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LYING-OVER PAIRS OF COMMUTATIVE RINGS

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(R, T) is said to be a lying-over pair in case $R \subset T$ is an extension of (commutative) rings each of whose intermediate extensions possesses the lying-over property. This paper treats several types of extensions, including lying-over pairs, which figure in some known characterizations of integrality. Several new characterizations of integrality are thereby obtained; as well, our earlier characterization of P-extensions is sharpened with the aid of a suitable weakening of the incomparability property. In numerous cases, a lying-over pair (R, T) must be an integral extension (for example, if R is quasisemilocal or if (R, T) is a coherent pair of overrings). However, any algebraically closed field F of positive characteristic has an infinitely-generated algebra T such that (F, T) is a lying-over pair. For any ring R, $(R, R[X])$ is a lying-over pair if and only if R has Krull dimension 0. An algebra T over a field F produces a lying-over pair (F, T) if and only if T is integral over each nonfield between F and T. Each lying-over pair (R, T) satisfies the going-up property and, as a consequence, sustains enough incomparability to establish the following inequalities for Krull dimensions:

$$\dim (R) \leqq \dim (T) \leqq \dim (R) + 1.$$

1. Introduction. This paper is a sequel to [**10**]. As in [**10**], we adopt the conventions that each ring considered is commutative, with unit; and an inclusion (extension) of rings signifies that the smaller ring is a subring of the larger and possesses the same multiplicative identity. Also as in [**10**], our work's principal motivation arises from the following characterization of integrality in terms of the lying-over (LO) and incomparability (INC) properties.

FOLKLORE THEOREM. *For rings $R \subset T$, the following are equivalent:*

(1) *T is an integral extension of R;*

(2) (a) *For any inclusions of rings $R \subset A \subset B \subset T$, the extension $A \subset B$ satisfies* INC, *and*

(b) *For any inclusions of rings $R \subset A \subset B \subset T$, the extension $A \subset B$ satisfies* LO.

Two proofs of the above theorem are discussed in Remark 2.5. Whereas much of [**10**] focussed on the study of condition (a), our concern here is

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primarily with condition (b). Our initial line of inquiry was suggested by a comparison between the "folklore theorem" and the next result (established in Theorem 2.1 below).

THEOREM. *For rings $R \subset T$, the following are equivalent:*

(1) *T is an integral extension of R;*

(2) (i) *Each element of T is a root of a suitable polynomial in $R[X]$ with unit content, and*

(ii) *For any inclusions of rings $R \subset A \subset T$ and prime ideal $P \in \mathrm{Spec}(A)$, one has $PT \neq T$.*

It was shown in [**10**, Corollary 4] that conditions (a) and (i) are equivalent, and so (a) characterizes the P-extensions introduced in [**19**]. This characterization is sharpened in Corollary 2.4, with the aid of the notion of a MINC-extension (defined in Section 2). Although we have not completely resolved the question of possible equivalence of conditions (b) and (ii), it is apparent that (b) $\Rightarrow$ (ii), with the converse holding at least in case T is quasisemilocal (see Theorem 2.7). Indeed, in that case, (b) actually guarantees integrality, a phenomenon encountered also in Remarks 2.8 and 2.12 for the cases of certain pseudovaluation domains (in the sense of [**20**]), adjacent extensions ([**6**], [**24**]), and coherent pairs ([**27**], [**28**], [**31**]).

(R, T) is called a *lying-over pair* (for short, an LO-pair) in case condition (b) is satisfied. As Section 2 details, examples of LO-pairs abound, the list thereof being augmented in Lemma 2.11 via localization and the $D + M$ construction. Despite the special cases noted earlier, an LO-pair need not amount to an integral extension, the simplest example being $F \subset F[X]$, for F any field (see Proposition 2.9). In fact, part of Section 4 is devoted to indicating how such polynomial extensions and related integral extensions lurk within any nonintegral LO-pair. For a decidedly nonintegral, infinitely generated LO-pair, see Example 4.3.

We develop new characterizations of integrality in terms of LO-pairs (in Corollaries 3.3, 3.5 and 3.6), the principal tool being the fact that LO-pairs exhibit going-up (GU) behavior. (See Corollary 3.2. Of course, an LO-extension need not satisfy GU.) Section 3 explores consequences of such GU-behavior, such as a bound on the transcendence degree of an LO-pair (in Proposition 3.7). Perhaps the most surprising of these consequences, Proposition 3.10, indicates that any LO-pair satisfies a weak form of INC. As a result (see Corollary 3.11), the Krull dimensions of any LO-pair (R, T) satisfy $\dim(R) \leqq \dim(T) \leqq \dim(R) + 1$. In short, the trust is not only that condition (b) of the "folklore theorem" implies part of (a), but that LO-pairs are in some ways much more similar to integral extensions than are P-extensions.

Any unexplained terminology is standard, as in [**17**] and [**22**].

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2. LO-**pairs and** MINC-**extensions.** We begin by recalling some terminology from [**19**] and [**10**]. Given rings $R \subset T$ and an element $u \in T$, we say that u is *primitive over* R in case u is a root of a polynomial $f \in R[X]$ with unit content, i.e., such that the coefficients of f generate the unit ideal of R. If each element of T is primitive over R, then $R \subset T$ is said to be a *P-extension*. Next, adapting some terminology from [**22**, p. 35], we call (R, T) a *survival pair* in case $PT \neq T$ whenever P is a prime ideal of an intermediate ring between R and T. We now proceed to recast and establish the theorem stated in the introduction.

THEOREM 2.1. *For rings $R \subset T$, the following are equivalent:*
(1) *T is an integral extension of R;*
(2) (i) *$R \subset T$ is a P-extension, and*
(ii) *(R, T) is a survival-pair.*

Proof. It is straightforward to show (1) $\Rightarrow$ (2). Indeed, (1) $\Rightarrow$ (i) since any monic polynomial has unit content; and (1) $\Rightarrow$ (ii) since any integral extension satisfies LO (cf. [**22**, Theorem 44]).

Conversely, assume (i) and (ii). If (1) fails, select $u \in T$ such that u is not integral over R. By replacing R if necessary, we may suppose that R is integrally closed in T. Accordingly, R_P is integrally closed in $T_{R \setminus P}$, for each maximal ideal P of R. Then, since $u/1 \in T_{R\setminus P}$ is primitive over R_P, a lemma of Seidenberg (cf. [**30**, Theorem 6], [**22**, Theorem 67]) guarantees that either $u/1 \in R_P$ or $(u/1)^{-1} \in R_P$. Now, the former possibility cannot hold for every P since $u \notin R$, and so there exists a maximal ideal M of R such that $(u/1) \notin R_M$ and $(u/1)^{-1} \in R_M$, whence $(u/1)^{-1} \in MR_M$. Since (ii) guarantees that $MT \neq T$, the usual modest boost from Zorn's lemma supplies a maximal ideal N of T containing MT. Of course, $N \cap R = M$, and so $NT_{R\setminus M}$ is a proper ideal of $T_{R \setminus M}$. However,

$$1 = (u/1)^{-1}(u/1) \in (MR_M)T_{R\setminus M} = MT_{R\setminus M} \subset NT_{R\setminus M},$$

the desired contradiction, to complete the proof.

Before we can sharpen the characterization of P-extensions in [**10**], the following definition is needed. An inclusion $R \subset T$ of rings is said to be a MINC-*extension* (the notation signifying "incomparability with respect to maximal ideals of R") if, whenever comparable prime ideals $Q_1 \subset Q_2$ of T each contract to the same maximal ideal ($Q_1 \cap R = Q_2 \cap R$) of R, one must then have $Q_1 = Q_2$. Evidently, each INC-extension is a MINC-extension, but as the next example shows, the converse fails.

Example 2.2. Consider a tower of rings $R \subset F \subset T$, where F is a field, the domain R is not a field, and T is not a field. Then $R \subset T$ is a MINC-extension which is not an INC-extension. First, note that T contains a nonmaximal prime ideal, since T is not a field. Accordingly, to prove the

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assertions, it is enough to show that each prime P of T contracts to the prime 0 in R. This, however, is immediate since $P \cap F = 0$, whence $P \cap R = (P \cap F) \cap R = 0$, as desired.

The extension in Example 2.2 is rather "large." In particular, it cannot be generated by a single element. One proof of this comes from the next assertion, which is a sharpening of the statement of the main result in [**10**].

THEOREM 2.3. *For rings $R \subset T$ and an element $u \in T$, the following are equivalent:*
(1) *u is primitive over R;*
(2) *$R \subset R[u]$ is a* MINC-*extension;*
(3) *$R \subset R[u]$ is an* INC-*extension.*

Proof. It is straightforward to check that the proof that (1) $\Leftrightarrow$ (3) in [**10**, Section 2] may be adapted to establish that (1) $\Leftrightarrow$ (2).

Directly from Theorem 2.3 and the pertinent definitions, we next infer the promised strengthening of [**10**, Corollary 4].

COROLLARY 2.4. *For rings $R \subset T$, the following are equivalent:*
(1) *$R \subset T$ is a P-extension;*
(2) *$R \subset R[u]$ is a* MINC-*extension, for each $u \in T$;*
(3) *$A \subset B$ satisfies* INC, *for each inclusion of rings $R \subset A \subset B \subset T$.*

We next introduce a useful terminological device. If $\mathscr{P}$ is a property which may be possessed by ring extensions then, by a *$\mathscr{P}$-pair* (R, T), we mean an extension $R \subset T$ such that $A \subset B$ satisfies $\mathscr{P}$ for all inclusions of rings $R \subset A \subset B \subset T$. For example, condition (a) [resp., (b)] of the "folklore theorem" is simply the requirement that (R, T) be an INC-pair [resp., LO-pair]. Taking $\mathscr{P}$ to be the "survives in" property indicated in [**22**, p. 35], we see that a survival-pair (R, T) is an extension such that $IB \neq B$ whenever one has rings $R \subset A \subset B \subset T$ and a proper ideal I of A. Evidently, this coincides with the earlier definition of "survival-pair." For other illustrations, if $\mathscr{P}$ is the going-down property GD (resp., the property that the associated contraction map of prime spectra is injective), then a domain R with quotient field K produces a $\mathscr{P}$-pair (R, K) if and only if R is a going-down ring in the sense of [**7**, p. 448] and [**11**, Theorem 1] (resp., an i-domain in the sense of [**25**]). Yet another use use of the "pair" terminology occurs in the following slight weakening of the preceding result.

COROLLARY 2.4 (bis). *For rings $R \subset T$, the following are equivalent:*
(1) *$R \subset T$ is a P-extension;*
(2) *(R, T) is a* MINC-*pair;*
(3) *(R, T) is an* INC-*pair.*

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Remark 2.5. By Corollary 2.4(bis), the assertion of the "folklore theorem" may be sharpened to the statement that a ring extension $R \subset T$ is integral if and only if (R, T) is both a MINC-pair and an LO-pair. Observe that any LO-pair must be a survival-pair, since any extension which satisfies LO must also satisfy the "survives in" property. Thus, one possible sharpening of the "folklore theorem" would assert that integrality of $R \subset T$ amounts to (R, T) being both a MINC-pair and a survival-pair. In fact, this is a valid sharpening: one merely needs to combine Theorem 2.1 and Corollary 2.4(bis). Thus, we have proved (a strengthening of) the "folklore theorem."

A somewhat different proof of the same result, namely that integrality results when (R, T) is both a MINC-pair and a survival-pair, will next be sketched. As in the proof of Theorem 2.1, we may suppose R integrally closed in T and (by passage to $R[u]$) T algebra-finite over R. Now, for each maximal ideal P of R, "survival" produces $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$; since $R \subset T$ satisfies MINC, Q is not comparable with any other prime of T which contracts to P. Thus, by Zariski's main theorem (as, e.g., in [**14**, Theorem]), the canonical map $R_f \to T_f$ is an isomorphism for some $f \in R \backslash P$. By varying P, we get finitely many elements f_i such that $R_{f_i} \xrightarrow{\cong} T_{f_i}$ for each i and $\sum R f_i = R$. Since $\prod R_{f_i}$ is a faithfully flat R-algebra [**3**, Proposition 3, p. 109], the inclusion map $R \to T$ is therefore an isomorphism [**3**, Proposition 1, p. 27]; that is, $R = T$, completing the proof.

Additional improvements of the "folklore theorem" will be given in Corollaries 3.3 and 3.5 below.

In comparing the corresponding conditions in the "folklore theorem" and Theorem 2.1, we are now led to the following question. Since the notion of P-extension (condition (i) in Theorem 2.1) coincides with the concept of (M)INC-pair ("folklore"'s condition (a)), it is natural to ask whether the notions of survival-pair (condition (ii)) and LO-pair (condition (b)) are equivalent. As noted in Remark 2.5, considering individual extensions shows that any LO-pair must be a survival-pair. However, as the next examples show, the converse is more subtle.

Examples 2.6. Two examples are given to show that a ring extension $R \subset T$ which satisfies the "survives in" property need not satisfy LO. (Note that we are not claiming that (R, T) is a survival-pair.) The first example presents Noetherian rings, and is due to Chevalley (cf. [**4**, Lemme 2]). For the example, let R be any local (Noetherian) domain of (Krull) dimension at least 2; by [**4**], R possesses a discrete (rank 1) valuation overring T whose maximal ideal N contracts to the maximal ideal M of R. Then R "survives in" T since each proper ideal I of R satisfies $IT \subset MT \subset N \subsetneqq T$. However, $R \subset T$ does not satisfy LO,

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since T has but two primes (0 and N), neither of which contracts to a nonzero nonmaximal prime of R.

The second example is nonNoetherian, and derives from W. J. Lewis' example [**13**, Example 4.4] of a treed domain which is not a going-down ring. In the example, $\dim(R) = 2$, $\dim(T) = 1$, T is a valuation domain, and R is quasilocal and contains a preassigned algebraically closed field. The reader is referred to [**13**] for details of the construction.

At this point, it seems appropriate to indicate a frequent benefit accruing from consideration of "pairs." To wit: it often happens that two inequivalent properties $\mathscr{P}$ and $\mathscr{S}$ of ring extensions sustain only a one-way implication, say $\mathscr{P} \Rightarrow \mathscr{S}$, while the concepts of $\mathscr{P}$-pair and $\mathscr{S}$-pair manage to be equivalent. One example is afforded by taking $\mathscr{P}$ = INC and $\mathscr{S}$ = MINC (see Example 2.2 and Corollary 2.4(bis)). For another example, consider $\mathscr{P}$ = mated (cf. [**7**]) and $\mathscr{S}$ = i(njective contraction map): for details, see [**25**, Example 2.3 and Corollary 2.11]. As will be shown in Corollary 3.2, a third example arises from $\mathscr{P}$ = GU and $\mathscr{S}$ = LO. Thus, somewhat undeterred by Examples 2.6, we shall continue to seek instances in which survival-pairs must be LO-pairs. (The general question remains open.) In this vein, the next result generalizes the observation that T is integral over R whenever (R, T) is a survival-pair for which T is a field.

THEOREM 2.7. *For rings $R \subset T$ such that T is quasisemilocal, the following are equivalent:*

(1) *T is integral over R;*

(2) *(R, T) is an* LO-*pair;*

(3) *(R, T) is a survival-pair.*

Proof. It remains only to prove that (3) $\Rightarrow$ (1). By replacing R if necessary, we may suppose R integrally closed in T. By Theorem 2.1, it is enough to show [given (3)] that each element of T is primitive over R. If the assertion fails, Theorem 2.3 provides $u \in T$ such that $R \subset R[u]$ is not a MINC-extension. Thus, there exist distinct comparable primes $Q_1 \subset Q_2$ of $R[u]$ and a maximal ideal M of R such that $Q_i \cap R = M$ for $i = 1, 2$. Now, let $e_u : R[X] \rightarrow R[u]$ be the R-algebra homomorphism sending X to u. Since $MR[u] \subset Q_1$ we have

$$MR[X] \subset e_u^{-1}(Q_1) \subsetneqq e_u^{-1}(Q_2),$$

whence by [**22**, Theorem 37], $MR[X] = e_u^{-1}(Q_1)$. In particular, $\ker(e_u) \subset MR[X]$, and so $MR[u] = Q_1$ is contained in infinitely many maximal ideals of $R[u]$.

Let $N_1, \ldots, N_k$ be all those maximal ideals of T which contract to M. (Note $k < \infty$ since T is quasisemilocal; possibly, $k = 0$.) Therefore, $R[u]$ has a maximal ideal Q such that $Q_1 \subset Q$ and $Q \neq N_i \cap R[u]$ for

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$i = 1, 2, \ldots, k$. We claim this contradicts (3), for $R[u]$ does not "survive in" T. Specifically, we claim that $QT = T$. If not, then $QT \subset N$ for some maximal ideal N of T, and

$$Q \subset QT \cap R[u] \subset N \cap R[u],$$

whence $Q = N \cap R[u]$ by the maximality of Q. However,

$$M = Q_1 \cap R \subset Q \cap R = N \cap R,$$

whence $M = N \cap R$, contradicting the choice of Q, to complete the proof.

The next result collects some additional instances in which the existence of LO-pairs guarantees integrality of the underlying extensions.

Remark 2.8. (a) It is a standard fact (cf. [**3**, Theorem 1, p. 376]) that if R is a valuation domain and T an overring of R such that R "survives in" T, then $R = T$. Along the same lines, now let R be a pseudovaluation domain, with quotient field K. (Recall, from [**20**], this means that, whenever elements x and y of K and a prime ideal P of R satisfy $xy \in P$, then either $x \in P$ or $y \in P$.) Let M be the (unique) maximal ideal of R, and T the (unique) valuation overring of R with maximal ideal M. (Such M and T exist, according to [**20**, Corollary 1.3 and Theorem 2.7]. In fact, $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets. If R is not a valuation domain, then [**20**, Theorem 2.10] shows $T = \{a \in K : aM \subset R\}$. In general, by [**1**, Proposition 2.5], $T = \{a \in K : aM \subset M\}$). We claim, under these conditions, that (R, T) is a survival-pair if and only if T is the integral closure of R.

For the proof, it suffices to attend to the "only if" half. However, when (R, T) is a survival-pair, one infers readily from the condition $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ that $\mathrm{Spec}(R) = \mathrm{Spec}(A)$ for all rings A between R and T. Then, by "folklore," T is integral over R, whence T is the integral closure of R, as claimed.

(b) Let $R \subset T$ be an adjacent extension in the sense of [**6**]; i.e., R and T are distinct rings such that each ring between R and T coincides with either R or T. (In the notation introduced in [**24**], $R \underset{\max}{\subset} T$.) We claim, under these conditions, that R "survives in" T (that is, (R, T) is a survival-pair) if and only if T is integral over R. For the proof, it suffices to appeal to [**6**, (2.5.3)], itself a translation of work on "minimal homomorphisms" in [**15**, Théorème 2.2(ii)].

In fact, the above appeal to [**6**] also shows that if $R \subset T$ is an integral adjacent extension, then T is module-finite over R and some maximal ideal of R is an ideal (necessarily maximal) of T. If, in addition, either R or T is quasilocal, then $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, a condition met earlier in (a). (For a far-reaching generalization, see [**1**, Theorem 3.10].) For an example of this phenomenon, let R be any field and $T = R[X]/(X^2)$.

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Additional references treating special cases of adjacent extensions include [**18**] and [**26**].

(c) If T is a flat overring of a domain R such that R "survives in" T, then $R = T$ (thus yielding another case in which the presence of a survival-pair guarantees integrality). For the proof, one needs merely to examine an argument of [**29**, Proposition 2].

(d) Recall from [**25**, p. 21] that an extension $A \subset B$ of rings has the *finite fiber property* in case each prime of A is the contraction of at most finitely many primes of B. By examining the proof of Theorem 2.7, one sees that T is integral over R if (R, T) is both a survival-pair and a finite fiber-pair.

(e) One type of "pair" which will be met later (in Remark 3.12(b)) treats the going-down property $\mathscr{P}$ = GD. A family of GD-pairs was explicitly constructed in [**7**, Corollary 4.4 (iii)]. For the present, we wish to note that whenever (R, T) is both a survival pair and GD-pair, then (R, T) is an LO-pair. (However, it will follow from Proposition 2.9 that T need not be integral over R in this case.) For the proof, it is enough to observe that whenever A "survives in" B and $A \subset B$ satisfies GD, then $A \subset B$ also satisfies LO: cf. [**22**, Exercise 38, p. 45].

Despite the special results in Theorem 2.7 and Remark 2.8(a), (b), (c) and (d), not all LO-pairs arise from integral extensions. The next result presents the archetypical example of this phenomenon. Suitably interpreted, its analysis will recur in the characterization in Theorem 4.1 of those LO-pairs whose first component is a field.

PROPOSITION 2.9. *If F is any field, then $(F, F[X])$ is an* LO-*pair.*

Proof. It suffices to show that $A \subset F[X]$ satisfies LO for all rings A between F and $F[X]$. If $A = F$, this is obvious, since 0 is the only prime ideal of F. If $A \neq F$, select $u \in A \setminus F$, and write

$$u = a_0 + a_1X + \ldots + a_nX^n$$

for suitable coefficients $a_i \in F$, with $a_n \neq 0$. Then X is a root of the monic polynomial

$$a_n^{-1}(-u + \sum a_iY^i) \in (F[u])[Y].$$

Hence, X is integral over $F[u]$ and, *a fortiori*, $F[X]$ is integral over A, from which the desired conclusion follows immediately.

In order to obtain "larger" examples of nonintegral LO-pairs, it will be convenient to collect (in Lemma 2.11) some facts about the behavior of LO-pairs under certain constructions. First, we pause to give a "simple overring" characterization of LO-pairs. It may be viewed as an analogue of the characterization of (M)INC-pairs via condition (2) in Corollary 2.4.

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PROPOSITION 2.10. *For any inclusion $R \subset T$ of rings, (R, T) is an* LO-*pair if and only if $A \subset A[u]$ satisfies* LO *whenever $u \in T$ and A is a ring between R and T.*

Proof. The "only if" half is trivial. If the "if" half fails, there exists a ring A between R and T, together with a prime $P \in \mathrm{Spec}(A)$, such that no prime of T contracts to P. Consider the set

$$S = \{(B, W) : B \text{ is a ring, } A \subset B \subset T, W \in \mathrm{Spec}(B), W \cap A = P\}$$

together with the partial order on S given by

$$(B_1, W_1) \leqq (B_2, W_2) \Leftrightarrow [B_1 \subset B_2 \text{ and } W_2 \cap B_1 = W_1].$$

It is straightforward to verify that S contains an upper bound for any given chain in S and so, by Zorn's lemma, there is a maximal element $(D, Q) \in S$. Since no prime of T contracts to P, we have $D \neq T$. Select $u \in T \backslash D$. By hypothesis, $D \subset D[u]$ satisfies LO, and so some prime W of $D[u]$ satisfies $W \cap D = Q$. Then $(D[u], W) \in S$, contradicting maximality of (D, Q), to complete the proof.

For part (a) of the next result, the reader is assumed to be familiar with the basic facts about the $D + M$ construction, as summarized in [**2**, Theorems 2.1 and 3.1]. Note that part (b) is an analogue of a result for P-extensions (that is, INC-pairs) established in [**19**, Theorem 4] and [**10**, Corollary 9]. Analogues of (c) abound: cf. [**6**, Theorem 2.7].

LEMMA 2.11. (a) *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let $R \subset T$ be subrings of F. Then $(R + M, T + M)$ is an* LO-*pair (resp., a survival pair) if and only if (R, T) is an* LO-*pair (resp., a survival pair).*

(b) *Let $R \subset A \subset T$ be rings. If (A, T) is an* LO-*pair (resp., a survival-pair) and $R \subset A$ is integral, then (R, T) is an* LO-*pair (resp., a survival-pair).*

(c) *For rings $R \subset T$, the following are equivalent:*

(1) *(R, T) is an* LO-*pair;*

(2) *(R_S, T_S) is an* LO-*pair for each multiplicative subset S of R;*

(3) *$(R_M, T_{R \backslash M})$ is an* LO-*pair for each maximal ideal M of R.*

Proof. (Sketch): (a) The typical ring which is contained between $R + M$ and $T + M$ is of the form $A + M$, for a ring A intermediate between R and T. Prime ideals of $A + M$ are of two types, viz., primes of V and primes of the form $P + M$, corresponding to $P \in \mathrm{Spec}(A)$. Given these facts, one finds that the key to the assertion concerning survival-pairs is the observation that $(P + M)(T + M) = PT + M$. For

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the assertion about LO-pairs, the key fact is that

$$(Q + M) \cap (A + M) = (Q \cap A) + M$$

for $Q \in \mathrm{Spec}(T)$.

(b) To establish the assertion about LO-pairs, it is enough to prove that $B \subset T$ satisfies LO whenever B is a ring between R and T. To do this, use a "rectangle argument", in the sense described in [**13**, p. 270]. Specifically, "factor" $B \subset T$ as the composite of the extensions $B \subset BA$ and $BA \subset T$. The first of these extensions is integral (since A is integral over R), and thus satisfies LO; the second also satisfies LO, since (A, T) is an LO-pair. Thus, $B \subset T$ satisfies LO, as desired. The proof carries over, *mutatis mutandis*, for the assertion about survival-pairs.

(c) The typical ring contained between R_S and T_S is of the form A_S, for a ring A between R and T. Verification of the assertions then reduces to checking the following two statements. $A_S \subset T_S$ satisfies LO whenever $A \subset T$ satisfies LO; and $A \subset T$ satisfies LO whenever $A_{R \setminus M} \subset T_{R \setminus M}$ satisfies LO for each maximal ideal M of R. Both of these facts are easy consequences of the description of primes in localizations (cf. [**3**, Proposition 1 1(ii), p. 70]).

Remark 2.12 and Proposition 2.13 present our initial applications of the preceding lemma.

Remark 2.12. (a) For each n, either a nonnegative integer or the symbol ∞, there exists an LO-pair (R, T) such that $\dim(R) = n$ and T is not integral over R. For such a construction, let F be a field. If $n = 0$, then Proposition 2.9 shows that $(F, F[X])$ is a satisfactory pair. In general, let $F(X) + M$ be a valuation domain of dimension n (with maximal ideal M); then, by Lemma 2.11(a) and the lore of the $D + M$ construction, $(F + M, F[X] + M)$ is a satisfactory (R, T).

Note that $\dim(T) = \dim(R) + 1$ for the above construction of (R, T). For an extension of this fact, see Corollary 3.11 below.

(b) For the above construction of $(R, T) = (F + M, F[X] + M)$, observe that T is an overring of R but, by [**12**, Theorem 3], R is not coherent (if $n > 0$). Indeed, by [**9**, Proposition 3.9], R is not even a locally finite-conductor domain. (Recall that a domain A, with quotient field K, is said to be *finite-conductor* if, for each $v \in K$, the conductor $\{a \in A : av \in A\}$ is a finitely generated ideal of A. Examples of finite-conductor domains include all coherent domains and GCD (pseudo-Bézout) domains. More generally, a domain A is called *locally finite-conductor* if A_M is finite-conductor for each maximal ideal M of A.) Another approach to this, in the same spirit as Theorem 2.7 and Remark 2.8, will next be given.

First, some terminology is needed. By a *locally finite-conductor pair* (A, B) is meant an extension $A \subset B$ of rings such that each ring con-

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tained between A and B is a locally finite-conductor domain. Examples include the "coherent pairs" recently introduced in [**27**] and [**28**]. As special cases of these, one has the "Noetherian pairs" studied in [**31**]; for an important instance of those, see [**16**, Theorem, p. 129]. Finally, the promised result may be stated: if T is an overring of R such that (R, T) is both an LO-pair and a locally finite-conductor pair, then T is integral over R.

For the proof, we may take R to be integrally closed in T. By globalization [**3**, Corollary 1, p. 88], it is enough to prove that $R_M = T_{R\setminus M}$ for each maximal ideal M of R. By Lemma 2.11(c), $(R_M, T_{R\setminus M})$ is an LO-pair. Moreover, R_M is a finite-conductor domain which is integrally closed in its overring $T_{R\setminus M}$, and so [**23**, Theorem 2] may be applied to show that $R_M = T_{R\setminus M}$, completing the proof.

Before the statement of the next result, it is convenient to recall (cf. [**3**, Exercise 16, p. 143; Corollary, p. 92; Exercise 17, p. 46]) that a ring A is von Neumann regular (absolutely flat) if and only if A_P is a field for each $P \in \operatorname{Spec}(A)$; and that $\dim(A) = 0$ if and only if the reduced ring associated to A is von Neumann regular.

PROPOSITION 2.13. *Let R be a ring. Then $(R, R[X])$ is an* LO-*pair if and only if* $\dim(R) = 0$.

Proof. First, a reduction is needed. Let $T = R/\sqrt{R}$, the reduced ring associated to R. By the above remarks, $\dim(R) = 0$ if and only if T is von Neumann regular. We shall need the fact that $(R, R[X])$ is an LO-pair if and only if $(T, T[X])$ is an LO-pair. (This is somewhat subtle to establish. The relevant techniques will appear in the proof of Lemma 3.1 below and so, for reasons of space, we omit details at this point.) Accordingly, we may assume henceforth that $R = T$, i.e., that R is reduced.

By Lemma 2.11(c) and the preceding remarks, R may be assumed quasilocal. Then Proposition 2.9 dispatches the "if" half. Next, if the "only if" half fails, select a nonzero element, b, in the maximal ideal M of R, and consider $B = R[bX]$. By hypothesis, $B \subset R[X]$ satisfies LO, and so some prime Q of $R[X]$ satisfies $Q \cap B = MB$. (Note that MB is a prime of R since b is not nilpotent.) As $bX \in MR[X] \cap B \subset Q \cap B$, it follows that

$$bX \in MB = M + bMX + b^2MX^2 + \ldots .$$

Hence $b = bm$ for some $m \in M$ and, since $1 - m$ is a unit of R, we have $b = 0$, contradicting the choice of b, to complete the proof.

Let R be a domain, with integral closure R' and quotient field K. Recall from [**19**, Theorem 5] and [**10**, Corollary 5], as translated with the aid of Corollary 2.4(bis), that (R, K) is a (M)INC-pair if and only if R' is a Prüfer domain. However, the situation for survival-pairs or LO-pairs is

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less interesting here, for (R, K) is such a pair if and only if $R = K$. The next result indicates that a more interesting prospect arises if one eliminates the overring K from consideration. It may be viewed as a generalization of the fact (cf. [**22**, Exercise 29, p. 43]) that R and K are the only overrings of R if and only if R is a valuation domain of dimension at most 1. First, we need to recall from [**25**, Proposition 2.34] that R is a quasilocal i-domain if and only if each overring of R is quasilocal; that is, by [**25**, Corollary 2.15] (see also [**8**, Corollary 2.5] and [**17**, Theorem 26.2]), if and only if R' is a valuation domain.

PROPOSITION 2.14. *For a domain R with quotient field K, the following are equivalent:*
(1) *R is a quasilocal i-domain and* $\dim(R) \leqq 1$*;*
(2) *(R, T) is a survival-pair for each ring T such that $R \subset T \subsetneqq K$;*
(3) *(R, T) is an* LO-*pair for each ring T such that $R \subset T \subsetneqq K$.*

Proof. (1) $\Rightarrow$ (3). By (1), R' is a valuation domain of dimension at most 1. But R and R' share the same valuative dimension (by [**17**, Proposition 30.13]), and $\dim_v(R') = \dim(R')$ since R' is a Prüfer domain. Hence each overring of R is quasilocal and has dimension at most 1, from which (3) is evident.

(3) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (1): Let A be any overring of R other than K. By (2), P "survives in" A_P for each nonzero prime ideal P of A. It follows that each maximal ideal M of A satisfies $M \subset P$, whence A is quasilocal, of dimension at most 1. Accordingly, (1) holds, and the proof is complete.

To close the section, note that Proposition 2.14 specializes, in the case of integrally closed R, to yield two new characterizations of the valuation domains of dimension at most 1.

3. LO-pairs are going-up.

The types of $\mathscr{P}$-pairs considered in Section 2 included the following choices for $\mathscr{P}$: INC, LO, and GD. Another property of extensions which is traditionally cited in such lists is the going-up property GU (cf. [**22**, p. 28]), and so one might wonder why GU-pairs were not treated earlier. The fact is (see Corollary 3.2) that the GU-pairs coincide with the LO-pairs, and this section is devoted to exploring consequences of this fact. Before beginning with a technical lemma, it is first convenient to recall from [**22**, Theorem 4.2; Exercise 3, p. 41] that GU $\Rightarrow$ LO and LO $\not\Rightarrow$ GU.

LEMMA 3.1. *Let $R \subset T$ be rings. Then:*
(a) *$R \subset T$ satisfies* GU *if and only if, for each $Q \in \operatorname{Spec}(T)$, $P = Q \cap R$ is such that $R/P \subset T/Q$ satisfies* LO.
(b) *(R, T) is an* LO-*pair if and only if, for each $Q \in \operatorname{Spec}(T)$, $P = Q \cap R$ is such that $(R/P, T/Q)$ is an* LO-*pair.*

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Proof. An easy, but key, observation is the following. If $Q_1 \subset Q_2$ are primes of T and we set $P_i = Q_i \cap R$ for $i = 1, 2$, then

$$(Q_2/Q_1) \cap (R/P_1) = P_2/P_1.$$

This much said, the remaining details for (a) may safely be left to the reader.

As for (b), the "if" half is straightforward. Indeed, to verify that (R, T) is an LO-pair, it is enough to show that $A \subset T$ satisfies LO for all rings A between R and T. Since GU $\Rightarrow$ LO, it therefore suffices, by part (a), to prove that $A/W \subset T/Q$ satisfies LO whenever $Q \in \mathrm{Spec}(T)$ and $W = Q \cap A$. However, by hypothesis, $P = Q \cap R$ is such that $(R/P, T/Q)$ is an LO-pair and, since $R/P \subset A/W \subset T/Q$, the desired result follows.

The proof of the "only if" half of (b) is a bit subtler. Indeed, in case (R, T) is an LO-pair, $Q \in \mathrm{Spec}(T)$ and $P = Q \cap R$, our task is to show that $D \subset T/Q$ satisfies LO for each ring D contained between R/P and T/Q. To this end, note that R/P is contained in T/Q by virtue of the canonical ring-isomorphism

$$R/P \overset{\cong}{\to} (R + Q)/Q,$$

and so D may be identified as $D = A/Q$, where A is a suitable ring contained between $R + Q$ and T. Now, $R + Q \subset T$ satisfies LO since (R, T) is an LO-pair. Thus, by applying the proof's first observation [with R replaced by $R + Q$, and $Q_1 = P_1 = Q$], we see that the extension $(R + Q)/Q \subset T/Q$ inherits LO from $R + Q \subset T$, which completes the proof.

We pause to reiterate that the techniques used in the preceding proof permit one to complete the proof of Proposition 2.13.

As was promised following Examples 2.6, the next result affords another instance in which the "pair" approach surmounts one-way implications.

COROLLARY 3.2. *For any extension $R \subset T$ of rings, (R, T) is a* GU-*pair if and only if (R, T) is an* LO-*pair.*

Proof. Of course, the "only if" half is immediate, since GU $\Rightarrow$ LO. For the "if" half, one need only note that the criterion in part (a) of Lemma 3.1 is a formal consequence of the criterion in part (b).

Recall from Remark 2.5 that the "folklore theorem" translates to the statement that an extension $R \subset T$ of rings is integral if and only if (R, T) is both a MINC-pair and an LO-pair. This result will be sharpened in Corollaries 3.3 and 3.5.

COROLLARY 3.3. *For rings $R \subset T$, the following are equivalent:*

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(1) *T is an integral extension of R;*

(2) (α)*T may be generated as an R-algebra by a set of elements each of which is primitive over R, and*

(b) *(R, T) is an* LO-*pair.*

Proof. By virtue of Proposition 2.10 and Corollary 3.2, the assertion translates to [**10**, Remark 8(c)].

Recall from [**5**, p. 176] that a *normal pair* consists of a domain A and an overring B such that each ring contained between A and B is integrally closed in B. The next result is in the spirit of Remarks 2.8 and 2.12(b).

COROLLARY 3.4. *If (R, T) is both an* LO-*pair and a normal pair, then $R = T$.*

Proof. By Lemma 2.11(c) and the remarks in [**5**, p. 176], R may be assumed quasilocal. If the assertion fails, select $u \in T \backslash R$. As $R \subset R[u]$ is not integral, Corollary 3.3 guarantees that u is not primitive over R. In particular, $1 \notin MR[u]$, where M denotes the maximal ideal of R. However, by [**5**, Proposition 1], either $u \in R$ or $u^{-1} \in R$, whence $u^{-1} \in M$ and $1 = u^{-1}u \in MR[u]$, the desired contradiction.

COROLLARY 3.5. *For rings $R \subset T$, the following are equivalent:*

(1) *T is integral over R;*

(2) ($\aleph$) *$R \subset T$ is a* MINC-*extension, and*

(b) *(R, T) is an* LO-*pair.*

Proof. By the above comments and Corollary 2.4, it suffices to show that ($\aleph$) and (b) jointly guarantee that $R \subset R[u]$ is a MINC-extension for each $u \in T$. However, this is a formal consequence of combining the facts that (thanks to Corollary 3.2) $R[u] \subset T$ satisfies GU (and LO) with the datum ($\aleph$). The proof is complete.

A theorem of Kaplansky [**21**] asserts an extension $R \subset T$ of rings is integral if (and only if) the extension $R[X] \subset T[X]$ of polynomial rings satisfies GU. In tandem with Corollary 3.2, this readily implies the next result.

COROLLARY 3.6. *An extension $R \subset T$ of rings is integral if and only if $(R[X], T[X])$ is an* LO-*pair.*

Recall from [**10**, Corollary 4] that if T is a (commutative) algebra over a field F, then (F, T) is a (M)INC-pair if and only if T is algebraic over F. The next result studies LO-pairs in this context. For motivation, see [**24**, Corollary 4, p. 10] and [**31**, Theorem 4].

PROPOSITION 3.7. *Let F be a field and T an F-algebra such that (F, T) is an* LO-*pair. Then* $\dim(T) \leqq 1$. *If, in addition, T is a domain, then*

$$\text{tr.}\deg_F(T) \leqq 1.$$

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Proof. If the first assertion fails, consider a chain $P_1 \subset P_2 \subset P_3$ of three distinct primes in T. Select $u \in P_2 \backslash P_1$ and $v \in P_3 \backslash P_2$. By passage to $F[u, v]$, with the aid of the primes $P_i \cap F[u, v]$, we may thus assume that T is algebra-finite over F. Hence, by Noether's normalization lemma, T contains two elements, X and Y, (transcendental and) algebraically independent over F. We next sketch two ways to complete the proof.

First, set $A = F[Y]$. Since $F \subset F[X] \subset A[X] \subset T$ and (F, T) is an LO-pair, Corollary 3.2 guarantees that $F[X] \subset A[X]$ satisfies GU. Thus, by the result of Kaplansky stated above, A is integral over F, a contradiction since Y is transcendental.

For a second proof, which makes no use of [**21**], we again let $A = F[Y]$. Since $F \subset A \subset A[X] \subset T$ and (F, T) is an LO-pair, $(A, A[X])$ must also be an LO-pair. Therefore, by Proposition 2.13, A is von Neumann regular, the desired absurdity.

If the final assertion fails, then the quotient field K of T satisfies $\text{tr. deg}_F\ (K) \geqq 2$. Select nonzero $a, b, c, d \in T$ such that $u = ab^{-1}$ and $v = cd^{-1}$ are algebraically independent indeterminates over F. If $A = F[a, b, c, d]$, the result that was just established shows $\dim(A) \leqq 1$, although

$$2 = \text{tr. deg}_F\ (F[u, v]) \leqq \text{tr. deg}_F\ (F(a, b, c, d)) = \text{tr. deg}_F\ (A) = \dim(A),$$

the desired contradiction.

Remark 3.8. We pause to show that Proposition 2.10 is a best-possible LO-analogue of Corollary 2.4, in the following sense. There exists a ring extension $R \subset T$ such that $R \subset R[u]$ satisfies LO for each $u \in T$ although (R, T) is not an LO-pair. For example, let X and Y be algebraically independent indeterminates over a field F, let $R = F$, and set $T = F[X, Y]$. It is trivial that each $R \subset R[u]$ satisfies LO. However, by Proposition 3.7, (R, T) is not an LO-pair since $\dim(T) = 2$.

Examples of LO-pairs have included, *inter alia*, integral extensions and certain polynomial ring extensions. The next result shows that arbitrary LO-pairs not only bring forth associated integral extensions but also exhibit a property shared by all polynomial extensions (cf. [**22**, Theorem 37]).

COROLLARY 3.9. *Let (R, T) be an* LO-*pair, and let $P \in$* Spec(R). *Then:*

(a) *If Q is a maximal ideal of T such that $Q \cap R = P$, then $R/P \subset T/Q$ is an algebraic extension of fields.*

(b) *There is no chain $Q_1 \subset Q_2 \subset Q_3$ of distinct primes of T such that $Q_i \cap R = P$ for $i = 1, 2, 3$.*

Proof. (a) As Q is maximal and (thanks to Corollary 3.2) $R \subset T$ satisfies GU, it follows that P is maximal. However, by Lemma 3.1(b),

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the fields R/P and T/Q form an LO-pair, and so the desired assertion is obtained by applying (the comment preceding) Theorem 2.7.

(b) Lemma 2.11(c) guarantees that (R_P, T_S) is an LO-pair, where $S = R\backslash P$. If $Q = Q_1 \subset Q_2 \subset Q_3$ are distinct primes of T contracting to P, it follows from Lemma 3.1(b) that $(R_P/PR_P, T_S/QT_S)$ is also an LO-pair. Thus, by Proposition 3.7,

$$\dim(T_S/QT_S) \leqq 1.$$

However, the chain formed by the primes Q_iT_S/QT_S reveals

$$\dim(T_S/QT_S) \geqq 2,$$

a contradiction, completing the proof.

On comparing corresponding conditions in the statements of Corollary 3.5 and the "folklore theorem", one might well ask whether LO-pairs satisfy that aspect of the INC property which is not addressed by MINC. The next result provides an affirmative answer.

PROPOSITION 3.10. *Let (R, T) be an LO-pair. If comparable prime ideals $Q_1 \subset Q_2$ of T contract to the same nonmaximal prime ideal $P(=Q_1 \cap R = Q_2 \cap R)$, then $Q_1 = Q_2$.*

Proof. By Lemma 3.1(b), $(R/P, T/Q_1)$ is an LO-pair of domains. Since P is nonmaximal, R/P is not von Neumann regular, and so Proposition 2.13 shows that T/Q_1 is an algebraic extension of R/P. Thus, any $u \in Q_2\backslash Q_1$ (if such exist) satisfies

$$u^n + r_1u^{n-1} + \ldots + r_{n-1}u + r_n \in Q_1$$

for finitely many suitable elements $r_i \in R$. Hence,

$$r_n \in (Tu + Q_1) \cap R \subset Q_2 \cap R = P \subset Q_1,$$

so that $v = u^{n-1} + r_1u^{n-2} + \ldots + r_{n-1}$ satisfies $uv \in -r_n + Q_1 = Q_1$. As $u \notin Q_1$ and Q_1 is prime, it follows that $v \in Q_1$. By repeating the argument enough times, we see that $u \in Q_1$, a contradiction. Hence, no such u exists; i.e., $Q_1 = Q_2$, as asserted.

COROLLARY 3.11. *If (R, T) is an LO-pair, then*

$$\dim(R) \leqq \dim(T) \leqq \dim(R) + 1.$$

Proof. As $R \subset T$ satisfies GU by virtue of Corollary 3.2, it follows that $\dim(R) \leqq \dim(T)$. (Cf. [**22**, Theorem 46]). On the other hand, the inequality $\dim(T) \leqq \dim(R) + 1$ follows directly by combining Corollary 3.9(b) and Proposition 3.10.

Remark 3.12. (a) As Proposition 2.13 shows, the condition $\dim(T) = \dim(R) + 1$ obtains for certain (nonintegral) LO-pairs (R, T). In view

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of Corollary 3.11, one might ask whether the remaining possibility, $\dim(R) = \dim(T)$, characterizes integral extensions amongst the LO-pairs. We show next that the answer is negative in general.

For a counterexample, let $(R, T) = (F + M, F[X] + M)$ as in Remark 2.12(a), such that the dimension of the ambient valuation domain $F(X) + M$ is $n = \infty$. Evidently, (R, T) is an LO-pair, $\dim(R) = \dim(T) = \infty$, but T is not integral over R.

(b) Before giving an affirmative counterpart of the result in (a), observe the following going-down behavior. For the specific (R, T) in (a), each $u \in T$ is of the form $v + m$ for suitable $v \in F[X]$ and $m \in M$, whence $R[u] = F[v] + M$, and so $R \subset R[u]$ satisfies GD by virtue of [**11**, Corollary]. Then, by considering the specific GD-pairs cited in Remark 2.8(e), we are led to conjecture the following result. Let (R, T) be an LO-pair and n a nonnegative integer such that each maximal ideal of R has height n. Assume that $\dim(T) = n(= \dim(R))$. If either (1) $R \subset R[u]$ satisfies GD for each $u \in T\backslash R$ or (2) $R[u]$ is treed for each $u \in T\backslash R$, then T is integral over R.

For the proof, deny, and argue as in the proof of Theorem 2.7. Then the supposed failure of MINC leads to a maximal ideal M of R and an element u of $T\backslash R$ such that $MR[u]$ is a nonmaximal prime ideal of $A = R[u]$. Select a chain $P_0 \subset P_1 \subset \ldots \subset P_n = M$ of $n + 1$ distinct primes of R. In case (1), we obtain (via GD) a chain $Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} \subset MA$ of primes of A such that $Q_i \cap R = P_i$ for $i = 0, 1, \ldots, n - 1$. Since MA is not maximal, it follows that $\dim(A) \geqq n + 1$. However, since (A, T) is an LO-pair, Corollary 3.11 yields $\dim(A) \leqq \dim(T) = n$, a contradiction. For case (2), Corollary 3.2 provides, via GU, a chain of primes $W_0 \subset W_1 \subset \ldots \subset W_n$ of A such that $W_i \cap R = P_i$ for $i = 0, 1, \ldots, n$. We may assume that W_n properly contains MA. Since A is supposed treed, MA and W_{n-1} are comparable. As

$$MA \cap R = M \not\subset P_{n-1} = W_{n-1} \cap R,$$

it follows that $W_{n-1} \subset MA$, so that the height of W_n in A is at least $n + 1$. However, as above, Corollary 3.11 shows that $\dim(A) \leqq n$, the desired contradiction.

It is an open question whether the above result may be extended to the case in which ($n < \infty$ and) neither (1) nor (2) is assumed.

4. LO-pairs with first coordinate a field. One upshot of Lemma 3.1(b) is that the study of LO-pairs (R, T) may be reduced to the case in which R and T are domains. As was apparent from the consequences in Section 3 of Proposition 3.7, the special case in which R is a field merits further consideration. We begin this brief, final section by characterizing this case.

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THEOREM 4.1. *Let $F \subset T$ be rings such that T is a domain and F is a field. Then the following are equivalent:*

(1) *(F, T) is an* LO-*pair;*

(2) *T is integral over each ring A contained between F and T such that A is not a field;*

(3) *T is integral over $F[u]$ for each element $u \in T$ which is transcendental over F.*

Moreover, if the above conditions hold, then T is integral over $F[v]$ for some element $v \in T$.

Proof. (1) $\Rightarrow$ (2). Assume (1) and consider a ring A, $F \subset A \subset T$, such that A is not a field. Of course, A is not algebraic over F (cf. [**22**, Theorem 48]). Thus, by Proposition 3.7,

$$\text{tr. deg}_F\,(A) = 1 = \dim(A).$$

Moreover, any ring B which is between A and T satisfies

$$\text{tr. deg}_F\,(A) \leqq \text{tr. deg}_F\,(B) \leqq 1$$

(again, by Proposition 3.7), so that $\text{tr. deg}_F\,(B) = 1 = \dim(B)$. The desired conclusion, that T is integral over A, may be obtained in two ways. Specifically, observe that (A, T) is a (M)INC-pair and apply the "folklore theorem" (or Remark 2.5 or Corollary 3.5); or apply the result in Remark 3.12(b) [both of whose auxiliary conditions hold in the present case].

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Assume (3), and consider a ring A between F and T. Our task is to show that $A \subset T$ satisfies LO. Without loss of generality, A is not a field. Hence, A is not algebraic over F; that is, there exists an element $u \in A$ which is transcendental over F. By (3), T is integral over $F[u]$ and, *a fortiori*, also integral over A, whence $A \subset T$ satisfies LO, as desired.

The final assertion follows from condition (3) unless T is algebraic over F, in which case setting $v = 0$ suffices. The proof is complete.

If T is a domain containing a field F then, by a *standard factorization of T (over F)*, we shall mean inclusions $F \subset F[u] \subset T$ arising from an element $u \in T$ such that T is integral over $F[u]$. It is clear that if a domain T which contains a field F happens to sustain a standard factorization, then $\text{tr. deg}_F(T) \leqq 1$; by Noether's normalization lemma, the converse holds in case T is algebra-finite over F. By Theorem 4.1, there is a standard factorization whenever the field F and domain T are such that (F, T) is an LO-pair. The next result furthers the preceding observations.

Remark 4.2. (a) If T is a domain containing a field F and $\text{tr. deg}_F\,(T) = 1$, it need not be the case that T has a standard factorization over F.

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Indeed, if $T = F[X]_{XF[X]}$, then one cannot find an element $u \in T$ and a field K between F and T such that T is integral over $K[u]$. (Sketch of proof: Deny. Since T is local, integrality forces $K[u]$ to be local, whence u is algebraic over K. By transitivity of integrality, T is integral over K, so that $1 = \dim(T) = \dim(K) = 0$, a contradiction.)

(b) The example in (a) also shows that the converse of the final assertion of Theorem 4.1 is false. Specifically, there exists a domain T containing a field F such that $\text{tr. deg}_F\,(T) = 1$ and (F, T) is not an LO-pair. (For a simpler example, consider $F \subset F(X) = A$. Observe, moreover, that A does not sustain a standard factorization over F. However, A does have the more general type of factorization treated in (a): it suffices to set $K = A$ and $u = 0$.) We next present an algebra-finite example of this phenomenon.

Let F be any field, and set $B = F[X, (X - 1)^{-1}]$, viewed inside $F(X)$. Then B is algebra-finite over F, $\text{tr. deg}_F\,(B) = 1$ and, since $(X - 1) \in \text{Spec}(F[X])$ does not "survive in" B, it follows that (F, B) is not an LO-pair.

(c) We next present an example of an LO-pair each of whose standard factorizations have nontrivial integral part. Specifically, we claim that $T = \mathbf{C}[X^2, X^3]$ is such that $(\mathbf{R}, T)$ is an LO-pair, but one cannot find a field K between $\mathbf{R}$ and T, together with an element $u \in T$, such that $T = K[u]$.

For the proof, apply Lemma 2.11(b) and Proposition 2.9 to the chain $\mathbf{R} \subset \mathbf{C} \subset \mathbf{C}[X]$, to conclude that $(\mathbf{R}, \mathbf{C}[X])$ is an LO-pair. It follows, *a fortiori*, that $(\mathbf{R}, T)$ is also an LO-pair, as asserted. If the final assertion fails, so that $\mathbf{R} \subset K \subset K[u] = T$, then T cannot be algebraic over the field K. Thus,

$$\text{tr. deg}_{\mathbf{R}}\, K = \text{tr. deg}_{\mathbf{R}}\, T - \text{tr. deg}_{\mathbf{K}}\, T = 0.$$

By the fundamental theorem of algebra, $K \subset \mathbf{C}$. Hence $T = \mathbf{C}[u]$, which is well-known to be impossible. The proof is complete.

The next example significantly extends the moral of Remark 4.2(c).

Example 4.3. Let F be a field of characteristic $p > 0$, and set

$$T = F[X, X^{1/p}, X^{1/p^2}, \ldots, X^{1/p^n}, \ldots]$$

viewed inside an algebraic closure of $F(X)$. Then (F, T) is an LO-pair each of whose standard factorizations have infinitely-generated integral part.

To begin the proof, we claim that there is no field L such that $F \subsetneqq L \subset T$. Indeed, if such L exists, select $v \in L \backslash F$. As $v^{-1} \in L$, there exists a positive integer n such that $v, v^{-1} \in F[X^{1/p^n}]$. However, $F[X^{1/p^n}]$ is (isomorphic to) the F-algebra of polynomials over F in one indeterminate,

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and hence all its units lie in F. In particular, $v \in F$, the desired contradiction.

Next, if A is any ring such that $F \subsetneq A \subset T$, we claim that $\dim(A) = 1$. Indeed, $\dim(A) > 0$, by the result of the preceding paragraph. On the other hand, if $\dim(A) > 1$, choose a chain of distinct primes $0 \subset P_1 \subset P_2$ of A. By arguing as at the start of the proof of Proposition 3.7, one produces a finite-type F-subalgebra B of A such that $\dim(B) \geqq 2$. Select a positive integer n such that $B \subset F[X^{1/p^n}]$. Then

$$\dim(B) = \text{tr. deg}_F\,(B) \leqq \text{tr. deg}_F\,(F[X^{1/p^n}]) = 1,$$

the desired contradiction.

To show that (F, T) is an LO-pair, Theorem 4.1 reduces the task to proving that T is integral over each ring A contained properly between F and T. Observe, by the result of the preceding paragraph, that (A, T) is a MINC-pair. Accordingly, by Remark 2.5, it suffices to show that (A, T) is a survival-pair. If this fails, some ring B such that $A \subset B \subset T$ and some prime P of B satisfy $PT = T$. One deduces an equation $1 = \sum b_i t_i$, where $b_i \in P$ and $t_i \in T$ for each i. Choose a positive integer n such that $b_i, t_i \in F[X^{1/p^n}]$ for each i. Set $D = F[\{b_i\}]$ inside B, and let $Q = P \cap D \in \operatorname{Spec}(D)$. Observe, from the above equation, that Q does not "survive in" $F[X^{1/p^n}]$, contradicting the fact that (thanks to Proposition 2.9), $(F, F[X^{1/p^n}])$ is an LO-pair. This completes the proof that (F, T) is an LO-pair.

To establish the final assertion, we shall actually show it is impossible to have an element $u \in T$ and a field K between F and T such that T is finitely-generated as a $K[u]$-algebra. Indeed, if such u and K existed, the first observation of the proof would force $K = F$. Then T would be a finite-type F-algebra, so that $T \subset F[X^{1/p^n}]$ for a suitable positive integer n, contradicting the presence in T of $X^{1/p^{n+1}}$. This completes the proof.

Recently, Zaks [**32**, Lemma 32] has shown that whenever domains $A \subset B$ satisfy $\dim(B) = \dim(A) + 1 < \infty$ and B is a subring of a polynomial ring over A in several (possibly, infinitely many) variables, then B is isomorphic with a suitable subring of $A[X]$, the polynomial ring over A in one variable. The next result uses the preceding example to address the question of a possible analogue for LO-pairs.

Remark 4.4. (a) Our first result is negative. Specifically, it is possible for an LO-pair (A, B) to satisfy $\dim(A) = 0$ and $\dim(B) = 1$ without forcing B to be isomorphic with a subring of $A[X]$. For example, let $(A, B) = (F, T)$ as in Example 4.3, and assume further that F is the finite field with p elements. For convenience, let Y denote an indeterminate over F. If the assertion fails, there exists an injective ring-homomorphism $h : T \to F[Y]$. Since $h(1) = 1$ and F is a prime field, h restricts to the identity map on F. Now if $f_n = h(X^{1/p^n})$ for each positive integer n,

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then the equations $(X^{1/p^{n+1}})^{p^n} = X^{1/p}$ entail $(f_{n+1})^{p^n} = f_1$, since h is a homomorphism. Comparing degrees, we find that

$$\deg(f_1) = p^n \deg(f_{n+1})$$

for each n. Hence, $\deg(f_1) = 0$, and so $f_1 \in F$, contradicting the fact that h is an injection.

(b) But, there is a positive side, too. Once again, let $(A, B) = (F, T)$ as in Example 4.3. This time, F will be chosen "large". To wit: let $Y, X, X_1, X_2, X_3, \ldots$ be denumerably many algebraically independent indeterminates over the prime field $\mathbf{Z}/p\mathbf{Z}$, and set

$$F = \mathbf{Z}/p\mathbf{Z}(\{X_i^{1/p^j}\}_{1 \leqq i,j}),$$

viewed inside the algebraic closure of $\mathbf{Z}/p\mathbf{Z}(X_1, X_2, X_3, \ldots)$. Then an injective ring-homomorphism $h: T \to F[Y]$ does exist. In fact, it is evident that, by sending X to X_1 and X_i to X_{i+1} for each $i \geqq 1$, one may construct such a $\mathbf{Z}/p\mathbf{Z}$-map h whose image lies within F.

We close with a return to the archetypical examples. The final result may be regarded as a companion to Corollary 3.9(a).

PROPOSITION 4.5. *Let (R, T) be an* LO-*pair such that T is not integral over R. Then there exist a maximal ideal M of R, a nonmaximal prime ideal Q of T satisfying $Q \cap R = M$, and an element $X \in T/Q$ such that X is transcendental over the field $F = R/M$ and T/Q is integral over $F[X]$. (Thus, $R/M \subset T/Q$ factors as the composite of an integral extension and a polynomial ring extension over a field.)*

Proof. By Corollary 3.5, $R \subset T$ is not a MINC extension. Thus, some nonmaximal prime Q of T contracts to a maximal prime M of R. Let $F = R/M$. By Lemma 3.1(b), $(F, T/Q)$ is an LO-pair, and so Theorem 4.1 supplies a standard factorization $F \subset F[u] \subset T/Q$; that is, $u \in T/Q$ and T/Q is integral over $F[u]$. It remains only to show that u is a satisfactory X, that is, that u is not algebraic over F. If the result fails, then $F \subset T/Q$ is a composite of integral extensions, hence integral, so that T/Q is a field, contradicting the nonmaximality of Q. This completes the proof.

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