e_n z^{-n-2}.$$

Then the condition above is equivalent to the OPE

$$Y(e,y)Y(e,z) \simeq Y(e,z)Y(e,y) \sim \frac{\partial Y(e,z)}{y-z} + \frac{2Y(e,z)}{(y-z)^2} + \frac{c_e/2}{(y-z)^4},$$

or to the Virasoro commutation relation

$$[L^e_m, L^e_n] = (m-n)L^e_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c_e.$$

Here we have identified a scalar with multiplication by it.

1.6.1.2 Conformal Vectors in Vertex Algebras

Let $\mathbf{V}$ be a vertex algebra and recall that the translation operator $T = T^{(1)}$ is given by

$$T\colon \mathbf{V} \longrightarrow \mathbf{V}, \quad a \mapsto Ta = a_{(-2)}\mathbf{1}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let ω be a Virasoro vector of $\mathbf{V}$. Denote the actions L_n^{ω} simply by L_n and the central charge c_{ω} by c:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c.$$

Such a vector ω is called a *conformal vector* of $\mathbf{V}$ if, for all $a \in \mathbf{V}$,

$$\omega_{(0)}a = a_{(-2)}\mathbf{1},$$

that is, L_{-1} agrees with the translation operator T as operators acting on $\mathbf{V}$:

$$L_{-1} = T \colon \mathbf{V} \longrightarrow \mathbf{V}.$$

Then, for a conformal vector ω, the translation property (VT) with $k = 1$ gives

$$(L_{-1}a)_{(n)}b = -na_{(n-1)}b. \tag{6.1}$$

Namely:

$$Y(L_{-1}a, z) = \partial_z Y(a, z). \tag{6.2}$$

The standard Virasoro vectors for the Heisenberg vertex algebras, the lattice vertex algebras, the affine vertex algebras, and the Virasoro vertex algebras are conformal vectors of the vertex algebras under consideration.

Note 6.1. 1. A conformal vector in our sense is often called a *Virasoro element* in the literatures (cf. [1], [32]), and a Virasoro vector in our sense is sometimes called a conformal vector (cf. [83]). 2. The Virasoro vector obtained by Sugawara construction is a conformal vector of the affine vertex algebra as long as it makes sense. 3. Under the presence of a conformal vector ω, left ideals of the vertex algebra become two-sided ideals by skew-symmetry; for $T = \omega_{(0)}$ is one of the left actions.

1.6.1.3 Grading by Conformal Weights

Let ω be a conformal vector of a vertex algebra $\mathbf{V}$ and consider the action of L_0 on $\mathbf{V}$. Then, for $a, b \in \mathbf{V}$,

$$L_0(a_{(n)}b) = (L_0a)_n b + a_{(n)}(L_0 b) - (n+1)a_n b \tag{6.3}$$

by the Borcherds identity (V1) with $(p, q, r) = (1, n, 0)$ and (6.1).

Now assume that the operator L_0 is semisimple on $\mathbf{V}$ with integral eigenvalues and consider the eigenspace decomposition:

$$\mathbf{V} = \bigoplus_d \mathbf{V}_d, \quad \text{where } \mathbf{V}_d = \operatorname{Ker}(L_0 - d)|_{\mathbf{V}}. \tag{6.4}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The eigenvalue of a homogeneous element a is called the *conformal weight* of a with respect to ω, which we denote by $\Delta(a)$:

$$a \in \mathbf{V}_d \iff \Delta(a) = d.$$

For example, since $L_0\mathbf{1} = \omega_{(1)}\mathbf{1} = 0$ and $L_0\omega = \omega_{(1)}\omega = 2\omega$,

$$\Delta(\mathbf{1}) = 0 \ \text{ and } \ \Delta(\omega) = 2.$$

The relation (6.3) implies

$$\mathbf{V}_{d\,(n)}\mathbf{V}_e \subset \mathbf{V}_{d+e-n-1}\,. \tag{6.5}$$

Namely:

$$\Delta(a_{(n)}b) = \Delta(a) + \Delta(b) - n - 1.$$

Note 6.2. A direct sum decomposition of $\mathbf{V}$ satisfying the relation (6.5) in general is called a *grading* of the vertex algebra $\mathbf{V}$.

1.6.1.4 Projective Linear Transformations

Let ω be a conformal vector and consider the actions L_m with $m = -1, 0, 1$. Then, as a part of the Virasoro commutation relations, we have

$$[L_0, L_{-1}] = L_{-1}, \ \ [L_0, L_1] = -L_1, \ \ [L_1, L_{-1}] = 2L_0.$$

Thus the actions L_{-1}, L_0, L_1 give rise to a representation of $\mathfrak{sl}_2$ on $\mathbf{V}$ by identifying $L_{-1} = E$, $L_0 = (1/2)H$, and $L_1 = -F$, for which

$$\exp xE = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \ \ \exp xH = \begin{bmatrix} e^x & 0 \\ 0 & e^{-x} \end{bmatrix}, \ \ \exp xF = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}.$$

The corresponding projective linear transformations are given by the substitutions

$$\exp xL_{-1} : z \mapsto z + x, \ \ \exp xL_0 : z \mapsto e^x z, \ \ \exp xL_1 : z \mapsto \frac{z}{1 - xz}.$$

Moreover, the operators L_{-1}, L_0, L_1 generate formal actions on the space of generating series $Y(a, z)$ with $a \in \mathbf{V}$ corresponding to projective linear transformations of the variable z. Indeed, by the commutator formula (VC),

$$[L_m, a_{(n)}] = \sum_{i=0}^{\infty} \binom{m+1}{i} (L_{i-1}a)_{(m+n-i)}.$$

In particular, for $m = -1, 0, 1$, we have, by the translation property (6.2),

$$\begin{aligned}
[L_{-1}, Y(a, z)] &= \partial_z Y(a, z), \\
[L_0, Y(a, z)] &= z\partial_z Y(a, z) + Y(L_0 a, z), \\
[L_1, Y(a, z)] &= z^2 \partial_z Y(a, z) + 2zY(L_0 a, z) + Y(L_1 a, z).
\end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Then, by exponentiation,

$$\begin{aligned} e^{xL_{-1}}Y(a,z)e^{-xL_{-1}} &= Y(a,z+x), \\ x^{L_0}Y(a,z)x^{-L_0} &= Y(x^{L_0}a,xz), \\ e^{xL_1}Y(a,z)e^{-xL_1} &= Y\left(e^{x(1-xz)L_1}(1-xz)^{-2L_0}a, \frac{z}{1-xz}\right). \end{aligned} \tag{6.6}$$

The first one is the translation covariance.

1.6.1.5 Formal Change of Coordinates

The transformations corresponding to L_0 and L_1, which send z to xz and to $z/(1-xz)$, respectively, are thought of as coordinate transformations that fix the origin, thus forming a part of more general transformations of the affine line $\mathbb{C}$ generated by vector fields in $z\mathbb{C}[[z]](d/dz)$ under the isomorphism

$$z\mathbb{C}[[z]]\frac{d}{dz} \longrightarrow \bigoplus_{n\geq 0}\mathbb{C}L_n, \quad z^{n+1}\frac{d}{dz} \mapsto -L_n \tag{6.7}$$

of Lie algebra.

Assume that $\mathbf{V}$ is graded by integral conformal weights by L_0 for a conformal vector, and that the actions of $L_1, L_2, \cdots$ are locally nilpotent on $\mathbf{V}$.

Let $\phi\colon t \mapsto \phi(t)$ be a formal change of variables such that

$$\phi(t) = \phi_1 t + \phi_2 t^2 + \cdots \quad \text{with} \quad \phi_1 \neq 0.$$

Let $c_0, c_1, c_2, \cdots$ be a sequence of complex numbers such that

$$\phi(t) = \left(\exp \sum_{j=1}^{\infty} c_j t^{j+1}\partial_t\right)c_0^{t\partial_t}t.$$

Accordingly, define an operator $R(\phi)$ acting on the vertex algebra $\mathbf{V}$ by the Virasoro action via the embedding (6.7) as

$$R(\phi) = \exp\left(-\sum_{j=1}^{\infty} c_j L_j\right)c_0^{-L_0}.$$

Here, by definition, $c_0^{t\partial_t}t^n = c_0^n t^n$ and $c_0^{-L_0}a = c_0^{-\Delta(a)}a$ for a homogeneous $a \in \mathbf{V}$.

Consider the series $\phi_z(t)$ in t with coefficients in $z\mathbb{C}[[z]]$ defined by

$$\phi_z(t) = \phi(z+t) - \phi(z).$$

Then, for $a \in \mathbf{V}$ and ϕ as above,

$$R(\phi)^{-1}Y(a,z)R(\phi) = Y(R(\phi_z)^{-1}a, \phi(z)). \tag{6.8}$$

This formula is called *Huang's formula.*

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.2 Vertex Operator Algebras and their Modules

This section is devoted to giving the precise definition of a vertex operator algebra (VOA) and the concepts of modules of various types for it. We will be selecting the types of modules to work with according to the purpose.

1.6.2.1 Vertex Operator Algebras

A *vertex operator algebra* (VOA) is a pair $(\mathbf{V}, \omega)$ of a vertex algebra $\mathbf{V}$ and a conformal vector ω of $\mathbf{V}$ satisfying the following conditions:

(1) The action of L_0 is semisimple on $\mathbf{V}$ with integral eigenvalues.
(2) The eigenvalues of L_0 are bounded below.
(3) The eigenspaces for L_0 are finite-dimensional.

The central charge of ω is called the *central charge* of the VOA $(\mathbf{V}, \omega)$.

As usual, we alternatively say that a VOA is a vertex algebra $\mathbf{V}$ *equipped with* a conformal vector ω satisfying (1)–(3), and often refer to $(\mathbf{V}, \omega)$ by $\mathbf{V}$ without mentioning ω explicitly.

By (1), the underlying vertex algebra $\mathbf{V}$ is graded by the conformal weights as in (6.4). By (3), the eigenspaces are finite-dimensional:

$$\dim \mathbf{V}_d < \infty \quad \text{for all} \quad d \in \mathbb{Z}.$$

By (2), the grading is of the following shape:

$$\mathbf{V} = \underbrace{\mathbf{V}_{-d_0} \oplus \cdots \oplus \mathbf{V}_{-1}}_{\text{negative degrees}} \oplus \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \cdots .$$

In other words, the conformal weights are bounded below.

A VOA $(\mathbf{V}, \omega)$ is said to be *of CFT type* if the grading satisfies $\mathbf{V}_d = 0$ for all $d < 0$ and $\mathbf{V}_0 = \mathbb{C}\mathbf{1}$:

$$\mathbf{V} = \mathbb{C}\mathbf{1} \oplus \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots .$$

Many typical examples of VOAs, such as Heisenberg VOAs, lattice VOAs associated with positive-definite even lattices, Virasoro VOAs, affine VOAs associated with finite-dimensional simple Lie algebras at general levels, etc., are of CFT type under the standard choices of the conformal vectors.

Note 6.3. A pair $(\mathbf{V}, \omega)$ of a vertex algebra $\mathbf{V}$ and a conformal vector $\omega \in \mathbf{V}$ is called a *conformal vertex algebra* if it satisfies (1).

1.6.2.2 Weak Modules and Ordinary Modules

A module over the underlying vertex algebra $\mathbf{V}$ of a VOA $(\mathbf{V}, \omega)$ is a called a *weak module* for the VOA.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

For a weak module $\mathbf{M}$, consider the action of L_0 on $\mathbf{M}$. Then, we have a counterpart of the property (6.3) for modules: for $a \in \mathbf{V}$ and $v \in \mathbf{M}$,

$$L_0(a_n v) = a_n(L_0 v) + (L_0 a)_n v - (n+1)a_n v. \tag{6.9}$$

Assume that $\mathbf{M}$ has an eigenspace decomposition

$$\mathbf{M} = \bigoplus_{\lambda \in \mathbb{C}} \mathbf{M}_\lambda, \quad \text{where} \quad \mathbf{M}_\lambda = \operatorname{Ker}(L_0 - \lambda)\big|_{\mathbf{M}}.$$

Let us call such a module a *weight module* for $(\mathbf{V}, \omega)$ and the eigenspace $\mathbf{M}_\lambda$ the *weight space* of *conformal weight* λ.

Let us write $\Delta(v) = \lambda$ when $v \in \mathbf{M}_\lambda$. Then, by (6.9), we have

$$\Delta(a_n v) = \Delta(a) + \Delta(v) - n - 1. \tag{6.10}$$

In other words, for $d, n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$,

$$\rho_n(\mathbf{V}_d)\,\mathbf{M}_\lambda \subset \mathbf{M}_{\lambda+d-n-1},$$

where ρ_n denotes the n-th action of $\mathbf{V}$ on $\mathbf{M}$.

A weight module $\mathbf{M}$ is called an *ordinary module* (or just a *module*) if the weight spaces are finite-dimensional and the real part of the conformal weight in every coset in $\mathbb{C}/\mathbb{Z}$ is bounded below. That is, for any coset in $\mathbb{C}/\mathbb{Z}$, there exists a representative λ such that

$$\bigoplus_{n \in \mathbb{Z}} \mathbf{M}_{\lambda+n} = \mathbf{M}_\lambda \oplus \mathbf{M}_{\lambda+1} \oplus \cdots \quad \text{and} \quad \dim \mathbf{M}_{\lambda+n} < \infty \quad (n \in \mathbb{Z}).$$

The adjoint module $\mathbf{V}$ is an ordinary module by the conditions (1)–(3).

If $\mathbf{M}$ is a simple ordinary module, then there exists a complex number λ, called the *lowest conformal weight* of $\mathbf{M}$, such that

$$\mathbf{M} = \mathbf{M}_\lambda \oplus \mathbf{M}_{\lambda+1} \oplus \cdots \quad \text{with} \quad \mathbf{M}_\lambda \neq 0,$$

and the whole space $\mathbf{M}$ is generated by $\mathbf{M}_\lambda$ as a module over $\mathbf{V}$.

Note 6.4. 1. For ordinary modules, we may consider the *graded dimensions* and the *conformal characters* (cf. Section 1.6.5). 2. Assume that $\mathbf{M}$ has a generalized eigenspace decomposition with respect to L_0 and write $\Delta(v) = \lambda$ when v is in the generalized eigenspace of conformal weight λ. Then (6.10) also holds true for generalized eigenspaces.

1.6.2.3 ℕ-Graded Modules

Let us set up an appropriate category of modules for a VOA to work with in practice for representation theory.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

An $\mathbb{N}$-*graded module* for a VOA $(\mathbf{V},\omega)$ is a weak module $\mathbf{M}$ equipped with a grading of the form

$$\mathbf{M} = \bigoplus_{k=0}^{\infty} \mathbf{M}(k) = \underbrace{\mathbf{M}(0)}_{\text{top}} \oplus \mathbf{M}(1) \oplus \mathbf{M}(2) \oplus \cdots$$

satisfying the condition that, for all $k \in \mathbb{N}$ and $d, n \in \mathbb{Z}$,

$$\rho_n(\mathbf{V}_d)\,\mathbf{M}(k) \subset \mathbf{M}(k+d-n-1).$$

A weak module is said to be $\mathbb{N}$-*gradable* if there exists a grading that makes it into an $\mathbb{N}$-graded module. Note that a simple $\mathbb{N}$-gradable module $\mathbf{M}$ can be given an $\mathbb{N}$-grading such that $\mathbf{M}$ is generated as a module over $\mathbf{V}$ by the top subspace $\mathbf{M}(0)$.

For example, a weight module $\mathbf{M}$ is $\mathbb{N}$-gradable if the real part of the conformal weight in every coset in $\mathbb{C}/\mathbb{Z}$ is bounded below. Indeed, such a module is $\mathbb{N}$-graded by

$$\mathbf{M}(k) = \bigoplus_{i \in I} \mathbf{M}_{\lambda_i + k}, \quad k = 0, 1, 2, \cdots,$$

where λ_i are the minimal representatives of the conformal weights modulo $\mathbb{Z}$. In particular, ordinary modules are $\mathbb{N}$-gradable.

The category of $\mathbb{N}$-graded modules is the category of which the objects are $\mathbb{N}$-graded modules and the morphisms are homomorphisms of modules over the underlying vertex algebra $\mathbf{V}$. Thus a morphism in this category need not respect the gradings.

Note 6.5. An $\mathbb{N}$-graded module is also called an *admissible module* in the literatures, although the same term is used in a different sense for affine Kac–Moody algebras.

1.6.2.4 Rationality, C_2-Cofiniteness, and Regularity

Let us briefly describe conditions on a VOA and their consequences that guarantee finiteness or semisimplicity of module categories and play prominent roles in representation theory of VOAs.

1. A VOA is said to be *rational* if any $\mathbb{N}$-graded module is a direct sum of simple $\mathbb{N}$-graded modules. In other words, the category of $\mathbb{N}$-graded module is semisimple,

   A rational VOA has only finitely many (isomorphism classes of) simple $\mathbb{N}$-gradable modules and they are ordinary.
2. A VOA $(\mathbf{V},\omega)$ is said to be C_2-*cofinite*, if the following condition holds:

$$\dim \mathbf{V}/\mathbf{V}_{(-2)}\mathbf{V} < \infty.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

A C_2-cofinite VOA *of CFT type* has only finitely many simple weak modules and they are ordinary. Moreover, any weak module is $\mathbb{N}$-gradable.

3. A VOA is said to be *regular* if any weak module is a direct sum of simple ordinary modules.

   A regular VOA is rational and C_2-cofinite. If a VOA *of CFT type* is rational and C_2-cofinite, then it is regular.

Since the Heisenberg VOA has infinitely many simple modules, it is neither rational nor C_2-cofinite, so not regular either. The lattice VOAs associated with positive-definite lattices, the affine VOAs associated with integrable representations of affine Kac–Moody algebras, and the VOAs associated with the Virasoro minimal models are known to be regular, hence rational and C_2-cofinite.

1.6.2.5 Contragredient Modules

Let $\mathbf{M}$ be an ordinary module for a VOA $(\mathbf{V}, \omega)$. By composition of the actions given by (6.6), the transformation corresponding to $z \mapsto 1/z$ is found as

$$Y_{\mathbf{M}}(a,z) \mapsto Y_{\mathbf{M}}(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1}).$$

We will use this to define the concept of the contragredient module.

To this end, consider the restricted dual of $\mathbf{M}$ defined by

$$\mathbf{M}' = \bigoplus_{\lambda \in \mathbb{C}} \mathbf{M}_\lambda^*$$

where $\mathbf{M}_\lambda^* = \mathrm{Hom}_{\mathbb{C}}(\mathbf{M}_\lambda, \mathbb{C})$ is the dual space. Denote the canonical pairing by

$$\mathbf{M}' \times \mathbf{M} \longrightarrow \mathbb{C}, \quad (\varphi, v) \mapsto \langle \varphi, v \rangle,$$

and the action of $\mathbf{V}$ on $\mathbf{M}'$ by

$$\langle Y_{\mathbf{M}'}(a,z)\varphi, v\rangle = \langle \varphi, Y_{\mathbf{M}}(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})v\rangle.$$

Then $\mathbf{M}'$ becomes an ordinary module for $(\mathbf{V}, \omega)$, called the *contragredient module* or the *dual module* of $\mathbf{M}$.

1.6.3 Simple $\mathbb{N}$-Graded Modules

In this section, we will explain a way to develop representation theory of VOAs in the category of $\mathbb{N}$-graded modules. Recall that an $\mathbb{N}$-graded module has the shape

$$\mathbf{M} = \underbrace{\mathbf{M}(0)}_{\text{top}} \oplus \mathbf{M}(1) \oplus \mathbf{M}(2) \oplus \cdots,$$

where the grading is not necessarily given by the L_0 eigenvalues.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Since the actions of homogeneous elements of **V** raise, preserve, or lower the degrees, we can apply an analogue of "highest weight theory" to our categories.

Thus, to classify $\mathbb{N}$-graded simple **V**-modules, we will classify the top subspace as a module over a certain algebra that induces actions preserving the degree, called the *zero-mode actions*, and reconstruct the whole **M** as a simple quotient of the induced module.

Such an algebra can be universally constructed, denoted $\mathbf{A}(\mathbf{V})$, and called *Zhu's algebra* associated with the VOA $(\mathbf{V}, \omega)$.

1.6.3.1 Lie Algebra of Fourier Modes

Let **V** be a vertex algebra and **M** a module over it. It follows from the commutator formula (MC),

$$[a_m, b_n] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{m+n-i},$$

that the actions of elements of **V** on **M** form a Lie subalgebra of End **M**, called the Lie algebra of *Fourier modes* on **M**.

Such a Lie algebra is actually the image of a universal one defined by taking the commutator formula as the defining relation. Indeed, consider the space

$$\hat{\mathbf{V}} = \mathbf{V} \otimes \mathbb{C}[t, t^{-1}],$$

and equip it with the bracket operation given by

$$[a \otimes t^m, b \otimes t^n] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b) \otimes t^{m+n-i}. \tag{6.11}$$

Let $\hat{\boldsymbol{T}}$ denote the operator on $\hat{\mathbf{V}}$ defined by

$$\hat{\boldsymbol{T}}(a \otimes t^n) = (Ta) \otimes t^n + na \otimes t^{n-1}.$$

Then the bracket operation on $\hat{\mathbf{V}}$ induces a Lie algebra structure on the quotient

$$\mathfrak{g}(\mathbf{V}) = \mathbf{V} \otimes \mathbb{C}[t, t^{-1}] / \hat{\boldsymbol{T}}(\mathbf{V} \otimes \mathbb{C}[t, t^{-1}])$$

and any module **M** over **V** can be regarded as a $\mathfrak{g}(\mathbf{V})$-module by

$$(a \otimes t^m)\, v = a_m v \quad (a \in \mathbf{V},\ v \in \mathbf{M}).$$

We will call the Lie algebra $\mathfrak{g}(\mathbf{V})$ the *Lie algebra of Fourier modes* associated with **V**, which covers the Lie algebra of Fourier modes on every module.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.3.2 Triangular Decomposition

Let $(\mathbf{V}, \omega)$ be a VOA. For a homogeneous element $a \in \mathbf{V}$ and an integer n, let $J_n(a)$ denote the image of $a \otimes t^{n+\Delta(a)-1}$ in the quotient $\mathfrak{g}(\mathbf{V})$:

$$J_n(a) = \pi(a \otimes t^{n+\Delta(a)-1}).$$

The definition of the Lie bracket (6.11) turns out to be

$$[J_m(a), J_n(b)] = \sum_{i=0}^{\infty} \binom{m + \Delta(a) - 1}{i} J_{m+n}(a_{(i)}b).$$

The action of $J_n(a)$ on an $\mathbb{N}$-graded module lowers the degree by n:

$$J_n(a)\colon \mathbf{M}(k) \longrightarrow \mathbf{M}(k-n).$$

The Lie algebra $\mathfrak{g}(\mathbf{V})$ is $\mathbb{Z}$-graded by the degree as

$$\mathfrak{g}(\mathbf{V}) = \bigoplus_n \mathfrak{g}_n(\mathbf{V}),$$

where $\mathfrak{g}_n(\mathbf{V})$ is the subspace of degree n:

$$\mathfrak{g}_n(\mathbf{V}) = \mathrm{Span}\,\{J_n(a) \,|\, a \in \mathbf{V}_d \;\; (d \in \mathbb{Z})\}\,.$$

We have the *triangular decomposition*

$$\mathfrak{g}(\mathbf{V}) = \mathfrak{g}_{<0}(\mathbf{V}) \oplus \mathfrak{g}_0(\mathbf{V}) \oplus \mathfrak{g}_{>0}(\mathbf{V}),$$

where $\mathfrak{g}_{<0}(\mathbf{V})$ and $\mathfrak{g}_{>0}(\mathbf{V})$ are the sums of subspaces of negative and positive degrees, respectively.

In particular, the action of an element of $\mathfrak{g}_0(\mathbf{V})$ is called the *zero-mode action*, usually denoted $o(a)$ for a homogeneous element $a \in \mathbf{V}$. Thus

$$o(a) = J_0(a) = a_{\Delta(a)-1} : \mathbf{M}(k) \longrightarrow \mathbf{M}(k),$$

where we have identified $J_0(a)$ in $\mathfrak{g}_0(\mathbf{V})$ with its action on $\mathbf{M}$.

1.6.3.3 Zhu's Algebra

For a VOA $(\mathbf{V}, \omega)$, the Borcherds identity (M1) for modules reads, for homogeneous elements $a, b \in \mathbf{V}$,

$$\begin{aligned}
&\sum_{i=0}^{\infty} \binom{m + \Delta(a) - 1}{i} J_{m+n+r}(a_{(r+i)}b) \\
&\quad = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} J_{m+r-i}(a) J_{n+i}(b) \\
&\qquad - \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} J_{n+r-i}(b) J_{m+i}(a).
\end{aligned} \tag{6.12}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Since the top space $\mathbf{M}(0)$ of an $\mathbb{N}$-graded module $\mathbf{M}$ is annihilated by $\mathfrak{g}_{>0}(\mathbf{V})$ and preserved by $\mathfrak{g}_0(\mathbf{V})$, the relation (6.12) with $(m,n,r) = (1,0,-1)$ reads

$$\sum_{i=0}^{\infty} \binom{\Delta(a)}{i} o(a_{(i-1)}b)\big|_{\mathbf{M}(0)} = o(a)o(b)\big|_{\mathbf{M}(0)}$$

and, with $(m,n,r) = (1,1,-2)$,

$$\sum_{i=0}^{\infty} \binom{\Delta(a)}{i} o(a_{(i-2)}b)\big|_{\mathbf{M}(0)} = 0.$$

We take these relations as defining a universal algebra acting on the top space $\mathbf{M}(0)$ for every $\mathbb{N}$-graded module $\mathbf{M}$.

We therefore define, for homogeneous $a, b \in \mathbf{V}$, the elements $a * b$ and $a \circ b$ by setting

$$a * b = \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(i-1)}b, \quad a \circ b = \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(i-2)}b. \tag{6.13}$$

Then it is not difficult to see that the operation $*$ induces a structure of an associative algebra on the quotient defined by the operation $\circ$ as

$$\mathbf{A}(\mathbf{V}) = \mathbf{V}/\mathbf{V} \circ \mathbf{V}.$$

The algebra $\mathbf{A}(\mathbf{V})$ thus obtained is called *Zhu's algebra.* By abuse of notation, we will denote the elements of $\mathbf{A}(\mathbf{V})$ by their representatives in $\mathbf{V}$.

The top space $\mathbf{M}(0)$ becomes an $\mathbf{A}(\mathbf{V})$-module by

$$\mathbf{A}(\mathbf{V}) \times \mathbf{M}(0) \longrightarrow \mathbf{M}(0), \quad (a, v) \mapsto o(a)v = J_0(a)v,$$

since $o(a * b)\big|_{\mathbf{M}(0)} = o(a)o(b)\big|_{\mathbf{M}(0)}$ and $o(a \circ b)\big|_{\mathbf{M}(0)} = 0$ holds by construction.

1.6.3.4 Zhu's One-to-One Correspondence

Let $\mathfrak{g}(\mathbf{V}) = \mathfrak{g}_{<0}(\mathbf{V}) \oplus \mathfrak{g}_0(\mathbf{V}) \oplus \mathfrak{g}_{>0}(\mathbf{V})$ be the Lie algebra of Fourier modes, and consider the Lie subalgebra $\mathfrak{g}_{\geq 0}(\mathbf{V}) = \mathfrak{g}_0(\mathbf{V}) \oplus \mathfrak{g}_{>0}(\mathbf{V})$. Note that the identity map induces a surjection

$$\pi_0 \colon \mathfrak{g}_0(\mathbf{V}) \longrightarrow \mathbf{A}(\mathbf{V}),$$

which sends the Lie bracket of $\mathfrak{g}_0(\mathbf{V})$ to the commutators in Zhu's algebra.

For a simple $\mathbb{N}$-graded $\mathbf{V}$-module $\mathbf{M} = \mathbf{M}(0) \oplus \mathbf{M}(1) \oplus \cdots$, the top $\mathbf{M}(0)$ is seen to be an $\mathbf{A}(\mathbf{V})$-module.

Conversely, for any simple $\mathbf{A}(\mathbf{V})$-module W, regard it as a $\mathbf{U}(\mathfrak{g}_0(\mathbf{V}))$-module via the surjection π_0 and further as a $\mathbf{U}(\mathfrak{g}_{\geq 0}(\mathbf{V}))$-module by letting $\mathfrak{g}_{>0}(\mathbf{V})$ act by 0, and consider the induced module:

$$\mathbf{M}(W) = \mathbf{U}(\mathfrak{g}(\mathbf{V})) \otimes_{\mathbf{U}(\mathfrak{g}_{\geq 0}(\mathbf{V}))} W.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Then a simple $\mathbb{N}$-graded module for $(\mathbf{V}, \omega)$ can be constructed as an appropriate quotient of $\mathbf{M}(W)$.

In this way, we arrive at the following result.

Theorem 6.6 (Zhu) *There exists a one-to-one correspondence between the isomorphism classes of simple $\mathbb{N}$-gradable modules for a VOA $(\mathbf{V}, \omega)$ and the isomorphism classes of simple $\mathbf{A}(\mathbf{V})$-modules.*

Note 6.7. Zhu's algebra plays an important role also in modular invariance of conformal characters (cf. Section 1.6.5).

1.6.4 Fusion Rules

Let $\mathbf{C}$ be a commutative associative algebra and let $\mathbf{L}, \mathbf{M}, \mathbf{N}$ be submodules of $\mathbf{C}$. Then the multiplication on $\mathbf{C}$ restricts to a map

$$\mathbf{L} \times \mathbf{M} \longrightarrow \mathbf{N}$$

satisfying properties coming from those of $\mathbf{C}$; that is, $(au)v = u(av) = a(uv)$ for $a \in \mathbf{C}$, $u \in \mathbf{L}$, and $v \in \mathbf{N}$. Let us now replace the submodules of $\mathbf{C}$ by arbitrary modules and consider a map $\mathcal{Y}\colon \mathbf{L} \times \mathbf{M} \longrightarrow \mathbf{N}$ satisfying

$$\mathcal{Y}(au)v = \mathcal{Y}(u)av = a\mathcal{Y}(u)v.$$

The first equality guarantees that $\mathcal{Y}$ induces a map $\mathbf{L} \otimes_{\mathbf{A}} \mathbf{M} \longrightarrow \mathbf{N}$, which becomes a homomorphism of modules over $\mathbf{C}$ by the second equality.

Such a map constructed for a triple of modules can be seen to motivate the concept of an *intertwining operator* for a VOA. For example, for the lattice VOA $\mathbf{V}_L$ associated with an even lattice L and $\lambda, \mu \in L$, consider the triple $\mathbf{F}_\lambda, \mathbf{F}_\mu, \mathbf{F}_{\lambda+\mu}$ of modules for the Heisenberg VOA $\mathbf{F}_0$. Then the generating series map $Y(-, z)$ of $\mathbf{V}_L$ restricts to a map

$$\mathbf{F}_\lambda \times \mathbf{F}_\mu \longrightarrow \mathbf{F}_{\lambda+\mu}((z)), \quad (u, v) \mapsto Y(u, z)v,$$

which is an intertwining operator for $\mathbf{F}_0$, and this generalizes to the case when λ, μ are replaced by arbitrary elements of $\mathfrak{h}^*$.

The dimension of the space of intertwining operators is in fact a mathematical formulation of the concept of *fusion rules* originally considered in physics, and the analogue of the tensor product is called the *fusion product.*

For a vector space V and an indeterminate z, consider series of the following form with coefficients in V:

$$v(z) = \sum_{\alpha \in \mathbb{C}} v_\alpha z^{-\alpha-1}.$$

The set of such series is denoted $V\{z\}$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.4.1 Intertwining Operators

Let $\mathbf{L}, \mathbf{M}, \mathbf{N}$ be ordinary modules for a VOA $(\mathbf{V}, \omega)$. An *intertwining operator* of type $\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}$ is a linear map

$$\mathcal{Y}(-,z)\colon \mathbf{L} \longrightarrow \mathrm{Hom}(\mathbf{M}, \mathbf{N}\{z\}),\ \ u \mapsto \big[v \mapsto \mathcal{Y}(u,z)v\big]$$

such that the coefficients in the expansion

$$\mathcal{Y}(u,z) = \sum_{\alpha} u_{\alpha} z^{-\alpha-1}$$

satisfy the following properties:

(I0) Local truncation. For any $u \in \mathbf{L}$, $v \in \mathbf{M}$, and $\alpha \in \mathbb{C}$,

$$\sum_{n} z^{-\alpha-n-1} u_{\alpha+n} v \in \mathbf{N}((z)) z^{-\alpha}.$$

(I1) Borcherds identity. For all $a \in \mathbf{V}$, $u \in \mathbf{L}$, $v \in \mathbf{M}$, $p, r \in \mathbb{Z}$, and $q \in \mathbb{C}$:

$$\begin{aligned}\sum_{i=0}^{\infty} \binom{p}{i} (a_{r+i}u)_{p+q-i} v = &\sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a_{p+r-i}(u_{q+i} v) \\ &- \sum_{i=0}^{\infty} (-1)^{r+i} \binom{r}{i} u_{q+r-i}(a_{p+i} v).\end{aligned}$$

(IT) Translation. For all $u \in \mathbf{L}$, $v \in \mathbf{M}$, and $\alpha \in \mathbb{C}$,

$$(L_{-1}u)_{\alpha} v = -\alpha u_{\alpha-1} v.$$

For example, for $\mathbf{L} = \mathbf{V}$, the generating series $Y_{\mathbf{M}}(-,z)$ giving a module structure on $\mathbf{M}$ is an intertwining operator of type $\binom{\mathbf{M}}{\mathbf{V}\,\mathbf{M}}$.

By (I1) with $a = \omega$, the property (IT) turns out to be given by

$$L_0(u_{\alpha} v) = (L_0 u)_{\alpha} v + u_{\alpha}(L_0 v) - (\alpha + 1) u_{\alpha} v.$$

Therefore, (IT) is equivalent to the following: for homogeneous $u \in \mathbf{L}$, $v \in \mathbf{M}$, and $\alpha \in \mathbb{C}$,

$$\Delta(u_{\alpha} v) = \Delta(u) + \Delta(v) - \alpha - 1.$$

The Borcherds identity (I1) is stable under shifting the index q. Therefore, if the coefficients of $\mathcal{Y}(u,z)$ satisfy (I1), then those of $\mathcal{Y}(u,z)z^{-\beta}$ also satisfy it for any scalar β, and this freedom is fixed by (IT).

By making use of this freedom, in turn, the series $\mathcal{Y}(u,z)v$ for simple ordinary modules is turned to a series with integral powers by setting

$$\mathcal{Y}^{\circ}(u,z)v = z^{\lambda+\mu-\nu} \mathcal{Y}(u,z)v, \tag{6.14}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

where λ, μ, ν are the lowest conformal weights of $\mathbf{L}, \mathbf{M}, \mathbf{N}$, respectively. Expanding the modified series as

$$\mathcal{Y}^{\circ}(u,z)v = \sum_{n} \mathcal{Y}^{\circ}_{n}(u)v\, z^{-n-1},$$

we have $\mathcal{Y}^{\circ}_{n}(u)\mathbf{M}_{(}m) \subset \mathbf{N}(m + \Delta(u) - n - 1)$, where $\mathbf{M}(k) = \mathbf{M}_{\mu+k}$ and $\mathbf{N}(l) = \mathbf{N}_{\nu+l}$ are the homogeneous subspaces with respect to the associated $\mathbb{N}$-gradings.

Note 6.8. 1. The property (IT) is often called the $L(-1)$-*derivative property* in the literatures. 2. The symbol $\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}$ exhibits that the intertwining operator is contravariant in the lower entries $\mathbf{L}$ and $\mathbf{M}$ and covariant in the upper $\mathbf{N}$.

1.6.4.2 Fusion Rules

Let $\mathbf{L}, \mathbf{M}, \mathbf{N}$ be ordinary modules for a VOA $(\mathbf{V}, \omega)$. Then the set of intertwining operators of type $\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}$ forms a vector space denoted

$$I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}} = \left\{\text{intertwining operators of type } \binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}\right\}.$$

The dimension of this space is called the *fusion rule*, and usually denoted.

$$N^{\mathbf{N}}_{\mathbf{L},\mathbf{M}} = \dim I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}.$$

The fusion rules satisfy the following properties:

$$N^{\mathbf{N}}_{\mathbf{L}\mathbf{M}} = N^{\mathbf{N}}_{\mathbf{M}\mathbf{L}} = N^{\mathbf{M}'}_{\mathbf{L}\mathbf{N}'},$$

where $\mathbf{M}'$ and $\mathbf{N}'$ denote the contragredient modules.

1. The equality $N^{\mathbf{N}}_{\mathbf{L}\mathbf{M}} = N^{\mathbf{N}}_{\mathbf{M}\mathbf{L}}$ holds by the actions corresponding to the skew-symmetry. Indeed, we have an isomorphism $* : I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}} \longrightarrow I\binom{\mathbf{N}}{\mathbf{M}\,\mathbf{L}}$ sending $\mathcal{Y}$ to $\mathcal{Y}^*$ defined by

$$\mathcal{Y}^*(v,z)u = e^{zL_{-1}}\mathcal{Y}(u,-z)v$$

   for $u \in \mathbf{L}$ and $v \in \mathbf{M}$,
2. The equality $N^{\mathbf{N}}_{\mathbf{L}\mathbf{M}} = N^{\mathbf{M}'}_{\mathbf{L}\mathbf{N}'}$ holds by the contragredient actions. Indeed, we have an isomorphism $' : I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}} \longrightarrow I\binom{\mathbf{M}'}{\mathbf{L}\,\mathbf{N}'}$ sending $\mathcal{Y}$ to $\mathcal{Y}'$ defined by

$$\langle \mathcal{Y}'(u,z)\varphi, v\rangle = \langle \varphi, \mathcal{Y}(e^{zL_1}(-z^2)^{-L_0}u, z^{-1})v\rangle$$

   for $u \in \mathbf{L}$, $v \in \mathbf{M}$, and $\varphi \in \mathbf{N}'$.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.4.3 Fusion Products

Let us first note that we may equivalently formulate intertwining operators as a map of the form

$$\mathcal{Y}\colon \mathbf{L}\otimes\mathbf{M} \longrightarrow \mathbf{N}\{z\},\quad u\otimes v \mapsto \mathcal{Y}(u\otimes v, z) = \mathcal{Y}(u,z)v.$$

In the sequel, we will freely switch from one to the other.

For a pair $\mathbf{L},\mathbf{M}$ of ordinary modules for a VOA $(\mathbf{V},\omega)$, their *fusion product* is a pair $(\mathbf{L}\boxtimes\mathbf{M}, \mathcal{Y}^{\mathbf{L}\boxtimes\mathbf{M}}_{\mathbf{L},\mathbf{M}})$ of an ordinary module $\mathbf{L}\boxtimes\mathbf{M}$ and an intertwining operator

$$\mathcal{Y}^{\mathbf{L}\boxtimes\mathbf{M}}_{\mathbf{L},\mathbf{M}}\colon \mathbf{L}\otimes\mathbf{M} \longrightarrow \mathbf{L}\boxtimes\mathbf{M}\{z\},$$

satisfying the following universal property:

For any ordinary module $\mathbf{N}$ for $(\mathbf{V},\omega)$ and any intertwining operator

$$\mathcal{Y}\colon \mathbf{L}\otimes\mathbf{M} \longrightarrow \mathbf{N}\{z\}$$

there exists a unique homomorphism $\phi\colon \mathbf{L}\boxtimes\mathbf{M} \longrightarrow \mathbf{N}$ of modules such that the diagram

$$\begin{tikzcd}
\mathbf{L}\otimes\mathbf{M} \arrow[r, "\mathcal{Y}"] \arrow[d, "\mathcal{Y}^{\mathbf{L}\boxtimes\mathbf{M}}_{\mathbf{L},\mathbf{M}}"'] & \mathbf{N}\{z\} \\
(\mathbf{L}\boxtimes\mathbf{M})\{z\} \arrow[ur, "\phi"'] &
\end{tikzcd}$$

commutes, where $\phi\colon (\mathbf{L}\boxtimes\mathbf{M})\{z\} \longrightarrow \mathbf{N}\{z\}$ is the obvious map induced by $\phi\colon \mathbf{L}\boxtimes\mathbf{M} \longrightarrow \mathbf{N}$.

If the fusion product $\mathbf{L}\boxtimes\mathbf{M}$ exists, then, for any ordinary module $\mathbf{N}$, the fusion rule is given by

$$N^{\mathbf{N}}_{\mathbf{L},\mathbf{M}} = \dim \mathrm{Hom}_V(\mathbf{L}\boxtimes\mathbf{M}, \mathbf{N}),$$

hence, under semisimplicity of the module category,

$$\mathbf{L}\boxtimes\mathbf{M} = \bigoplus_{\mathbf{W}} N^{\mathbf{W}}_{\mathbf{L},\mathbf{M}}\mathbf{W},$$

where the direct sum runs over the isomorphism classes of simple modules, which we identify with their representatives.

Note 6.9. When the VOA is good enough, the fusion products satisfy *associativity* in an appropriate sense and, under semisimplicity of the module category, the free abelian group generated by the simple modules becomes a commutative ring by the fusion product, called the *fusion ring*, for which the fusion rules are the structure constants.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.4.4 Determination of fusion rules

Let $\mathbf{L}$ be a simple ordinary module for a VOA $(\mathbf{V}, \omega)$. For homogeneous $a \in \mathbf{V}$ and $u \in \mathbf{L}$, consider the elements $a * u$ and $a \circ u$ given by the same formula as (6.13). That is,

$$a * u = \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{i-1}u, \quad a \circ u = \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{i-2}u.$$

Then, as in the case of Zhu's algebra, the operation $*$ induces an action of Zhu's algebra $\mathbf{A}(\mathbf{V})$ on the quotient defined by $\circ$ as

$$\mathbf{A}(\mathbf{L}) = \mathbf{L}/\mathbf{V} \circ \mathbf{L}.$$

Moreover, the space $\mathbf{A}(\mathbf{L})$ admits a right action of $\mathbf{A}(\mathbf{V})$ induced by

$$u * a = \sum_{i=0}^{\infty} \binom{\Delta(a) - 1}{i} a_{i-1}u.$$

It is not difficult to see that $\mathbf{A}(\mathbf{L})$ becomes an $\mathbf{A}(\mathbf{V})$-bimodule by the left and the right operations $*$, which is called the *Frenkel–Zhu bimodule.*

Let $\mathbf{L}, \mathbf{M}, \mathbf{N}$ be simple ordinary modules with the lowest conformal weights λ, μ, ν, respectively, and equip them with $\mathbb{N}$-grading by, for $k \in \mathbb{N}$,

$$\mathbf{L}(k) = \mathbf{L}_{\lambda+k}, \;\; \mathbf{M}(k) = \mathbf{M}_{\mu+k}, \;\; \mathbf{N}(k) = \mathbf{N}_{\nu+k}.$$

For an intertwining operator $\mathcal{Y}$ of type $\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}}$, consider the following maps for $k \in \mathbb{N}$ by taking the series $\mathcal{Y}^{\circ}(u, z)$ given by (6.14):

$$\mathbf{L}(k) \times \mathbf{M}(0) \longrightarrow \mathbf{N}(0), \;\; (u, v) \mapsto \mathcal{Y}^{\circ}_{k-1}(u)v.$$

Then it induces

$$\pi(\mathcal{Y})\colon \mathbf{A}(\mathbf{L}) \otimes_{\mathbf{A}(\mathbf{V})} \mathbf{M}(0) \longrightarrow \mathbf{N}(0),$$

giving rise to a map

$$\pi\colon I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}} \longrightarrow \mathrm{Hom}_{\mathbf{A}(\mathbf{V})}(\mathbf{A}(\mathbf{L}) \otimes_{\mathbf{A}(\mathbf{V})} \mathbf{M}(0), \mathbf{N}(0)).$$

Theorem 6.10 (Frenkel–Zhu, Li) *Let $\mathbf{L}, \mathbf{M}, \mathbf{N}$ be simple ordinary modules for a VOA $(\mathbf{V}, \omega)$. Then*

$$\dim I\binom{\mathbf{N}}{\mathbf{L}\,\mathbf{M}} \leq \dim \mathrm{Hom}_{\mathbf{A}(\mathbf{V})}(\mathbf{A}(\mathbf{L}) \otimes_{\mathbf{A}(\mathbf{V})} \mathbf{M}(0), \mathbf{N}(0)).$$

Moreover, if the VOA is rational, then the equality holds.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

1.6.4.5 Examples of Fusion Rules

Let us list some examples of fusion rules.

1. Heisenberg VOAs. Consider the Heisenberg algebra $\hat{\mathfrak{h}}$ and the associated vertex algebra $\mathbf{V} = \mathbf{F}_0$, which is a VOA with the standard conformal vector ω. Consider the Fock modules $\mathbf{F}_\lambda$ of charge $\lambda \in \mathfrak{h}^*$. The fusion rules among them are described as follows: for $\lambda, \mu, \nu \in \mathfrak{h}^*$,

$$\dim I\binom{\mathbf{F}_\nu}{\mathbf{F}_\lambda\ \mathbf{F}_\mu} = \begin{cases} 1 & if \lambda + \mu = \nu, \\ 0 & otherwise. \end{cases}$$

   In terms of the fusion product, it is expressed as

$$\mathbf{F}_\lambda \boxtimes \mathbf{F}_\mu = \mathbf{F}_{\lambda+\mu}.$$

2. Lattice VOAs. For a positive-definite even lattice L, consider the lattice VOA $(\mathbf{V}_L, \omega)$ with ω the standard conformal vector. The simple ordinary modules are classified by the cosets in L°/L as

$$\mathbf{V}_{L+\lambda} = \bigoplus_{\mu \in L} \mathbf{F}_{\lambda+\mu}, \quad (\lambda \in L^\circ),$$

   and the fusion rules are described as, for representatives λ, μ of L°/L,

$$\mathbf{V}_{L+\lambda} \boxtimes \mathbf{V}_{L+\mu} = \mathbf{V}_{L+(\lambda+\mu)}.$$

3. Simple affine VOAs at integrable levels. Consider the simple affine VOA $(\mathbf{L}(k,0),\omega)$ associated with $\mathfrak{sl}_2$ at level $k = 1, 2, \cdots$. The simple ordinary modules are the module $\mathbf{L}(k,j)$ with spin $j = 0, 1/2, 1, \ldots, k/2$. The fusion rules are described as follows: for half integers i, j with $0 \le i, j \le k/2$,

$$\mathbf{L}(k,i) \boxtimes \mathbf{L}(k,j) = \bigoplus_{l} \mathbf{L}(k,l),$$

   where the sum is over the half integers l with $1 \le l \le k/2$ satisfying

   (1) $|i-j| \le l \le i+j$ and $i+j+l \in \mathbb{Z}$, and (2) $i+j+l \le k$.

   The condition (1) is just the ordinary Clebsch–Gordan rules of decomposition of tensor products.

   For example, if $k = 1$, then

$$\begin{aligned} \mathbf{L}(1,0) &\boxtimes \mathbf{L}(1,0) = \mathbf{L}(1,0), \\ \mathbf{L}(1,0) &\boxtimes \mathbf{L}(1,1/2) = \mathbf{L}(1,1/2), \\ \mathbf{L}(1,1/2) &\boxtimes \mathbf{L}(1,0) = \mathbf{L}(1,1/2), \\ \mathbf{L}(1,1/2) &\boxtimes \mathbf{L}(1,1/2) = \mathbf{L}(1,0). \end{aligned}$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Compare the result with that for the lattice VOA $\mathbf{V}_L$ associated with the one-dimensional lattice of type A_1, which is isomorphic to $\mathbf{L}(1,0)$.

The result is generalized to simple affine VOAs associated with integrable representations of affine Kac–Moody algebras. See [96] for $\mathfrak{sl}_2$, and [60] and [97] for the general case.

4. Virasoro minimal models. Consider the simple Virasoro VOA $\mathbf{L}(1/2,0)$ of central charge $1/2$. The list of simple modules is

$$\mathbf{L}(1/2,0), \mathbf{L}(1/2,1/2), \mathbf{L}(1/2,1/16),$$

and the fusion rules are described as

$$\begin{aligned}
\mathbf{L}(1/2,0) &\boxtimes \mathbf{L}(1/2,h) &&= \mathbf{L}(1/2,h) \ \ (h = 0, 1/2, 1/16),\\
\mathbf{L}(1/2,1/2) &\boxtimes \mathbf{L}(1/2,1/2) &&= \mathbf{L}(1/2,0),\\
\mathbf{L}(1/2,1/2) &\boxtimes \mathbf{L}(1/2,1/16) &&= \mathbf{L}(1/2,1/16),\\
\mathbf{L}(1/2,1/16) &\boxtimes \mathbf{L}(1/2,1/16) &&= \mathbf{L}(1/2,0) \oplus \mathbf{L}(1/2,1/2).
\end{aligned}$$

The other cases are obtained by the symmetry of fusion rules.

The results are generalized to the case of the Virasoro minimal models, where the fusion rules are given by

$$\mathbf{L}(c_{p,q}, h_{k,l}) \boxtimes \mathbf{L}(c_{p,q}, h_{m,n}) = \bigoplus_{(r,s)} \mathbf{L}(c_{p,q}, h_{r,s}),$$

where the sum is over the pairs (r,s) of integers with $1 \le r \le q-1$ and $1 \le s \le p-1$, up to identification $(r,s) \sim (q-r, p-s)$, satisfying the following conditions:

(1) $|k-m| < r < k+m$ and $|l-n| < s < l+n$.

(2) $k+m+r \in 2\mathbb{Z}+1$ and $l+n+s \in 2\mathbb{Z}+1$.

(3) $k+m+r \le 2q$ and $l+n+s \le 2p$.

See Subsection 1.3.3.4 for the notations and [15] and [99] for details.

5. Fixed-point VOAs $\mathbf{V}_L^+$. Let $\mathbf{V}_L$ be the VOA associated with a positive-definite even lattice L and $\mathbf{V}_L^+$ the vertex subalgebra of fixed-points by the automorphism θ, a lift of (-1)-involution of L to $\mathbf{V}_L$.

For simplicity, consider the case when L is unimodular. Recall the twisted module $\mathbf{V}_L^{\mathrm{tw}}$, which becomes an (untwisted) module for $\mathbf{V}_L^+$ and decomposes into the direct sum of simple components $\mathbf{V}_L^{\mathrm{tw},\pm}$. Now the list of simple modules for $\mathbf{V}_L^+$ is

$$\mathbf{V}_L^+, \ \mathbf{V}_L^-, \ \mathbf{V}_L^{\mathrm{tw},+}, \ \mathbf{V}_L^{\mathrm{tw},-}.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

The fusion rules are

$$\begin{aligned}\mathbf{V}_L^- \boxtimes \mathbf{V}_L^- &= \mathbf{V}_L^+, \ \mathbf{V}_L^- \boxtimes \mathbf{V}_L^{\mathrm{tw},\pm} = \mathbf{V}_L^{\mathrm{tw},\mp},\\ \mathbf{V}_L^{\mathrm{tw},\pm} \boxtimes \mathbf{V}_L^{\mathrm{tw},\pm} &= \mathbf{V}_L^+, \ \mathbf{V}_L^{\mathrm{tw},\pm} \boxtimes \mathbf{V}_L^{\mathrm{tw},\mp} = \mathbf{V}_L^-.\end{aligned}$$

See [27] for details.

1.6.5 Modular Invariance

Let $\mathbf{M}$ be a simple ordinary module for a VOA $(\mathbf{V}, \omega)$ of central charge c. The graded dimension with respect to the grading by conformal weight multiplied by $q^{-c/24}$ is called the *conformal character* of $\mathbf{M}$ and denoted as.

$$\mathrm{ch}_{\mathbf{M}}(q) = q^{-c/24}\mathrm{Tr}_{\mathbf{M}} q^{L_0} = q^{-c/24}\sum_{n=0}^{\infty} q^{\lambda+n} \dim \mathbf{M}_{\lambda+n},$$

where λ is the lowest conformal weight of $\mathbf{M}$. We are interested in the behavior of the conformal characters as functions of τ, where $q = e^{2\pi\sqrt{-1}\tau}$, when $\mathbf{M}$ varies over the ordinary simple modules.

For simple affine VOAs $\mathbf{L}(k, 0)$ associated with integrable representations of affine Kac–Moody algebras, the conformal characters agree with specialization of the characters in the sense of Kac–Moody algebras, and they satisfy certain modular transformation properties.

On the other hand, for the moonshine module $\mathbf{V}^\natural$, the conformal character is given by

$$J(\tau) - 744 = q^{-1} + 196884q + \cdots,$$

where $J(\tau)$ is the elliptic modular function, which is certainly invariant under modular transformations.

Such nice modular transformation properties can be uniformly described as a consequence of a general fact, modular invariance of conformal characters, which holds not only for conformal characters, but also for functions called *torus one-point functions* under suitable conditions.

In this section, we will briefly describe the theory of modular invariance established by Y. C. Zhu by combining ideas from conformal field theory in physics with C_2-cofiniteness and Zhu's algebra.

1.6.5.1 Torus One-Point Functions

For a simple ordinary module $\mathbf{M}$ with the lowest conformal weight $\lambda \in \mathbb{C}$ for a VOA $(\mathbf{V}, \omega)$, consider the series in $q = e^{2\pi\sqrt{-1}\tau}$ defined for a homogeneous $a \in \mathbf{V}$ by

$$\chi_{\mathbf{M}}(a, \tau) = q^{-c/24}\mathrm{Tr}\, o(a) q^{L_0}\big|_{\mathbf{M}} = q^{-c/24}\sum_{n=0}^{\infty} \mathrm{Tr}\, o(a)\big|_{\mathbf{M}_{\lambda+n}} q^{\lambda+n},$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

where $o(a) = a_{\Delta(a)-1}$ is the zero-mode action on the weight spaces $\mathbf{M}_{\lambda+n}$. This series is called the *one-point function* (on the torus) associated with $\mathbf{M}$ at $a \in \mathbf{V}$.

In particular, when the element a is the vacuum $\mathbf{1}$, the one-point function agrees with the conformal character $\mathrm{ch}_{\mathbf{M}}(q)$:

$$\chi_{\mathbf{M}}(\mathbf{1},\tau) = \mathrm{ch}_{\mathbf{M}}(q) = q^{-c/24} \sum_{\alpha\in\mathbb{C}} \dim \mathbf{M}_\alpha q^\alpha.$$

Now assume that $(\mathbf{V},\omega)$ is rational; that is, the category of $\mathbb{N}$-graded modules is semisimple. Then there are only finitely many isomorphism classes of simple modules, say $\mathbf{M}^1,\ldots,\mathbf{M}^n$, which are ordinary. We are interested in the behavior of the one-point functions

$$\chi_{\mathbf{M}^1}(a,\tau),\ldots,\chi_{\mathbf{M}^n}(a,\tau).$$

under transformations of weight k by the full modular group $\mathrm{SL}_2(\mathbb{Z})$,

$$f(\tau) \mapsto (c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big),$$

where $a,b,c,d \in \mathbb{Z}$ and $ad-bc=1$.

1.6.5.2 Eisenstein Series and Serre Derivative

In describing the properties of one-point functions as defined here, the Eisenstein series naturally arise.

For $k \in \mathbb{N}$, let $G_{2k}(\tau)$ denote the Eisenstein series in its q-expansion:

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2\big(2\pi\sqrt{-1}\big)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where ζ is the Riemann zeta function and $\sigma_m(n)$ the sum of mth powers of divisors of n. For $k \geq 2$, the Eisenstein series is a modular form of weight $2k$:

$$G_{2k}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{2k}G_{2k}(\tau).$$

For $k=2,3$, we have

$$G_4(\tau) = \frac{\pi^4}{45}\Big(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big),\ G_6(\tau) = \frac{2\pi^6}{945}\Big(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big).$$

Consider the ring of modular forms $\mathbb{C}[G_4(\tau),G_6(\tau)]$, which is a Noetherian ring. For $k=1$, we have

$$G_2(\tau) = \frac{\pi^2}{3}\Big(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\Big),$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

whose transformation property is

$$G_2\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 G_2(\tau) - 2\pi\sqrt{-1}c(c\tau+d).$$

Although it is not a modular form, for a modular form $f(\tau)$ of weight k, the Serre derivative

$$D_k f(\tau) = (2\pi\sqrt{-1})\frac{d}{d\tau}f(\tau) + kG_2(\tau)f(\tau)$$

becomes a modular form of weight $k+2$.

1.6.5.3 Space of One-Point Functions

Let $\mathbf{M}$ be an ordinary module for a VOA $(\mathbf{V}, \omega)$ and consider the associated one-point function $\chi_{\mathbf{M}}(-,\tau)$. For homogeneous elements $a, b \in \mathbf{V}$, the following formulas hold by the axioms for vertex algebras and invariance of trace under cyclic permutation of entries: for $a, b \in \mathbf{V}$,

(1) $\chi_{\mathbf{M}}(a_{[0]}b,\tau) = 0$,

(2) $\chi_{\mathbf{M}}(a_{[-2]}b,\tau) + \sum_{k=2}^{\infty}(2k-1)G_{2k}(\tau)\chi_{\mathbf{M}}(a_{[2k-2]}b,\tau)$,

(3) $\chi_{\mathbf{M}}(L_{[-2]}a,\tau) = D_{\Delta[a]}\,\chi_{\mathbf{M}}(a,\tau) + \sum_{k=2}^{\infty} G_{2k}(\tau)\chi_{\mathbf{M}}(L_{[2k-2]}a,\tau)$,

where $a_{[n]}b$ and $L_{[n]}$ are defined by

$$Y(e^{2\pi\sqrt{-1}zL_0}a, e^{2\pi\sqrt{-1}z}-1)b = \sum_n a_{[n]}bz^{-n-1},$$
$$L_{[n]} = (2\pi\sqrt{-1})^2(\omega - (c/24)\mathbf{1})_{[n+1]},$$

respectively, and $D_{\Delta[a]}\chi_{\mathbf{M}}(a,\tau)$ is the Serre derivative with $\Delta[a]$ the conformal weight of a with respect to $L_{[0]}$.

Here the assignment $a \mapsto Y(e^{2\pi\sqrt{-1}zL_0}a, e^{2\pi\sqrt{-1}z}-1)$ gives a new VOA structure on $\mathbf{V}$ with the conformal vector $\tilde{\omega} = (2\pi\sqrt{-1})^2(\omega-(c/24)\mathbf{1})$, which is seen to be achieved by a particular case of Huang's formula (6.8) corresponding to a coordinate on the cylinder, and this is where the factor $q^{-c/24}$ of the conformal character arises.

Now the properties (1), (2), and (3) do not depend on the choice of the module $\mathbf{M}$. We therefore take them as conditions on a functional $\chi(-,\tau)$ on $\mathbf{V} \otimes \mathbb{C}[G_4(\tau), G_6(\tau)]$ valued in series in q, and call such a functional an *abstract one-point function* if it satisfies the conditions.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Let $\mathbf{O}_q(\mathbf{V})$ be the $\mathbb{C}[G_4(\tau), G_6(\tau)]$-submodule of $\mathbf{V} \otimes \mathbb{C}[G_4(\tau), G_6(\tau)]$ generated by the elements of the following form with $a, b \in \mathbf{V}$:

$$a_{[0]}b \;\text{ or }\; a_{[-2]}b + \sum_{k=2}^{\infty}(2k-1)G_{2k}(\tau)a_{[2k-2]}b. \tag{6.15}$$

Then an abstract one-point function $\chi(-, \tau)$ induces a functional on the quotient

$$\mathbf{V} \otimes \mathbb{C}[G_4(\tau), G_6(\tau)]/\mathbf{O}_q(\mathbf{V})$$

by (1) and (2).

1.6.5.4 Consequence of C_2-Cofiniteness

Recall that a VOA $(\mathbf{V}, \omega)$ is said to be C_2-*cofinite* if the quotient $\mathbf{V}/\mathbf{V}_{(-2)}\mathbf{V}$ is finite-dimensional.

If $(\mathbf{V}, \omega)$ is C_2-cofinite, $\mathbf{V} \otimes \mathbb{C}[G_4(\tau), G_6(\tau)]/\mathbf{O}_q(\mathbf{V})$ is a finitely generated $\mathbb{C}[G_4(\tau), G_6(\tau)]$-module, thus a Noetherian module since the ring $\mathbb{C}[G_4(\tau), G_6(\tau)]$ is Noetherian. Therefore, for any $a \in \mathbf{V}$, there exist $s \in \mathbb{N}$ and $g_i(\tau) \in \mathbb{C}[G_4(\tau), G_6(\tau)]$ such that

$$(L_{[-2]})^s a + \sum_{i=0}^{s-1} g_i(\tau)(L_{[-2]})^i a \in \mathbf{O}_q(\mathbf{V}).$$

After some algebra, the relation yields the following result.

Theorem 6.11 (Zhu) *Let $(\mathbf{V}, \omega)$ be a C_2-cofinite VOA, $\chi(-, \tau)$ a one-point function and $a \in V$ satisfy $L_{[n]}a = 0$ for $n > 0$. Then, the value of the one-point function at $a \in \mathbf{V}$ satisfies a differential equation of the form*

$$\left(q\frac{d}{dq}\right)^s \chi(a, \tau) + \sum_{i=0}^{s-1} h_i(\tau)\left(q\frac{d}{dq}\right)^i \chi(a, \tau) = 0,$$

where $h_i(\tau) \in \mathbb{C}[G_2(\tau), G_4(\tau), G_6(\tau)]$.

Note 6.12. 1. Theorem 6.11 only guarantees existence of a differential equation, and finding it explicitly is a separate problem. 2. By the property (1) in the preceding subsection, it follows that the result of Theorem 6.11 holds under a weaker assumption. See [28] and [85].

1.6.5.5 Modular Invariance

Assume that $(\mathbf{V}, \omega)$ is C_2-cofinite. Then the values of one-point functions $\chi(a, \tau)$ at any $a \in \mathbf{V}$ satisfy a linear ordinary differential equation of regular singular type for which $q = 0$ is a the only regular singular point, so the series solutions at $q = 0$ converge. Therefore, one-point functions can be viewed as taking

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

values in holomorphic functions of τ on the upper half-plane, and the space of abstract one-point functions is invariant under the modular transformations since the conditions that characterize it are modular invariant. Moreover, the initial terms of one-point functions give rise to linear functionals F on Zhu's algebra $\mathbf{A}(\mathbf{V})$ satisfying $F(a * b) = F(b * a)$.

Assume further that $(\mathbf{V}, \omega)$ is rational. Then Zhu's algebra $\mathbf{A}(\mathbf{V})$ is semisimple, and the simple $\mathbf{A}(\mathbf{V})$-modules are in one-to-one correspondence with the top spaces of the simple ordinary modules. It then follows that the modular invariant space of abstract one-point functions is spanned by the one-point functions associated with simple ordinary modules.

Theorem 6.13 (Zhu, Dong-Li-Mason) *Let $(\mathbf{V}, \omega)$ be rational and C_2-cofinite and let $\mathbf{M}^1, \ldots, \mathbf{M}^n$ be the list of simple ordinary modules. Then, for any $k \in \mathbb{N}$ and any element $a \in \mathbf{V}$ of weight k with respect to $L_{[0]}$, the series*

$$\chi_{\mathbf{M}^1}(a, \tau), \ldots, \chi_{\mathbf{M}^n}(a, \tau) \tag{6.16}$$

define holomorphic functions on the upper half-plane that span a vector space invariant under the weight k action of the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

In other words, the transformation of $\chi_{\mathbf{M}^i}(a, \tau)$ under a modular transformation of weight k becomes a linear combination of the functions (6.16) with constant coefficients.

Note 6.14. When the VOA and the module category are good enough, the matrix representing the modular transformation $\tau \mapsto -1/\tau$ is related to the fusion rules by a famous formula called the *Verlinde formula.*

Bibliographic Notes

General references are Frenkel, Huang, and Lepowsky [3], Lepowsky and Li [10], and Frenkel and Ben-Zvi [8]. For related models in physics, consult Di Francesco et al. [13] (cf. [19]).

For various transformation formulas, see Frenkel, Huang, and Lepowsky [3] and Huang [5] (cf. [8], [97], [100].).

Classification of $\mathbb{N}$-graded modules by Zhu's algebra is due to Y. C. Zhu in [24] (cf. [60], [100]). See Matsuo, Nagatomo, and Tsuchiya [82] for an alternative approach.

Interpretation of fusion rules by means of intertwining operators was initiated by Tsuchiya and Kanie [96] for affine Lie algebras and further developed geometrically in Tsuchiya, Ueno, and Yamada [97]. For general theory of fusion products, see [72] and references therein (cf. [65] and subsequent papers). For geometric interpretations, see Huang [5] and Nagatomo and Tsuchiya [86].

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Fusion rules for lattice VOAs are described in Dong and Lepowsky [4]. For determination of fusion rules by Frenkel and Zhu bimodules, see [60] and [74]. Fusion rules for affine vertex algebras were treated in [60]. See [96] and [97] for early treatments. Fusion rules for Virasoro minimal models were determined by Wang [99] (cf. Dong et al. [49]) based on Feigin and Fuchs [53]. Fusion rules for $\mathbf{V}_L^+$ are determined by Abe, Dong and Li [27].

General theory of modular invariance of conformal characters as well as the concept of C_2-cofiniteness of VOAs was established by Y. C. Zhu in [24] (cf. [100]), and generalized to twisted modules by Dong et al. [46]. See Mason and Tuite [21] for a good survey. For earlier studies on modular invariant characters for affine Kac–Moody algebras, see Kac and Peterson [66].

For regularity and rationality of VOAs, see Dong, Li, and Mason [47], Li [73], and Abe, Buhl, and Dong [26] (cf. [82]). For rationality of $\mathbf{V}_L^+$ for positive-definite lattices L, see Dong, Jiang, and Lin [44].

Finding the automorphism group of a VOA is an interesting problem. See Dong and Nagatomo [50] for $\operatorname{Aut}\mathbf{V}_L$ and Shimakura [94] for $\operatorname{Aut}\mathbf{V}_L^+$.

Epilogue

Let Λ be the Leech lattice, the unique even unimodular positive-definite lattice of rank 24 without roots. Consider the lattice VOA $\mathbf{V}_\Lambda$ with the standard conformal vector. It is a regular VOA of CFT type and the conformal character is given by

$$\begin{aligned}\mathrm{ch}_{\mathbf{V}_\Lambda}(q) &= j(\tau) - 720\\ &= q^{-1} + \underbrace{24} + 196884q + 21493760q^2 + \cdots.\end{aligned}$$

In particular, the dimension of the degree 1 subspace is 24, which is nonzero. Since Λ is unimodular, the VOA $\mathbf{V}_\Lambda$ is holomorphic, that is, the adjoint module $\mathbf{V}_\Lambda$ is the only simple module.

Consider the decomposition of $\mathbf{V}_\Lambda$ under a lift θ of the (-1)-involution of the lattice:

$$\mathbf{V}_\Lambda = \mathbf{V}_\Lambda^+ \oplus \mathbf{V}_\Lambda^-.$$

Then the fixed-point subVOA $\mathbf{V}_\Lambda^+$ is again of CFT type but with the degree 1 subspace now being 0. The VOA $\mathbf{V}_\Lambda^+$ is still regular, but not holomorphic, and the list of simple modules

$$\mathbf{V}_\Lambda^+,\ \mathbf{V}_\Lambda^-,\ \mathbf{V}_\Lambda^{\mathrm{tw},+},\ \mathbf{V}_\Lambda^{\mathrm{tw},-},$$

where $\mathbf{V}_\Lambda^{\mathrm{tw},\pm}$ are simple components of $\mathbf{V}_\Lambda^{\mathrm{tw}}$ as a $\mathbf{V}_\Lambda^+$-module and let $\mathbf{V}_\Lambda^{\mathrm{tw},+}$ be the one with integral grading with respect to L_0.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Now the moonshine module $\mathbf{V}^\natural$ is constructed as a module over $\mathbf{V}_\Lambda^+$ as

$$\mathbf{V}^\natural = \mathbf{V}_\Lambda^+ \oplus \mathbf{V}_\Lambda^{\mathrm{tw},+},$$

of which the conformal character is given as described in the Introduction by

$$\begin{aligned}\mathrm{ch}_{\mathbf{V}^\natural}(q) &= j(\tau) - 744 \\ &= q^{-1} + \underset{\sqcup}{0} + 196884q + 21493760q^2 + \cdots.\end{aligned}$$

The moonshine module $\mathbf{V}^\natural$ actually becomes a regular holomorphic VOA (cf. [42], [47]), which is of CFT type with the degree 1 subspace now being 0.

Existence of a vertex algebra structure on $\mathbf{V}^\natural$ that extends the structure of a module over $\mathbf{V}_\Lambda^+$ is stated in [32] and its detailed proof by hard calculations, heavily based on group theoretical consideration, is given in [1]. Alternatively, note that $\mathbf{V}_\Lambda^+$ and $\mathbf{V}_\Lambda^{\mathrm{tw},+}$ are closed under the fusion products as

$$\mathbf{V}_\Lambda^{\mathrm{tw},+} \boxtimes \mathbf{V}_\Lambda^{\mathrm{tw},+} = \mathbf{V}_\Lambda^+,$$

the intertwining operators are series with integral powers in z, and the fusion rules are at most one-dimensional. The vertex algebra structure on $\mathbf{V}^\natural$ can actually be obtained by intertwining operators among $\mathbf{V}_\Lambda^+$ and $\mathbf{V}_\Lambda^{\mathrm{tw},+}$, multiplied by appropriate constant scalar factors.

Nowadays, many constructions of $\mathbf{V}^\natural$ are known. See [95] for a nice construction from $(\mathbf{V}_{\sqrt{2}E_8}^+)^{\otimes 3}$ and [36] for constructions from $\mathbf{V}_\Lambda$. See [37] for general theory of constructing vertex algebras by intertwining operators.

Let $\mathbf{V}$ be a VOA of CFT type with the degree 1 subspace being 0, such as the moonshine module.

$$\mathbf{V} = \mathbb{C}\mathbf{1} \oplus 0 \oplus \mathbf{B} \oplus \cdots.$$

Then the degree 2 subspace $\mathbf{B}$ satisfies

$$\mathbf{B}_{(1)}\mathbf{B} \subset \mathbf{B}, \quad \mathbf{B}_{(3)}\mathbf{B} \subset \mathbf{V}_0 = \mathbb{C}\mathbf{1}.$$

Let us equip $\mathbf{B}$ with a product and a bilinear form:

$$\begin{aligned}\cdot \ &: \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{B}, \ (a,b) \mapsto a \cdot b, \\ (\,|\,) &: \mathbf{B} \times \mathbf{B} \longrightarrow \mathbb{C}, \ (a,b) \mapsto (a|b),\end{aligned}$$

in such a way that

$$a \cdot b = a_{(1)}b \ \text{ and } \ (a|b)\mathbf{1} = a_{(3)}b.$$

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

It is easy to see that the axioms for vertex algebras and their consequences imply, for all $a, b, c \in \mathbf{B}$,

$$a \cdot b = b \cdot a, \quad (a|b) = (b|a), \quad \text{and} \quad (a \cdot b|c) = (a|b \cdot c).$$

In other words, $\mathbf{B}$ is a commutative algebra with symmetric invariant bilinear form, often called the *Griess algebra* of $\mathbf{V}$. The Griess algebra $\mathbf{B}^\natural$ of the moonshine module $\mathbf{V}^\natural$ is the Griess–Conway algebra mentioned in the Introduction.

One of the specific features of $\mathbf{V}^\natural$ is the existence of an *Ising frame*, often called a *Virasoro frame*; that is, a full subVOA of a VOA isomorphic to a tensor product of the simple Virasoro VOA $\mathbf{L}(1/2, 0)$ of central charge $1/2$ (cf. [49], [43]). Here a subVOA of a VOA is said to be *full* if their conformal vectors agree. The moonshine module $\mathbf{V}^\natural$ inherits such a frame of length 48 from $\mathbf{V}_\Lambda^+ \subset \mathbf{V}_\Lambda$:

$$\underbrace{\mathbf{L}(1/2,0) \otimes \cdots \otimes \mathbf{L}(1/2,0)}_{48 \text{ times}} \subset \mathbf{V}_\Lambda^+.$$

In other words, the Griess–Conway algebra $\mathbf{B}^\natural$ has a set of 48 Virasoro vectors of central charge $1/2$, orthogonal to each other in the sense that the corresponding Virasoro actions commute, such that each generates a subVOA isomorphic to $\mathbf{L}(1/2, 0)$.

In general, let $\mathbf{V}$ be a VOA of CFT type with $\mathbf{V}_1 = 0$. Let e be a Virasoro vector in the Griess algebra $\mathbf{B}$ of central charge $1/2$; that is,

$$\omega \cdot e = e \cdot e = 2e, \quad (e|e) = \frac{1}{4}.$$

Such an e is called an *Ising vector* if it generates a subVOA $\mathbf{V}_e$ isomorphic to $\mathbf{L}(1/2, 0)$. Since $\mathbf{L}(1/2, 0)$ is regular, $\mathbf{V}$ decomposes as

$$\mathbf{V} = \mathbf{V}(e,0) \oplus \mathbf{V}(e,1/2) \oplus \mathbf{V}(e,1/16), \tag{2}$$

where $\mathbf{V}(e, h)$ denotes the sum of components isomorphic to $\mathbf{L}(1/2, h)$ for each $h = 0, 1/2, 1/16$. Note that $\mathbf{V}_e$ itself is one of the simple components in $\mathbf{V}(e, 0)$.

Consider the map

$$Y(-,z)\colon \mathbf{V} \longrightarrow \mathrm{Hom}(\mathbf{V}, \mathbf{V}((z))), \quad a \mapsto Y(a,z).$$

Then it induces an intertwining operator for the VOA $\mathbf{L}(1/2, 0)$ for each triple of appropriate simple components of $\mathbf{V}$ when multiplied by a rational power of z depending on the triple. Consequently, the fusion rules for the Ising model imply that the product operations of the vertex algebra $\mathbf{V}$ satisfy, for $n \in \mathbb{Z}$,

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\begin{aligned}
\mathbf{V}(e,0)_{(n)}\mathbf{V}(e,h) &\subset \mathbf{V}(e,h) \ \ (h=0,1/2,1/16),\\
\mathbf{V}(e,1/2)_{(n)}\mathbf{V}(e,1/2) &\subset \mathbf{V}(e,0),\\
\mathbf{V}(e,1/2)_{(n)}\mathbf{V}(e,1/16) &\subset \mathbf{V}(e,1/16),\\
\mathbf{V}(e,1/16)_{(n)}\mathbf{V}(e,1/16) &\subset \mathbf{V}(e,0)\oplus\mathbf{V}(e,1/2),
\end{aligned}$$

which imply that the following map τ_e gives rise to an automorphism of the whole VOA $\mathbf{V}$ (cf. [19] and [83]):

$$\tau_e : \mathbf{V} \longrightarrow \mathbf{V}, \ \ \tau_e v = \begin{cases} v & (v \in \mathbf{V}(e,0)\oplus\mathbf{V}(e,1/2)),\\ -v & (v \in \mathbf{V}(e,1/16)).\end{cases}$$

This automorphism is called the *Miyamoto involution* in the literatures.

Now let a be the half of an Ising vector e in $\mathbf{B}$. Then a is an idempotent of $\mathbf{B}$ with squared norm $1/16$:

$$a\cdot a = a, \ \ (a|a) = \frac{1}{16}.$$

Consider the eigenspace with eigenvalue λ for the action of the idempotent a on $\mathbf{B}$ by multiplication and write

$$\mathbf{B}(a,\lambda) = \{x \in \mathbf{B} \,|\, a\cdot x = \lambda x\}.$$

Then, since $\mathbf{V}$ is of CFT type with $\mathbf{V}_1 = 0$ and the highest weight vector of $\mathbf{V}_e$ is the vacuum $\mathbf{1} \in \mathbf{V}$, we have $\mathbf{V}_e \cap \mathbf{B} = \mathbb{C}e$, and it follows that the other simple components of $\mathbf{V}(e,0)$ can be chosen so that they intersect $\mathbf{B}$ with the span of the highest weight vectors. Therefore, we have $\mathbf{B}(a,1) = \mathbb{C}a$ and

$$\mathbf{B} = \mathbf{B}(a,1)\oplus\mathbf{B}(a,0)\oplus\mathbf{B}(a,1/4)\oplus\mathbf{B}(a,1/32), \tag{3}$$

where the eigenspaces are related to the decomposition (2) by

$$\begin{gathered}
\mathbf{B}(a,1)\oplus\mathbf{B}(a,0) = \mathbf{B}\cap\mathbf{V}(e,0),\\
\mathbf{B}(a,1/4) = \mathbf{B}\cap\mathbf{V}(e,1/2), \ \ \mathbf{B}(a,1/32) = \mathbf{B}\cap\mathbf{V}(e,1/16).
\end{gathered}$$

Regarding the decomposition (3), the fusion rules turn out to be described by Table 12. The information given to the idempotent a is strong enough to ensure that the structures of subalgebras generated by a pair of such idempotents actually fall into nine types ([90], [62]) corresponding to the conjugacy classes of the products of pairs of $2A$ involutions of the Monster (see [16]).

The properties of the idempotents a in the Griess algebra as here were axiomatized in [16], and a general framework in dealing with such algebras was formulated in [62] under the term *axial algebras* (cf. [80], [63] and [23]).

Let $\mathbf{A}$ be a commutative nonassociative algebra and consider a set of distinguished idempotents, called the *axes.* Assume that the actions of axes a by multiplication are semisimple with eigenvalues from a fixed set Λ as

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Table 12 Ising fusion rules for axes

	1	0	1/4	1/32
1	1	$\emptyset$	1/4	1/32
0	$\emptyset$	0	1/4	1/32
1/4	1/4	1/4	1,0	1/32
1/32	1/32	1/32	1/32	1,0,1/4

$$\mathbf{A} = \bigoplus_{\lambda \in \Lambda} \mathbf{A}(a, \lambda),$$

for which the multiplication of **A** obeys a prescribed set of fusion rules, recently called a *fusion law*, which sends a pair (λ, μ) of eigenvalues in Λ to a subset $\lambda * \mu$ of Λ, in such a way that

$$\mathbf{A}(a, \lambda)\mathbf{A}(a, \mu) \subset \sum_{\nu \in \lambda * \mu} \mathbf{A}(a, \nu).$$

Since **A** is commutative, we may and do assume that the fusion rules are symmetric; that is, $\lambda * \mu = \mu * \lambda$ for all $\lambda, \mu \in \Lambda$. An axial algebra is an algebra equipped with axes that satisfy the properties discussed here and generate the algebra.

The concept of axial algebras was further generalized in [39] to a class of algebras called *axial decomposition algebras.* These algebras have been extensively studied in recent years with fruitful outcomes.

Let us finally recall that the moonshine module $\mathbf{V}^\natural$ is of CFT type, regular, hence C_2-cofinite, and holomorphic of central charge $c = 24$. Note that the lattice VOAs associated with Niemeier lattices and many other VOAs also satisfy the same properties. For recent progress on the classification of such VOAs, initiated by Schellekens [91], see [98].

Let **V** be a VOA satisfying the properties listed. By Zhu's theory of modular invariance, the one-dimensional vector space spanned by the conformal character $\mathrm{ch}_{\mathbf{V}}(q) = \chi_{\mathbf{V}}(\mathbf{1}, \tau)$ is invariant under modular transformations. After inspecting the possible characters of the full modular group for the transformation, it turns out that the conformal character itself is modular invariant and it must be identical to the elliptic modular function $j(\tau)$ up to shifting the constant term (cf. [25], [21]). In particular, the dimension of the degree 2 subspace $\mathbf{V}_2$ of such a VOA must be 196884.

Let a be any element of **V** that is homogeneous of conformal weight k with respect to $L_{[0]}$, and consider the one-point function:

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

$$\chi_{\mathbf{V}}(a,\tau) = q^{-1}\sum_{n=0}^{\infty} \operatorname{Tr} o(a)\big|_{\mathbf{V}_n} q^n.$$

Then it is also invariant under modular transformations, but now of weight k. Moreover, if a is a singular vector with respect to the Virasoro actions, then $\chi_{\mathbf{V}}(a,\tau)$ is a cusp form (cf. [48]), which is 0 if the weight is less than 12.

Let us apply this observation to the case with $\mathbf{V}_1 = 0$, when the conformal character equals $j(\tau)-744$ and the shape of $\mathbf{V}$ agrees with that of the moonshine module:

$$\begin{array}{lcccccccccc} \mathbf{V} & = \mathbb{C}\mathbf{1} & \oplus & 0 & \oplus & \mathbf{B} & \oplus & \mathbf{V}_3 & \oplus & \mathbf{V}_4 & \oplus \cdots. \\ \text{dim} & 1 & & 0 & & 196884 & & & & & \end{array}$$

Then the absence of cusp forms, as mentioned, combined with general properties of the VOA such as the Borcherds identity and their consequences, implies the following trace formulae for the multiplication operators on the Griess algebra $\mathbf{B}$ for up to five elements of $\mathbf{B}$:

$$\begin{aligned}
\operatorname{Tr} o(a_1) &= 32814\,(a_1|\omega),\\
\operatorname{Tr} o(a_1)o(a_2) &= 4620\,(a_1|a_2) + 5084\,(a_1|\omega)(a_2|\omega),\\
\operatorname{Tr} o(a_1)o(a_2)o(a_3) &= 900\,(a_1|a_2|a_3) + 620\operatorname{Cyc}(a_1|a_2)(a_3|\omega)\\
&\quad + 744\,(a_1|\omega)(a_2|\omega)(a_3|\omega),\\
\operatorname{Tr} o(a_1)o(a_2)o(a_3)o(a_4) &\\
&= 166\,(a_1a_2|a_3a_4) - 116\,(a_1a_3|a_2a_4) + 166\,(a_1a_4|a_2a_3)\\
&\quad + 114\operatorname{Sym}(a_1|a_2|a_3)(a_4|\omega) + 52\operatorname{Sym}(a_1|a_2)(a_3|a_4)\\
&\quad + 80\operatorname{Sym}(a_1|a_2)(a_3|\omega)(a_4|\omega) + 104\,(a_1|\omega)(a_2|\omega)(a_3|\omega)(a_4|\omega),\\
\operatorname{Tr} o(a_1)o(a_2)o(a_3)o(a_4)o(a_5) &= 30\operatorname{Cyc}(a_1a_2|a_3|a_4a_5) + \cdots.
\end{aligned}$$

Note that for $a \in \mathbf{B}$, the zero-mode action $o(a)$ restricted to $\mathbf{B}$ agrees with multiplication by a. See [79] for details and [67] for a recent application.

These formulae were originally obtained by S. P. Norton for the Griess–Conway algebra $\mathbf{B}^\natural$ in [87] by investigating detailed structures of the algebra (cf. [38]).

The derivation of the formulae by modular invariance as described remarkably shows that such detailed properties of the algebra $\mathbf{B}^\natural$ are revealed as a consequence of general properties of the whole $\mathbf{V}^\natural$ endowed with the structure of a VOA, which is in accordance with the uniqueness of $\mathbf{V}^\natural$ conjectured by I. B. Frenkel et al. in [1] as mentioned in the Introduction.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Bibliography

Textbooks and Monographs

[1] I. B. Frenkel, J. Lepowsky, and A. Meurman: *Vertex operator algebras and the Monster*. Academic Press, Boston, 1988.

[2] A. J. Feingold, I. B. Frenkel, and J. F. X Ries: *Spinor construction of vertex operator algebras, triality, and* $E_8^{(1)}$. Contemporary Mathematics, 121. Amer. Math. Soc., Providence, 1991.

[3] I. B. Frenkel, Y. Huang, and J. Lepowsky: *On axiomatic approaches to vertex operator algebras and modules*. Mem. Amer. Math. Soc. 104, Amer. Math. Soc., Providence, 1993.

[4] C. Y. Dong and J. Lepowsky: *Generalized vertex algebras and relative vertex operators*. Birkhäuser, Boston, 1993.

[5] Y. Z. Huang: *Two-dimensional conformal geometry and vertex operator algebras*. Progr. Math., 148, Birkhäuser, Boston, 1997.

[6] V. G. Kac: *Vertex algebras for beginners*. Second edition. Amer. Math. Soc., Providence, 1998.

[7] A. Matsuo and K. Nagatomo: *Axioms for a vertex algebra and the locality of quantum fields*. MSJ Memoirs 4. Math. Soc. Japan, Tokyo, 1999.

[8] E. Frenkel and D. Ben-Zvi: *Vertex algebras and algebraic curves*. Second edition. Amer. Math. Soc., Providence, 2004.

[9] A. Beilinson and V. Drinfeld: *Chiral algebras*. Amer. Math. Soc. Colloquium Publications 51. Amer. Math. Soc., Providence, 2004.

[10] J. Lepowsky and H. S. Li: *Introduction to vertex operator algebras and their representations*. Progr. Math., 227. Birkhäuser, Boston, 2004.

[11] M. Rosellen: *A course in vertex algebra*. Preprint, arXiv:math/0607270.

Related Books

[12] J. H. Conway and N. J. A. Sloane: *Sphere packings, lattices and groups*. Third edition. Grundlehren der mathematischen Wissenschaften 290. Springer-Verlag, New York, 1999.

[13] P. Di Francesco, P. Mathieu, and D. Sénéchal: *Conformal field theory*. Springer, New York, 1997.

[14] W. Ebeling: *Lattices and codes. A course partially based on lectures by Friedrich Hirzebruch*. Third edition. Advanced Lectures in Mathematics. Springer Spektrum, Wiesbaden, 2013.

[15] K. Iohara and Y. Koga: *Representation theory of the Virasoro algebra*. Springer Monographs in Mathematics. Springer, London, 2011.

[16] A. A. Ivanov: *The Monster group and Majorana involutions*. Cambridge Tracts in Mathematics 176. Cambridge University Press, Cambridge, 2009.

[17] V. G. Kac: *Infinite-dimensional Lie algebras*. Third edition. Cambridge University Press, Cambridge, 1990.

[18] V. G. Kac, A. K. Raina, and N. Rozhkovskaya: *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*. Second edition. Advanced Series in Mathematical Physics 29. World Scientific, Hackensack, 2013.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

Lecture Notes and Surveys

[19] P. Ginsparg: Applied conformal field theory. In: *Champs, cordes et phénomènes critiques* (Les Houches, 1988), 1–168, North-Holland, Amsterdam, 1990.

[20] P. Goddard: Meromorphic conformal field theory. In: *Infinite-dimensional Lie algebras and groups*, 556–587, Adv. Ser. Math. Phys., 7, World Scientific, Teaneck, 1989.

[21] G. Mason and M. Tuite: Vertex operators and modular forms. In: *A window into zeta and modular physics*, 183–278, Math. Sci. Res. Inst. Publ., 57, Cambridge University Press, Cambridge, 2010.

[22] M. Tuite: Vertex algebras according to Isaac Newton. *J. Phys. A*, **50**, (2017), no. 41.

[23] J. I. Hall: Transposition algebras. *Bull. Inst. Math. Acad. Sin. (N.S.)*, **14**, (2019), 155–187.

Theses

[24] Y. C. Zhu: Vertex operator algebras, elliptic functions and modular forms. Yale University, 1990.

[25] G. Höhn: Selbstduale Vertexoperatorsuperalgebren und das Babymonster. 1995, Bonn. arXiv:0706.0236.

Articles

[26] T. Abe, G. Buhl, and C. Dong: Rationality, regularity and C_2-cofiniteness. *Trans. Amer. Math. Soc.*, **356**, (2004), 3391–3402.

[27] T. Abe, C.Y. Dong, and H.S. Li: Fusion rules for the vertex operator algebras $M(1)^+$ and V_L^+. *Comm. Math. Phys.*, **253**, (2005), 171–219.

[28] T. Arakawa and K. Kawasetsu: Quasi-lisse vertex algebras and modular linear differential equations. In: *Lie groups, geometry, and representation theory*, 41–57, Progr. Math., 326, Birkhäuser/Springer, Cham, 2018.

[29] B. Bakalov and V. G. Kac: Field algebras. *Int. Math. Res. Not.*, **2003**, 123–159.

[30] B. Bakalov and V. G. Kac: Twisted modules over lattice vertex algebras. In: *Lie theory and its applications in physics V*, 3–26, World Scientific, River Edge, 2004.

[31] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov: Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys.*, **B241**, (1984), 333–380.

[32] R. B. Borcherds: Vertex algebras, Kac–Moody algebras, and the Monster. *Proc. Natl. Acad. Sci. USA*, **83**, (1986), 3068–3071.

[33] R. E. Borcherds: Monstrous moonshine and monstrous Lie superalgebras. *Invent, Math.*, **109**, (1992), 405–444.

[34] R. E. Borcherds: Vertex algebras. In: *Topological field theory, primitive forms and related topics* (Kyoto, 1996), 35–77, Progr. Math., 160, Birkhäuser, Boston, 1998.

[35] R. E. Borcherds and A. J. E. Ryba: Modular moonshine. II. *Duke Math. J.*, **83**, (1996), 435–459.

[36] S. Carnahan: 51 constructions of the moonshine module. *Comm. Number Theory Phys.*, **12**, (2018), 305–334.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

[37] S. Carnahan: Building vertex algebras from parts. *Comm. Math. Phys.*, **373**, (2020), 1–43.

[38] J. H. Conway: A simple construction for the Fischer-Griess monster group. *Invent. Math.*, **79**, (1985), 513–540.

[39] T. De Medts, S. F. Peacock, S. Shpectorov, and M. Van Couwenberghe: Decomposition algebras and axial algebras. *J. Algebra*, **556**, (2020), 287–314.

[40] C. Y. Dong: Vertex algebras associated with even lattices. *J. Algebra*, **160**, (1993), 245–265.

[41] C. Y. Dong: Twisted modules for vertex algebras associated with even lattices *J. Algebra*, **165**, (1994), 91–112.

[42] C. Y. Dong: Representations of the moonshine module vertex operator algebra. In: *Mathematical aspects of conformal and topological field theories and quantum groups* (South Hadley, MA, 1992), 27–36, Contemp. Math., 175, Amer. Math. Soc., Providence, 1994.

[43] C. Y. Dong, R. L. Griess, and G. Höhn: Framed vertex operator algebras, codes and Moonshine module. *Comm. Math. Phys.*, **193**, (1998), 407–448.

[44] C. Y. Dong, C. Jiang, and X. Lin: Rationality of vertex operator algebra V_L^+: higher rank. *Proc. London Math. Soc.*, (3), **104**, (2012), 799–826.

[45] C. Dong and J. Lepowsky: The algebraic structure of relative twisted vertex operators. *J. Pure Appl. Algebra*, **110**, (3), (1996), 259–295.

[46] C. Y. Dong, H. S. Li, and G. Mason: Modular-invariance of trace functions in orbifold theory and generalized moonshine. *Comm. Math. Phys.*, **214**, (1), (2000), 1–56.

[47] C. Y. Dong, H. S. Li, and G. Mason: Regularity of rational vertex operator algebras. *Adv. Math.*, **132**, (1997), 148–166.

[48] C. Y. Dong and G. Mason: Monstrous moonshine of higher weight. *Acta Math.*, **185**, (2000), 101–121.

[49] C. Y. Dong, G. Mason, and Y.C. Zhu: Discrete series of the Virasoro algebra and the moonshine module. In: *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), 295–16, Proc. Sympos. Pure Math., 56, Part 2, Amer. Math. Soc., Providence, 1994.

[50] C. Y. Dong and K. Nagatomo: Automorphism groups and twisted modules for lattice vertex operator algebras. In: *Recent developments in quantum affine algebras and related topics* (Raleigh, NC, 1998), 117–133, Contemp. Math., 248, Amer. Math. Soc., Providence, 1999.

[51] B. Doyon, J. Lepowsky, and A. Milas: Twisted vertex operators and Bernoulli polynomials. *Comm. Contemp. Math.*, **8**, (2006), 247–307.

[52] P. Etingof and D. Kazhdan: Quantization of Lie bialgebras. V. Quantum vertex operator algebras. *Selecta Math.*, (N.S.), **6**, (2000), 105–130.

[53] B. L. Feigin and D. B. Fuchs: Cohomology of some nilpotent subalgebras of the Virasoro and Kac-Moody Lie algebras. *J. Geom. Phys.*, **5**, (1988), 209–235.

[54] A. J. Feingold, J. F. Ries, and M. D. Weiner: Spinor construction of the c=1/2 minimal model. In: *Moonshine, the Monster, and related topics* (South Hadley, MA, 1994), 45–92, Contemp. Math., 193, Amer. Math. Soc., Providence, 1996.

[55] E. Frenkel, V. Kac, A. Radul, and W. Wang: $\mathcal{W}_{1+\infty}$ and $\mathcal{W}(gl_\infty)$ with central charge N. *Comm. Math. Phys.*, **170**, (1995), 337–357

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

[56] E. Frenkel and M. Szczesny: Twisted modules over vertex algebras on algebraic curves. *Adv. Math.*, **187**, (2004), 195–227.

[57] I. B. Frenkel and V. G. Kac: Basic representations of affine Lie algebras and dual resonance models. *Inventiones Math.*, **62**, (1980), 23–66.

[58] I. B. Frenkel, J. Lepowsky, and A. Meurman: An E_8-approach to F_1. In: *Finite groups—coming of age* (Montreal, Que., 1982), 99–120, Contemp. Math., 45, Amer. Math. Soc., Providence, 1985.

[59] I. B. Frenkel, J. Lepowsky, and A. Meurman: A natural representation of the Fischer-Griess Monster with the modular function J as character. *Proc. Nati. Acad. Sci. USA*, **81**, (1984), 3256–3260.

[60] I. B. Frenkel and Y. C. Zhu: Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, **66**, (1992), 123–168.

[61] R. L. Griess, Jr.: The friendly giant. *Invent. Math.*, **69**, (1982), 1–102.

[62] J. I. Hall, F. Rehren, and S. Shpectorov: Universal axial algebras and a theorem of Sakuma. *J. Algebra*, **421**, (2015), 394–424.

[63] J. I. Hall, F. Rehren, and S. Shpectorov: Primitive axial algebras of Jordan type. *J. Algebra*, **437**, (2015), 79–115.

[64] Y. Z. Huang: Differential equations and intertwining operators. *Comm. Contemp. Math.*, **7**, (2005), 375–400.

[65] Y. Z. Huang and J. Lepowsky: A theory of tensor products for module categories for a vertex operator algebra. *Selecta Math.* (N.S.), **1**, (1995), 699–756.

[66] V. G. Kac and D. H. Peterson: Infinite-dimensional Lie algebras, theta functions and modular forms. *Adv. Math.*, **53**, (1984), 125–264

[67] C. H. Lam and H. Yamauchi: The Conway–Miyamoto correspondences for the Fischer 3-transposition groups. *Trans. Amer. Math. Soc.*, **375**, (2022), 2025–2067.

[68] J. Lepowsky: Calculus of twisted vertex operators, *Proc. Natl. Acad. Sci. USA*, **82**, (1985), 8295–8299,

[69] J. Lepowsky and R. L. Wilson: Construction of the affine Lie algebra $A_1^{(1)}$. *Comm. Math. Phys.*, **62**, (1978), 43–53.

[70] H. S. Li: Local systems of vertex operators, vertex superalgebras and modules. *J. Pure Appl. Algebra*, **109**, (1996), 143–195.

[71] H. S. Li: Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules. In: *Moonshine, the Monster, and related topics* (South Hadley, MA, 1994), 203–236, Contemp. Math., 193, Amer. Math. Soc., Providence, 1996.

[72] H. S. Li: An analogue of the Hom functor and a generalized nuclear democracy theorem. *Duke Math. J.*, **93**, (1998), 73–114.

[73] H. S. Li: Some finiteness properties of regular vertex operator algebras. *J. Algebra*, **212**, (1999), 495–514.

[74] H. S. Li: Determining fusion rules by $A(V)$-modules and bimodules. *J. Algebra*, **212**, (1999), 515–556.

[75] H. S. Li: Axiomatic G_1-vertex algebras. *Comm. Contemporary Mathematics*, **5**, (2003), 281–327.

[76] H. S. Li and X. P. Xu: A characterization of vertex algebras associated to even lattices. *J. Algebra*, **173**, (1995), 253–270.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

[77] B. H. Lian and G. J. Zuckerman: Some classical and quantum algebras. In: *Lie theory and geometry*, 509–529, Progr. Math., 123, Birkhäuser, Boston, 1994.

[78] G. Mason: Vertex rings and their Pierce bundles. In: *Vertex algebras and geometry*, 45–104, Contemp. Math., 711, Amer. Math. Soc., Providence, 2018.

[79] A. Matsuo: Norton's trace formulae for the Griess algebra of a vertex operator algebra with larger symmetry. *Comm. Math. Phys.*, **224**, (2001), 565–591.

[80] A. Matsuo: 3-transposition groups of symplectic type and vertex operator algebras. *J. Math. Soc. Japan*, **57**, (2005), 639–649.

[81] A. Matsuo and M. Matsuo: The automorphism group of the Hamming code vertex operator algebra. *J. Algebra*, **228**, (2000), 204–226.

[82] A. Matsuo, K. Nagatomo, and A. Tsuchiya: Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. In: *Moonshine: the first quarter century and beyond*, 282–329, London Math. Soc. Lecture Note Ser., 372, Cambridge Univ. Press, Cambridge, 2010.

[83] M. Miyamoto: Griess algebras and conformal vectors in vertex operator algebras. *J. Algebra*, **179**, (1996), 523–548.

[84] M. Miyamoto: Binary codes and vertex operator (super)algebras. *J. Algebra*, **181**, (1996), 207–222.

[85] M. Miyamoto: Vertex operator algebras and modular invariance. In: *Vertex operator algebras, number theory and related topics*, 233–250, Contemp. Math., 753, Amer. Math. Soc., Providence, 2020.

[86] K. Nagatomo and A. Tsuchiya: Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line. *Duke Math. J.*, **128**, (2005), 393–471.

[87] S. P. Norton: The Monster algebra: some new formulae. In: *Moonshine, the Monster, and related topics* (South Hadley, MA, 1994), 297–306, Contemp. Math., 193, Amer. Math. Soc., Providence, 1996.

[88] M. Primc: Vertex algebras generated by Lie algebras. *J. Pure Appl. Algebra*, **135**, (1999), 253–293.

[89] M. Roitman: On twisted representations of vertex algebras. *Adv. Math.*, **176**, (2003), 53–88.

[90] S. Sakuma: 6-transposition property of τ-involutions of vertex operator algebras. *Int. Math. Res. Not.*, **2007**, Art. ID rnm 030, 19 pp.

[91] A. N. Schellekens: Meromorphic $c = 24$ conformal field theories. *Comm. Math. Phys.*, **153**, (1993), 159–185.

[92] G. Segal: Unitary representations of some infinite-dimensional groups. *Comm. Math. Phys.*, **80**, (1981), 301–342.

[93] H. Shimakura: Decompositions of the Moonshine module with respect to subVOAs associated to codes over $\mathbb{Z}_{2k}$. *J. Algebra*, **251**, (2002), 308–322.

[94] H. Shimakura: The automorphism groups of the vertex operator algebras V_L^+: general case. *Math. Z.*, **252**, (2006), 849–862.

[95] H. Shimakura: An E_8-approach to the moonshine vertex operator algebra *J. London Math. Soc.*, (2) **83**, (2011), 493–516.

[96] A. Tsuchiya and Y. Kanie: Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group. In: *Conformal field theory and solvable lattice models* (Kyoto, 1986), 297–372, Adv. Stud. Pure Math., 16, Academic Press, Boston, 1988.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press

[97] A. Tsuchiya, K. Ueno, and Y. Yamada: Conformal field theory on universal family of stable curves with gauge symmetries. In: *Integrable systems in quantum field theory and statistical mechanics*, 459–566, Adv. Stud. Pure Math., 19, Academic Press, Boston, 1989.

[98] J. van Ekeren, C. H. Lam, S. Möller, and H. Shimakura: Schellekens' list and the very strange formula. *Adv. Math.*, **380**, (2021).

[99] W. Q. Wang: Rationality of Virasoro vertex operator algebras. *Int. Math. Res. Not.*, **1993**, (1993), 197–211.

[100] Y. C. Zhu: Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, **9**, (1996), 237–302.

https://doi.org/10.1017/9781009338073.002 Published online by Cambridge University Press