THE NONEXISTENCE OF A FACTORIZATION FORMULA FOR CAYLEY NUMBERS

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Let C be the Cayley algebra defined over the real field. If, for given elements α, β, and γ of a quaternion subalgebra of C, $\alpha = \beta\gamma$, it follows, by associativity, that for any nonzero element δ of the same quaternion subalgebra, $\alpha = (\beta\delta)(\delta^{-1}\gamma)$. For Cayley numbers ζ, ξ, and η with $\zeta = \xi\eta$, the relation $\zeta = (\xi\delta)(\delta^{-1}\eta)$ in general only holds when δ is a nonzero real number. Because of the existence of factorization results [**1**, **2**] in the orders of C, the question naturally arises of whether it is possible to choose one-to-one mappings, θ and ϕ, of C onto itself such that $\zeta = \theta\xi \,.\, \phi\eta$ whenever $\zeta = \xi\eta$. To discuss this question, we make the following definition.

An ordered triple of one-to-one mappings (θ, ϕ, ψ) of C onto itself is defined to be an *isotopism* of C if

$$\theta\xi \,.\, \phi\eta = \psi\xi\eta \quad \text{for all} \quad \xi, \eta \text{ in } C.$$

An isotopism in which the mappings θ, ϕ, ψ denote multiplication by reals is called *trivial*. For example, the isotopism $(\iota, -\iota, -\iota)$ where ι is the identity mapping and $-\iota$ maps every element onto its negative is a trivial isotopism of C.

The identity

$$u[(\alpha\beta)u] = (u\alpha)(\beta u) \tag{1}$$

in C provides a convenient example of an isotopism of C which is nontrivial.

The triple of mappings (θ, ϕ, ι) of C upon itself, if an isotopism, is called a *principal isotopism* of C. We prove

THEOREM. *There does not exist a nontrivial principal isotopism of C.*

Proof. Suppose that (θ, ϕ, ι) is a principal isotopism of C. Let i_s $(s = 0, 1, \ldots, 7)$ be as usual the basic units of C. Write

$$\theta i_s = u_s, \; \phi i_s = w_s \quad \text{for} \quad 0 \le s \le 7.$$

Then the sets u_s, w_s for $s = 0, 1, \ldots, 7$ are not necessarily units of C. Let $Nw_s = c_s$. Then $u_s w_t = \theta i_s,\; \phi i_t = i_s i_t$ and $u_0 w_0 = i_0 i_0 = 1$. Hence

$$c_0 u_0 = \bar{w}_0. \tag{2}$$

Also $u_0 w_t = i_0 i_t = i_t$ and $u_t w_0 = i_t i_0 = i_t$. Therefore

$$c_0 u_t = i_t \bar{w}_0. \tag{3}$$

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For $1 \leq t \leq 7$, we have $u_t w_t = i_t i_t = -1$. Thus

$$c_t u_t = -\bar{w}_t. \tag{4}$$

For $1 \leq s \leq 7$, $u_t w_s = i_t i_s$ and therefore by (4) we have

$$c_s u_t = (i_t i_s)\bar{w}_s = -(i_t i_s) c_s u_s.$$

Hence

$$u_t = -(i_t i_s) u_s. \tag{5}$$

It follows by (2) and (3) that for $1 \leq t \leq 7$

$$c_0 u_t = i_t \bar{w}_0 = c_0 i_t u_0.$$

Thus

$$u_t = i_t u_0. \tag{6}$$

By (5) and (6) for $1 \leq t$, $s \leq 7$

$$i_t u_0 = -(i_t i_s) u_s = -(i_t i_s)(i_s u_0). \tag{7}$$

Now let $u_0 = \xi_0 + \xi_1\ v$ where ξ_0, ξ_1 are quaternions in $(i_t, i_s, i_t i_s)$ and v is another unit different from ± 1. We assume that $t \neq s$. Then the right hand side of (7) equals

$$\begin{aligned}
-(i_t i_s)(i_s \xi_0 + i_s(\xi_1 v)) &= -(i_t i_s)(i_s \xi_0 + (\xi_1 i_s) v) \\
&= i_t \xi_0 - (i_t i_s)((\xi_1 i_s) v) \\
&= i_t \xi_0 - (\xi_1 i_t) v \\
&= i_t \xi_0 - i_t(\xi_1 v) \\
&= i_t(\xi_0 - \xi_1 v).
\end{aligned}$$

But the left hand side of (7) equals $i_t(\xi_0 + \xi_1 v)$. Thus $\xi_0 + \xi_1 v = \xi_0 - \xi_1 v$. Therefore $\xi_1 = 0$. Now i_t, i_s can be chosen as any pair of units of C. Thus u_0 is real. The result follows by (2), (3) and (4). i.e. all principal isotopisms of Cayley's algebra C are trivial.

REFERENCES

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