COMPOSITIO MATHEMATICA

# Spinors and essential dimension

Skip Garibaldi and Robert M. Guralnick

Compositio Math. **153** (2017), 535–556.

doi:10.1112/S0010437X16008162

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 153 (2017) 535–556
doi:10.1112/S0010437X16008162

# Spinors and essential dimension

Skip Garibaldi and Robert M. Guralnick

*With an appendix by Alexander Premet*

Abstract

We prove that spin groups act generically freely on various spinor modules, in the sense of group schemes and in a way that does not depend on the characteristic of the base field. As a consequence, we extend the surprising calculation of the essential dimension of spin groups and half-spin groups in characteristic zero by Brosnan *et al.* [*Essential dimension, spinor groups, and quadratic forms*, Ann. of Math. (2) **171** (2010), 533–544], and Chernousov and Merkurjev [*Essential dimension of spinor and Clifford groups*, Algebra Number Theory **8** (2014), 457–472] to fields of characteristic different from two. We also complete the determination of generic stabilizers in spin and half-spin groups of low rank.

## 1. Introduction

The essential dimension of an algebraic group $G$ is, roughly speaking, the number of parameters needed to specify a $G$-torsor. Since the notion was introduced in [BR97] and [RY00], there have been many papers calculating the essential dimension of various groups, such as [KM03, CS06, Flo08, KM08, GR09, Mer10, BM12, LMMR13], etc. (See [Mer16, Mer13] or [Rei10] for a survey of the current state of the art.) For connected groups, the essential dimension of $G$ tends to be less than the dimension of $G$ as a variety; for semisimple groups this is well known.[1] Therefore, the discovery by Brosnan *et al.* in [BRV10] that the essential dimension of the spinor group $\mathrm{Spin}_n$ grows exponentially as a function of $n$ (whereas $\dim \mathrm{Spin}_n$ is quadratic in $n$), was startling. Their results, together with refinements for $n$ divisible by 4 in [Mer09] and [CM14], determined the essential dimension of $\mathrm{Spin}_n$ for $n > 14$ over algebraically closed fields of characteristic zero. One goal of the present paper is to extend this result to all characteristics except 2.

### Generically free actions

The source of the characteristic zero hypothesis in [BRV10] is that the upper bound relies on a fact about the action of spin groups on spinors that is only available in the literature in case the field $k$ has characteristic zero. Recall that a group $G$ acting on a vector space $V$ is said to act *generically freely* if there is a dense open subset $U$ of $V$ such that, for every $K \supseteq k$ and every $u \in U(K)$, the stabilizer in $G$ of $u$ is the trivial group scheme. We prove the following theorem.

Received 23 March 2016, accepted in final form 18 July 2016, published online 2 March 2017.
*2010 Mathematics Subject Classification* 11E72 (primary), 11E88, 20G10 (secondary).
*Keywords:* spin module, spin group, characteristic 2, essential dimension, generically free, half-spin representation.

Guralnick was partially supported by NSF grants DMS-1265297, DMS-1302886 and DMS-1600056.


[1] See [GG16] for a proof that works regardless of the characteristic of the field.

Table 1. Stabilizer subgroup scheme in $\mathrm{Spin}_n$ of a generic vector in an irreducible (half-)spin representation for small $n$.

| $n$ | char $k \neq 2$ | char $k = 2$ | $n$ | char $k \neq 2$ | char $k = 2$ |
|---|---|---|---|---|---|
| 6 | $(\mathrm{SL}_3) \cdot (\mathbb{G}_a)^3$ | Same | 11 | $\mathrm{SL}_5$ | $\mathrm{SL}_5 \rtimes \mathbb{Z}/2$ |
| 7 | $G_2$ | Same | 12 | $\mathrm{SL}_6$ | $\mathrm{SL}_6 \rtimes \mathbb{Z}/2$ |
| 8 | $\mathrm{Spin}_7$ | Same | 13 | $\mathrm{SL}_3 \times \mathrm{SL}_3$ | $(\mathrm{SL}_3 \times \mathrm{SL}_3) \rtimes \mathbb{Z}/2$ |
| 9 | $\mathrm{Spin}_7$ | Same | 14 | $G_2 \times G_2$ | $(G_2 \times G_2) \rtimes \mathbb{Z}/2$ |
| 10 | $(\mathrm{Spin}_7) \cdot (\mathbb{G}_a)^8$ | Same | | | |

Theorem 1.1. *Suppose $n > 14$. Then $\mathrm{Spin}_n$ acts generically freely on the spin representation if $n \equiv 1, 3 \bmod 4$; a half-spin representation if $n \equiv 2 \bmod 4$; or a direct sum of the vector representation and a half-spin representation if $n \equiv 0 \bmod 4$. Furthermore, if $n \equiv 0 \bmod 4$ and $n \geqslant 20$, then $\mathrm{HSpin}_n$ acts generically freely on a half-spin representation.*

(We also compute the stabilizer of a generic vector for the values of $n$ not covered by Theorem 1.1. See below for precise statements.)

Throughout, we write $\mathrm{Spin}_n$ for the split spinor group, which is the simply connected cover (in the sense of linear algebraic groups) of the split group $\mathrm{SO}_n$. To be precise, the *vector representation* is the map $\mathrm{Spin}_n \to \mathrm{SO}_n$, which is uniquely defined up to equivalence unless $n = 8$. For $n$ not divisible by 4, the kernel $\mu_2$ of this representation is the unique central $\mu_2$ subgroup scheme of $\mathrm{Spin}_n$.

For $n$ divisible by 4, the natural action of $\mathrm{Spin}_n$ on the spinors is a direct sum of two inequivalent representations, call them $V_1$ and $V_2$, each of which is called a *half-spin* representation. The center of $\mathrm{Spin}_n$ in this case contains two additional copies of $\mu_2$, namely the kernels of the half-spin representations $\mathrm{Spin}_n \to \mathrm{GL}(V_i)$, and we write $\mathrm{HSpin}_n$ for the image of $\mathrm{Spin}_n$ (the isomorphism class of which does not depend on $i$). For $n \geqslant 12$, $\mathrm{HSpin}_n$ is not isomorphic to $\mathrm{SO}_n$.

Theorem 1.1 is known under the additional hypothesis that char $k = 0$, see [AP71, Theorem 1] for $n \geqslant 29$ and [Pop88] for $n \geqslant 15$. The proof below is independent of the characteristic zero results, and so gives an alternative proof.

To simplify some statements, we write 'an irreducible (half-)spin representation of $\mathrm{Spin}_n$' to mean a fundamental minuscule (hence, irreducible) representation of dimension $2^{\lfloor (n-1)/2 \rfloor}$ which is the spin representation for $n$ odd, whereas for $n$ even it is one of two inequivalent half-spin representations, compare [Che97, II.4.3, II.5.1].

We note that Guerreiro proved that the generic stabilizer in the Lie algebra $\mathfrak{spin}_n$, acting on a (half-)spin representation, is central for $n = 22$ and $n \geqslant 24$, see [Gue97, Tables 6 and 9]. At the level of group schemes, this gives the weaker result that the generic stabilizer is finite étale. Regardless, we recover these cases quickly, see § 3; the longest part of the proof of Theorem 1.1 concerns the cases $n = 18$ and 20.

**Generic stabilizer in $\mathrm{Spin}_n$ for small $n$**

For completeness, we list the stabilizer in $\mathrm{Spin}_n$ of a generic vector for $6 \leqslant n \leqslant 14$ in Table 1. The entries for $n \leqslant 12$ and char $k \neq 2$ are from [Igu70]; see §§ 7–9 for the remaining cases. The case $n = 14$ is particularly important due to its relationship with the structure of 14-dimensional quadratic forms with trivial discriminant and Clifford invariant (see [Ros99a, Ros99b, Gar09] and [Mer17]), so we calculate the stabilizer in detail in that case.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

For completeness, we also record the following.

Theorem 1.2. *Let $k$ be an algebraically closed field. The stabilizer in* $\mathrm{HSpin}_{16}$ *of a generic vector in a half-spin representation is isomorphic to* $(\mathbb{Z}/2)^4 \times (\mu_2)^4$.

The proof when $\mathrm{char}\, k \neq 2$ is short, see Lemma 4.2. The case of $\mathrm{char}\, k = 2$ is treated in an appendix by Alexander Premet. (Eric Rains has independently proved this result.)

**Essential dimension**

We recall the definition of essential dimension. For an extension $K$ of a field $k$ and an element $x$ in the Galois cohomology set $H^1(K, G)$, we define $\mathrm{ed}(x)$ to be the minimum of the transcendence degree of $K_0/k$ for $k \subseteq K_0 \subseteq K$ such that $x$ is in the image of $H^1(K_0, G) \to H^1(K, G)$. The *essential dimension* of $G$, denoted $\mathrm{ed}(G)$, is defined to be $\max \mathrm{ed}(x)$ as $x$ varies over all extensions $K/k$ and all $x \in H^1(K, G)$. There is also a notion of *essential $p$-dimension* for a prime $p$. The essential $p$-dimension $\mathrm{ed}_p(x)$ is the minimum of $\mathrm{ed}(\mathrm{res}_{K'/K}\, x)$ as $K'$ varies over finite extensions of $K$ such that $p$ does not divide $[K' : K]$, where $\mathrm{res}_{K'/K} \colon H^1(K, G) \to H^1(K', G)$ is the natural map. The essential $p$-dimension of $G$, $\mathrm{ed}_p(G)$, is defined to be the minimum of $\mathrm{ed}_p(x)$ as $K$ and $x$ vary; trivially, $\mathrm{ed}_p(G) \leqslant \mathrm{ed}(G)$ for all $p$ and $G$, and $\mathrm{ed}_p(G) = 0$ if for every $K$ every element of $H^1(K, G)$ is killed by some finite extension of $K$ of degree not divisible by $p$.

Our Theorem 1.1 gives upper bounds on the essential dimension of $\mathrm{Spin}_n$ and $\mathrm{HSpin}_n$ regardless of the characteristic of $k$. Combining these with the results of [BRV10, Mer09, CM14, Lot13] quickly gives the following, see § 6 for details.

Corollary 1.3. *For $n > 14$ and $\mathrm{char}\, k \neq 2$,*[2]

$$\mathrm{ed}_2(\mathrm{Spin}_n) = \mathrm{ed}(\mathrm{Spin}_n) = \begin{cases} 2^{(n-1)/2} - \dfrac{n(n-1)}{2} & \text{if } n \equiv 1, 3 \bmod 4; \\ 2^{(n-2)/2} - \dfrac{n(n-1)}{2} & \text{if } n \equiv 2 \bmod 4; \text{ and} \\ 2^{(n-2)/2} - \dfrac{n(n-1)}{2} + 2^m & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

*where $2^m$ is the largest power of 2 dividing $n$ in the final case. For $n \geqslant 20$ and divisible by 4,*

$$\mathrm{ed}_2(\mathrm{HSpin}_n) = \mathrm{ed}(\mathrm{HSpin}_n) = 2^{(n-2)/2} - \frac{n(n-1)}{2}.$$

Although Corollary 1.3 is stated and proved for split groups, it quickly implies analogous results for nonsplit forms of these groups, see [Lot13, § 4] for details.

Combining the corollary with the calculation of $\mathrm{ed}(\mathrm{Spin}_n)$ for $n \leqslant 14$ by Markus Rost in [Ros99a, Ros99b] (see also [Gar09]), we find for $\mathrm{char}\, k \neq 2$:

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{ed}(\mathrm{Spin}_n)$ | 0 | 0 | 4 | 5 | 5 | 4 | 5 | 6 | 6 | 7 | 23 | 24 | 120 | 103 | 341 | 326 |

**Notation**

Let $G$ be an affine group scheme of finite type over a field $k$, which we assume is algebraically closed. (If $G$ is additionally smooth, then we say that $G$ is an *algebraic group*.) If $G$ acts on a variety $X$, the stabilizer $G_x$ of an element $x \in X(k)$ is a subgroup scheme of $G$ with $R$-points

$$G_x(R) = \{g \in G(R) \mid gx = x\}$$

for every $k$-algebra $R$.

[2] Added in proof: Totaro has recently shown that the same result holds also in characteristic 2, see his paper *Essential dimension of the spin groups in characteristic 2*, arXiv:1701.05959.

If $\mathrm{Lie}(G) = 0$, then $G$ is finite and étale. If additionally $G(k) = 1$, then $G$ is the trivial group scheme $\mathrm{Spec}\, k$.

For a representation $\rho\colon G \to \mathrm{GL}(V)$ and elements $g \in G(k)$ and $x \in \mathrm{Lie}(G)$, we denote the fixed spaces by $V^g := \ker(\rho(g) - 1)$ and $V^x := \ker(\mathrm{d}\rho(x))$.

We use fraktur letters such as $\mathfrak{g}$, $\mathfrak{spin}_n$, etc., for the Lie algebras $\mathrm{Lie}(G)$, $\mathrm{Lie}(\mathrm{Spin}_n)$, etc.

## 2. Fixed spaces of elements

The main purpose of this section is to prove the following.

Proposition 2.1. *Let $V$ be an irreducible (half-)spin representation for $\mathrm{Spin}_n$ over an algebraically closed field $k$. Then for $n \geqslant 6$:*

(i) *for all noncentral $x \in \mathfrak{spin}_n$, $\dim V^x \leqslant \frac{3}{4} \dim V$;*
(ii) *if $n$ is divisible by 4, then for all noncentral $x \in \mathfrak{hspin}_n$, $\dim V^x \leqslant \frac{3}{4} \dim V$;*
(iii) *for all noncentral $g \in \mathrm{Spin}_n(k)$, $\dim V^g \leqslant \frac{3}{4} \dim V$;*
(iv) *if $n > 8$ and $g \in \mathrm{Spin}_n(k)$ is noncentral semisimple, then $\dim V^g \leqslant \frac{5}{8} \dim V$.*

Before we proceed with the proof, consider the general situation where $G$ is a split semisimple algebraic group with a representation $\rho : G \to \mathrm{GL}(V)$ over $k$. *For $x, y \in \mathfrak{g}$, if $y$ is in the Zariski-closure of $G(k) \cdot x$, then* $\dim V^x \leqslant \dim V^y$. This is clear, because the set of $z \in \mathfrak{g}$ with $\dim V^z > \dim V^y$ is Zariski-closed and stable under $G(k)$. We refer to this substitution principle as specializing $x$ to $y$.

Recall that $\mathrm{Lie}(Z(G))$ is the center of $\mathrm{Lie}(G) = \mathfrak{g}$. The previous observation shows that, *among noncentral $x \in \mathfrak{g}$, the maximum of $\dim V^x$ is achieved for a root element*, i.e. a generator of a one-dimensional root subalgebra. To see this, note that in the Jordan decomposition $x = s+n$ where $s$ is semisimple, $n$ is nilpotent, and $[s, n] = 0$, we have $V^x \subseteq V^s \cap V^n$, so it suffices to prove the result when $x$ is nonzero nilpotent and when $x$ is noncentral semisimple. In the former case, there is a root element $y \in \overline{G(k) \cdot x}$. If $x$ is noncentral semisimple, choose a root subgroup $U_\alpha$ of $G$ belonging to a Borel subgroup $B$ such that $x$ lies in $\mathrm{Lie}(B)$ and does not commute with $U_\alpha$. Then for all $y \in \mathrm{Lie}(U_\alpha)$ and all scalars $\lambda$, $x + \lambda y$ is in the same $\mathrm{Ad}(G)$ orbit as $x$ and $y$ is in the closure of the set of such elements; replace $x$ with $y$.

A similar analysis for elements of $G(k)$ shows that it suffices to consider root elements and semisimple elements $g$ such that $\rho(g)$ has prime order.

Lemma 2.2. *Suppose $g \in \mathrm{Spin}_8(k)$ is semisimple and $x$ is a graph automorphism of order 3. If $g^x$ is conjugate to $g$, then $g$ is conjugate to an element of $G_2(k)$.*

*Proof.* Some maximal torus $T$ is normalized by $x$, and we may assume that $T$ contains $g$. Let $W$ be a finite group inducing the Weyl group on $T$ (so $W/(T \cap W) = 2^3 S_3$). We can certainly choose $W$ so that 9 does not divide the order of $W$.

Since $g^x$ is conjugate to $g$, there is some $w \in W$ with $g^x = g^w$ and $g$ centralizes $y = xw^{-1}$. Raising $y$ to a power prime to 3, we see that $g$ centralizes an element of order 3 in the coset $xG$. The centralizer of any such element is contained in $G_2$. (If $\mathrm{char}\, k \neq 3$, the centralizers are $A_2$ or $G_2$ and $A_2 < G_2$. If $\mathrm{char}\, k = 3$, the centralizers are $G_2$ and a nonreductive subgroup of $G_2$.) □

Lemma 2.3. *Let $1 \neq g \in G_2(k)$ be semisimple. For each of the three eight-dimensional irreducible representations of $\mathrm{Spin}_8$, every eigenspace of $g$ has dimension at most 4.*



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

*Proof.* The weights of the representation $V$ are zero with multiplicity 2 and, with multiplicity 1, six nonzero weights $\pm\chi_i$ for $i = 1, 2, 3$ such that $\chi_1 + \chi_2 + \chi_3 = 0$.

Consider the eigenspace for $g$ with eigenvalue $\lambda \in k$. If $\lambda \neq \pm 1$, then the claim is obvious since $V$ is self-dual. As $g \neq 1$, $g$ cannot lie in the kernel of all three of the $\chi_i$. If $\lambda = 1$, then $g$ is in the kernel of at most one of the $\chi_i$, proving the claim. If $\lambda = -1$, then $g$ is in the kernel of at most two of the $\chi_i$, again proving the claim. □

*Proof of Proposition 2.1.* For (i), by the discussion above it suffices to check it in the case $x$ is a root element. If $n = 6$, then $\mathfrak{spin}_n \cong \mathfrak{sl}_4$ and $V$ is the natural representation of $\mathfrak{sl}_4$, so we have the desired equality. For $n > 6$, the module restricted to $\mathfrak{spin}_{n-1}$ is either irreducible or the direct sum of two half-spins and so the result follows.

For (ii), the natural map $\mathfrak{spin}_n \to \mathfrak{hspin}_n$ is a bijection on root subalgebras, so the claim follows from (i).

For (iii), we may assume that $g$ is unipotent or semisimple. If $g$ is unipotent, then by taking closures, we may pass to root elements and argue as for $x$ in the Lie algebra.

If $g$ is semisimple, we actually prove a slightly stronger result: *all* eigenspaces have dimension at most $\frac{3}{4}\dim V$. Note that this is the correct bound for $n = 6$, as $\mathrm{Spin}_6 \cong \mathrm{SL}_4$.

Suppose now that $n$ is even. The image of $g$ in $\mathrm{SO}_n$ can be viewed as an element of $\mathrm{SO}_{n-2} \times \mathrm{SO}_2$, where it has eigenvalues $(a, a^{-1})$ in $\mathrm{SO}_2$. Replacing if necessary $g$ with a multiple by an element of the center of $\mathrm{Spin}_n$, we may assume that $g$ is in the image of $\mathrm{Spin}_{n-2} \times \mathrm{Spin}_2$. Then $V = V_1 \oplus V_2$ where the $V_i$ are distinct half-spin modules for $\mathrm{Spin}_{n-2}$ and the $\mathrm{Spin}_2$ acts on each (since they are distinct and $\mathrm{Spin}_2$ commutes with $\mathrm{Spin}_{n-2}$). By induction, every eigenspace of $g$ has dimension at most $\frac{3}{4}\dim V_i$ and the $\mathrm{Spin}_2$ component of $g$ acts as a scalar, so this is preserved.

If $n$ is odd, then the image of $g$ in $\mathrm{SO}_n$ has eigenvalue 1 on the natural module, so is contained in a $\mathrm{SO}_{n-1}$ subgroup. Replacing if necessary $g$ with $gz$ for some $z$ in the center of $G$, we may assume that $g$ is in the image of $\mathrm{Spin}_{n-1}$ and the claim follows by induction.

For (iv), the crux case is where $n = 10$. As in the proof of (iii), we may assume that $g$ is the image of some $(g_8, a) \in \mathrm{Spin}_8 \times \mathrm{Spin}_2$ for some $a \in k^\times$, so $V = V_1 \oplus V_2$ is a sum of two inequivalent eight-dimensional representations of $\mathrm{Spin}_8$ and $g$ acts on $V$ as $\rho(g) = (a\rho_1(g_8), a^{-1}\rho_2(g_8))$ and $\rho_i\colon \mathrm{Spin}_8 \to \mathrm{GL}(V_i)$.

We bound the dimension of the space $\ker(\rho(g) - b) = \ker(\rho_1(g_8) - b/a) \oplus \ker(\rho_2(g_8) - ba)$ for $b \in k^\times$. If $\rho_i(g_8)$ is a scalar for some $i$, then $\rho_1(g_8) = \rho_2(g_8) = \pm 1$; as $g$ is noncentral, $a \neq \pm 1$, and this case is trivial.

Suppose $b/a \neq \pm 1$, so $\dim\ker(\rho_1(g_8) - b/a) \leqslant 4$ because $(V_i, \rho_i)$ is self-dual. As $\rho_2(g_8)$ is not a scalar, the dimension of its $ba$ eigenspace is at most 6. The case $ab \neq \pm 1$ is similar, so we may assume that $ab, b/a = \pm 1$, hence $a^4 = 1$ and $b = \pm a$. After replacing $g$ by the image of $(g_8^2, 1)$ if necessary, we are reduced to considering $\pm 1$ eigenspaces of $g$ the image of $(g_8, 1)$ so that $\rho_i(g_8)$ has order two.

If $g_8$ is in a $G_2$ subgroup, then this dimension is at most 8 (Lemma 2.3). If $g_8$ has order 2 (necessarily $\operatorname{char} k \neq 2$), then the conjugacy class of $g_8$ is invariant under the full group of graph automorphisms and so lives in $G_2$ (Lemma 2.2).

If $g_8$ has order 4 and has order 2 mod the center, then $g_8$ has no fixed space in two of the representations (since the square is $-1$) and at most a 6-space in one. Similarly for the $-1$ eigenspace. This completes the proof for $n = 10$.

The result for $n = 9$ follows, because $\mathrm{Spin}_9$ is contained in $\mathrm{Spin}_{10}$ and the module is the same. For $n > 10$, up to multiplying $g$ by an element of the center, it is the image of some

$(g_{n-2}, a) \in \mathrm{Spin}_{n-2} \times \mathrm{Spin}_2$, and the restriction of $V$ to $\mathrm{Spin}_{n-2}$ is a direct sum of irreducible (half-)spin representations as in the $n = 10$ case. The claim follows by induction. □

*Example* 2.4. The upper bound in Proposition 2.1(iv) is sharp. To see this, suppose $\mathrm{char}\, k \neq 2$. We can view $\mathrm{SO}_n$ as the group of matrices

$$\mathrm{SO}_n(k) = \{A \in \mathrm{SL}_n(k) \mid SA^\top S = A^{-1}\},$$

where $S$ is the matrix of 1 on the 'second diagonal', i.e. $S_{i,n+1-i} = 1$ and the other entries of $S$ are zero. The intersection of the diagonal matrices with $\mathrm{SO}_n$ is a maximal torus. For $n$ even, one finds elements of the form $(t_1, t_2, \ldots, t_{n/2}, t_{n/2}^{-1}, \ldots, t_1^{-1})$, and we abbreviate these as $(t_1, t_2, \ldots, t_{n/2}, \ldots)$.

We may identify $\mathrm{Spin}_8$, via a direct sum of its three inequivalent eight-dimensional irreducible representations, with a subgroup of $\mathrm{SO}_8 \times \mathrm{SO}_8 \times \mathrm{SO}_8$. In this sense, the triple $g_8 := (g_0, g_1, g_2)$ for

$$g_0 = (a^2, a^2, a^2, a^{-2}, \ldots), \quad g_1 = (a^4, 1, 1, 1, \ldots), \quad \text{and} \quad g_2 = (a^2, a^2, a^2, a^2, \ldots) \tag{2.5}$$

belongs to $\mathrm{Spin}_8$, see [Gar98, Example 1.6]. In the notation of the proof of Proposition 2.1(iv), take $g \in \mathrm{Spin}_{10}(k)$ to be the image of $(g_8, a) \in \mathrm{Spin}_8 \times \mathrm{Spin}_2$ such that $\rho_i(g_8) = g_i$ for $i = 1, 2$ and $a \in k^\times$ is not a root of unity. The $a$-eigenspace of $g$ has dimension 10, six of which comes from $a\rho_1(g_8)$ and four from $a^{-1}\rho_2(g_8)$. (Although the formulas in [Gar98] assume $\mathrm{char}\, k \neq 2$, the conclusion of this example holds also when $\mathrm{char}\, k = 2$, because the conclusion concerns the weights of the three representations $\rho_i$, which are independent of the characteristic.)

One can also find semisimple elements of $\mathrm{Spin}_{12}$ that have a 20-dimensional fixed space on a (32-dimensional) half-spin representation.

The proposition will feed into the following elementary lemma, which resembles [AP71, Lemma 4] and [Gue97, § 3.3].

Lemma 2.6. *Let $V$ be a representation of a semisimple algebraic group $G$ over an algebraically closed field $k$.*

(i) *If for every unipotent $g \in G$ and every noncentral semisimple $g \in G$ whose image in $\mathrm{GL}(V)$ has prime order we have*

$$\dim V^g + \dim g^G < \dim V, \tag{2.7}$$

*then for generic $v \in V$, $G_v(k)$ is central in $G(k)$.*

*For the next two statements, suppose $\mathrm{char}\, k = p > 0$ and let $\mathfrak{h}$ be a $G$-invariant subspace of $\mathfrak{g}$.*

(ii) *If, for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ such that $x^{[p]} = x$ or $x^{[p]^n} = 0$ for some $n$, we have*

$$\dim V^x + \dim(\mathrm{Ad}(G)x) < \dim V, \tag{2.8}$$

*then for generic $v \in V$, $\mathfrak{g}_v \subseteq \mathfrak{h}$.*

(iii) *If $\mathfrak{h}$ consists of semisimple elements and equation (2.8) holds for every $x \in \mathfrak{g} \setminus \mathfrak{h}$ with $x^{[p]} \in \{0, x\}$, then for generic $v$ in $V$, $\mathfrak{g}_v \subseteq \mathfrak{h}$.*

We will apply this to conclude that $G_v$ is the trivial group scheme for generic $v$, using that $\mathrm{Lie}(G_v) \subseteq \mathfrak{g}_v$. Note that the hypothesis that $\mathrm{char}\, k \neq 0$ in (ii) and (iii) is harmless: when $\mathrm{char}\, k = 0$, the conclusion of (i) suffices.

*Proof.* For (i), see [GG15, § 10] or adjust slightly the following proof of (ii). For $x \in \mathfrak{g}$, define

$$V(x) := \{v \in V \mid \text{there is } g \in G(k) \text{ such that } xgv = 0\} = \bigcup_{g \in G(k)} gV^x.$$

Define $\alpha\colon G \times V^x \to V$ by $\alpha(g, w) = gw$, so the image of $\alpha$ is precisely $V(x)$. The fiber over $gw$ contains $(gc^{-1}, cw)$ for $\mathrm{Ad}(c)$ fixing $x$, and so $\dim V(x) \leqslant \dim(\mathrm{Ad}(G)x) + \dim V^x$.

Let $X \subset \mathfrak{g}$ be the set of $x \in \mathfrak{g}\backslash\mathfrak{h}$ such that $x^{[p]} = x$ or $x^{[p]^n} = 0$ for some $n$; it is a union of finitely many $G$-orbits. (Every toral element, i.e. $x$ with $x^{[p]} = x$, belongs to $\mathrm{Lie}(T)$ for a maximal torus $T$ in $G$ by [BS66], and it is obvious that there are only finitely many $G$-orbits of toral elements in $\mathrm{Lie}(T)$.) Now $V(x)$ depends only on the $G$-orbit of $X$ (because $V^{\mathrm{Ad}(g)x} = gV^x$), so the union $\bigcup_{x \in X} V(x)$ is a finite union. As $\dim V(x) < \dim V$ by the previous paragraph, the union $\bigcup V(x)$ is contained in a proper closed subvariety $Z$ of $V$, and for every $v$ in the (nonempty, open) complement of $Z$, $\mathfrak{g}_v$ does not meet $X$.

For each $v \in (V \backslash Z)(k)$ and each $y \in \mathfrak{g}_v$, we can write $y$ as

$$y = y_n + \sum_{i=1}^{r} \alpha_i y_i, \quad [y_n, y_i] = [y_i, y_j] = 0 \text{ for all } i, j \tag{2.9}$$

such that $y_1, \ldots, y_r \in \mathfrak{g}_v$ are toral, $y_n \in \mathfrak{g}_v$ is nilpotent, and $y_n$ and the $y_i$ are in $\mathfrak{g}_v$, see [SF88, p. 82, Theorem 2.3.6(2)]. Thus, $y_n$ and the $y_1, \ldots, y_r$ are in $\mathfrak{h}$ by the previous paragraph, completing the proof of (ii).

For (iii), repeat the argument of (ii) above, changing $X$ to be the set of $x \in \mathfrak{g}\backslash\mathfrak{h}$ such that $x^{[p]} \in \{0, x\}$. In (2.9), the $y_i$ belong to $\mathfrak{h}_v$, hence we may assume that $y = y_n$. If $y^{[p]} = 0$, then $y$ is a nilpotent element of $\mathfrak{h}$, therefore zero, and we are done. Otherwise, there would exist $q \geqslant p$ the largest power of $p$ with $y^{[q]} \neq 0$, in which case $y^{[q]} \in \mathfrak{g}_v$ and $(y^{[q]})^{[p]} = 0$, hence $y^{[q]}$ is a nonzero nilpotent element of $\mathfrak{h}$, a contradiction. $\square$

Note that, in proving Theorem 1.1, we may assume that $k$ is algebraically closed (and so this hypothesis in Lemma 2.6 is harmless). Indeed, suppose $G$ is an algebraic group acting on a vector space $V$ over a field $k$. Fix a basis $v_1, \ldots, v_n$ of $V$ and consider the element $\eta := \sum t_i v_i \in V \otimes k(t_1, \ldots, t_n) = V \otimes k(V)$ for indeterminates $t_1, \ldots, t_n$; it is a sort of generic point of $V$. Certainly, $G$ acts generically freely on $V$ over $k$ if and only if the stabilizer $(G \times k(V))_v$ is the trivial group scheme, and this statement is unchanged by replacing $k$ with an algebraic closure. That is, $G$ acts generically freely on $V$ over $k$ if and only if $G \times K$ acts generically freely on $V \otimes K$ for $K$ an algebraic closure of $k$.

## 3. Proof of Theorem 1.1 for $n > 20$

Suppose $n > 2$, and put $V$ for an irreducible (half-)spin representation of $\mathrm{Spin}_n$. Recall that

$$\dim \mathrm{Spin}_n = r(2r-1) \quad \text{and} \quad \dim V = 2^{r-1} \quad \text{if } n = 2r$$

whereas

$$\dim \mathrm{Spin}_n = 2r^2 + r \quad \text{and} \quad \dim V = 2^r \quad \text{if } n = 2r+1$$

and in both cases $\mathrm{rank}\, \mathrm{Spin}_n = r$. Proposition 2.1 gives an upper bound on $\dim V^g$ for noncentral $g$, and certainly the conjugacy class of $g$ has dimension at most $(\dim \mathrm{Spin}_n) - r$. If we assume $n > 20$ and apply these, we obtain (2.7) and consequently the stabilizer $S$ of a generic $v \in V$ has



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

$S(k)$ central in $\mathrm{Spin}_n(k)$. Repeating this with the Lie algebra $\mathfrak{spin}_n$ (and $\mathfrak{h}$ the center of $\mathfrak{spin}_n$) we find that $\mathrm{Lie}(S)$ is central in $\mathfrak{spin}_n$. For $n$ not divisible by 4, the representation $\mathrm{Spin}_n \to \mathrm{GL}(V)$ restricts to a closed embedding on the center of $\mathrm{Spin}_n$, so $S$ is the trivial group scheme as claimed in Theorem 1.1.

For $n$ divisible by four, we conclude that $\mathrm{HSpin}_n$ acts generically freely on $V$, using Proposition 2.1(ii). As the kernel $\mu_2$ of $\mathrm{Spin}_n \to \mathrm{HSpin}_n$ acts faithfully on the vector representation $W$, it follows that $\mathrm{Spin}_n$ acts generically freely on $V \oplus W$, completing the proof of Theorem 1.1 for $n > 20$.

## 4. Proof of Theorem 1.1 for $n \leqslant 20$ and characteristic $\neq 2$

In this section we assume that $\mathrm{char}\, k \neq 2$, and in particular the Lie algebra $\mathfrak{spin}_n$ (and $\mathfrak{hspin}_n$ in case $n$ is divisible by four) is naturally identified with $\mathfrak{so}_n$.

### Case $n = 18$ or $20$

Take $V$ to be a half-spin representation of $G = \mathrm{Spin}_n$ (if $n = 18$) or $G = \mathrm{HSpin}_n$ (if $n = 20$). To prove Theorem 1.1 for these $n$, it suffices to prove that $G$ acts generically freely on $V$, which we do by verifying the inequalities in Lemma 2.8(i) and (ii).

*Nilpotents and unipotents.* Let $x \in \mathfrak{g}$ with $x$ nilpotent. The argument for unipotent elements of $G$ is essentially identical (as we assume $\mathrm{char}\, k \neq 2$) and we omit it.

If, for a particular $x$, we find that the centralizer of $x$ has dimension greater than 89 (if $n = 18$) or greater than 62 (if $n = 20$), then $\dim(\mathrm{Ad}(G)x) < \frac{1}{4} \dim V$ and we are done by Proposition 2.1.

The most interesting case is where the Jordan form of $x$ has partition $(2^{2t}, 1^{n-2t})$ for some $t$, where exponents denote multiplicity. If $n = 20$, then such a class has centralizer of dimension at least 100, and we are done. If $n = 18$, we may assume by similar reasoning that $t = 3$ or 4. The centralizer of $x$ has dimension at least 81, so $\dim(\mathrm{Ad}(G)x) \leqslant 72$. We claim that $\dim V^x \leqslant 140$; it suffices to prove this for an element with $t = 3$, as the element with $t = 4$ specializes to it. View it as an element in the image of $\mathfrak{so}_9 \times \mathfrak{so}_9 \to \mathfrak{so}_{18}$ where the first factor has partition $(2^4, 1)$ and the second has partition $(2^2, 1^5)$. Now, triality on $\mathfrak{so}_8$ sends elements with partition $2^4$ to elements with partition $2^4$ and $(3, 1^5)$ (see, for example, [CM93, p. 97]) consequently the $(2^4, 1)$ in $\mathfrak{so}_9$ acts on the spin representation of $\mathfrak{so}_9$ as a $(3, 2^4, 1^5)$. Similarly, the $(2^2, 1^5)$ acts on the spin representation of $\mathfrak{so}_9$ as $(2^4, 1^8)$. The action of $x$ on the half-spin representation of $\mathfrak{so}_{18}$ is the tensor product of these, and we find that $\dim V^x \leqslant 140$ as claimed.

Suppose $x$ is nilpotent and has a Jordan block of size at least five. An element with partition $(5, 1)$ in $\mathfrak{so}_6$ is a regular nilpotent in $\mathfrak{sl}_4$ with one-dimensional kernel. Using the tensor product decomposition as in the proof of Proposition 2.1, we deduce that an element $y \in \mathfrak{so}_n$ with partition $(5, 1^{n-5})$ has $\dim V^y \leqslant \frac{1}{4} \dim V$, and consequently by specialization $\dim V^x \leqslant \frac{1}{4} \dim V$. As $\dim(\mathrm{Ad}(G)x) \leqslant \dim G - \mathrm{rank}\, G < \frac{3}{4} \dim V$, the inequality is verified for this $x$.

Now suppose that $x$ is nilpotent and the largest Jordan block for $x$ has size 4. Thus, there are at least two Jordan blocks of size 4. We claim that $\dim V^x \leqslant \frac{1}{4} \dim V$. This reduces to computing in $\mathrm{Spin}_8$ where the result is clear for all three of the eight-dimensional representations. The largest such class will have four Jordan blocks of size 4 (for $n = 18$ or 20) and it is straightforward to compute that $\dim \mathrm{Ad}(G)x < \frac{3}{4} \dim V$.

If $x$ has at least two Jordan blocks of size at least 3, then $x$ specializes to $(3^2, 1^{n-6})$; as triality sends elements with partition $(3^2, 1^2)$ to elements with the same partition, we find $\dim V^x \leqslant \frac{1}{2} \dim V$. We are left with the case where $x$ has partition $(3, 2^{2t}, 1^{n-2t-3})$ for some $t$. If $t = 0$, then the centralizer of $x$ has dimension 121 or 154 and we are done. If $t > 0$, then $x$ specializes



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

to $y$ with partition $(3, 2^2, 1^{n-7})$. As triality on $\mathfrak{so}_8$ leaves the partition $(3, 2^2, 1)$ unchanged, we find $\dim V^x \leqslant \dim V^y \leqslant \frac{1}{2} \dim V$, as desired, completing the verification of (2.8) for $x$ nilpotent.

*Semisimple elements in* Lie$(G)$. For $x \in \mathfrak{so}_n$ semisimple, the most interesting case is when $x$ is diagonal with entries $(a^t, (-a)^t, 0^{n-2t})$ where exponents denote multiplicity and $a \in k^\times$. The centralizer of $x$ is $\mathrm{GL}_t \times \mathrm{SO}_{n-2t}$, so $\dim(\mathrm{Ad}(\mathrm{SO}_n)x) = \binom{n}{2} - t^2 - \binom{n-2t}{2}$. This is less than $\frac{1}{4} \dim V$ for $n = 20$, settling that case. For $n = 18$, if $t = 1$ or 2, $x$ is in the image of an element $(a, -a, 0, 0)$ or $(a/2, a/2, -a/2, -a/2)$ in $\mathfrak{sl}_4 \cong \mathfrak{so}_6$, and the tensor product decomposition gives that $\dim V^x \leqslant \frac{1}{2} \dim V$ and again we are done. If $t > 2$, we consider a nilpotent $y = \left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right)$ not commuting with $x$ where $Y$ is nine-by-nine and $y$ specializes to a nilpotent $y'$ with partition $(2^4, 1^8)$. Such a $y'$ acts on $V$ as 16 copies of $(3, 2^4, 1^5)$, hence $\dim V^{y'} = 160$. By specializing $x$ to $y$, we find $\dim V^x \leqslant 160$ and again we are done.

*Semisimple elements in G.* Let $g \in G(k)$ be semisimple, noncentral and of prime order. If $n = 20$, then $\dim g^G \leqslant 180 < \frac{3}{8} \dim V$ and we are done by Proposition 2.1(iv). So assume $n = 18$. If we find that the centralizer of $g$ has dimension greater than 57, then $\dim g^G < \frac{3}{8} \dim V$ and again we are done.

If $g^2$ is central but nontrivial, then $g$ has no fixed points (and every eigenspace is at most $\frac{1}{2} \dim V$). If $g^2 = 1$ but $g$ is not central, then $g$ maps to an involution in $\mathrm{SO}_{18}$ whose centralizer is no smaller than $\mathrm{SO}_8 \times \mathrm{SO}_{10}$ of dimension 73, and we are done. So assume $g$ has odd prime order. We divide into cases depending on the image $\overline{g} \in \mathrm{SO}_{18}$ of $g$.

If $\overline{g}$ has at least five distinct eigenvalues, then either it has at least six distinct eigenvalues $a, a^{-1}, b, b^{-1}, c, c^{-1}$, or it has four distinct eigenvalues that are not equal to 1, and the remaining eigenvalue is 1. In the latter case set $c = 1$. View $g$ as the image of $(g_1, g_2) \in \mathrm{Spin}_6 \times \mathrm{Spin}_{12}$ where $g_1$ maps to a diagonal $(a, b, c, c^{-1}, b^{-1}, a^{-1})$ in $\mathrm{SO}_6$, a regular semisimple element. Therefore, the eigenspaces of the image of $g_1$ under the isomorphism $\mathrm{Spin}_6 \cong \mathrm{SL}_4$ are all one-dimensional and the tensor decomposition argument shows that $\dim V^g \leqslant \frac{1}{4} \dim V$. As $\dim g^G \leqslant 144 < \frac{3}{4} \dim V$, we are done in this case.

If $\overline{g}$ has exactly four eigenvalues, then the centralizer of $\overline{g}$ is at least as big as $\mathrm{GL}_4 \times \mathrm{GL}_5$ of dimension 41, so $\dim g^G \leqslant 112 < \frac{1}{2} \dim V$. Viewing $g$ as the image of $(g_1, g_2) \in \mathrm{Spin}_8 \times \mathrm{Spin}_{10}$ such that the image $\overline{g}_1$ of $g_1$ in $\mathrm{SO}_8$ exhibits all four eigenvalues, then $\overline{g}_1$ has eigenspaces all of dimension 2 or of dimensions 3, 3, 1, 1. The images of $\overline{g}_1$ in each of the eight-dimensional representations are written in [Gar98, Example 1.6] and each has eigenspaces that are at most four-dimensional, so $\dim V^g \leqslant \frac{1}{2} \dim V$ and this case is settled.

In the remaining case, $\overline{g}$ has exactly two nontrivial (i.e. not 1) eigenvalues $a, a^{-1}$. If 1 is not an eigenvalue of $\overline{g}$, then the centralizer of $\overline{g}$ is $\mathrm{GL}_9$ of dimension 81, and we are done. If the eigenspaces for the nontrivial eigenvalues are at least four-dimensional, then we can take $g$ to be the image of $(g_1, g_2) \in \mathrm{Spin}_{10} \times \mathrm{Spin}_8$ where $g_1$ maps to $(a, a, a, a, 1, \ldots) \in \mathrm{SO}_{10}$. (See Example 2.4 for this notation.) The images of $(a, a, a, a, \ldots) \in \mathrm{SO}_8$ as in (2.5) are $(a, a, a, a^{-1}, \ldots)$ and $(a^2, 1, 1, 1, \ldots)$, so the largest eigenspace of $g_1$ on a half-spin representation is 6 and $\dim V^g \leqslant \frac{3}{8} \dim V$. As the conjugacy class of a regular element has dimension $144 < \frac{5}{8} \dim V$, this case is complete. Finally, if $\overline{g}$ has eigenspaces of dimension at most 2 for $a$, $a^{-1}$, then $\dim g^G \leqslant 58 < \frac{3}{8} \dim V$ and the $n = 18$ case is complete.

**Case $n = 17$ or 19**

For $n = 17$ or 19, the spin representation of $\mathrm{Spin}_n$ can be viewed as the restriction of a half-spin representation of the overgroup $\mathrm{HSpin}_{n+1}$. We have already proved that this representation of $\mathrm{HSpin}_{n+1}$ is generically free.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

**Case $n = 15$ or $16$**
We use the following general fact.

Lemma 4.1. *Let $G$ be a quasi-simple algebraic group and $H$ a proper closed subgroup of $G$ and $X \subset G(k)$ finite. Then for generic $g \in G(k)$, $H(k) \cap gXg^{-1} = H(k) \cap X \cap Z(G)(k)$.*

*Proof.* It suffices to check $\supseteq$. For each $x \in X \setminus Z(G)(k)$, note that $W(x) := \{g \in G \mid x^g \in H\}$ is a proper closed subvariety of $G$ and, since $X$ is finite, $\bigcup W(x)$ is also proper closed. Thus, for an open subset of $g$ in $G$, $g(X\setminus Z(G)(k))g^{-1}$ does not meet $H(k)$. □

Lemma 4.2. *Let $G = \mathrm{HSpin}_{16}$ and $V$ a half-spin representation over an algebraically closed field $k$ of characteristic $\neq 2$. The stabilizer of a generic vector in $V$ is isomorphic to $(\mathbb{Z}/2)^8$, as a group scheme.*

*Proof.* Consider $\mathrm{Lie}(E_8) = \mathrm{Lie}(G) \oplus V$ where the summands are the eigenspaces of an involution in $E_8$. That involution inverts a maximal torus $T$ of $E_8$ and so there is maximal Cartan subalgebra $\mathfrak{t} = \mathrm{Lie}(T)$ on which the involution acts as $-1$. As $E_8$ is smooth and adjoint, for a generic element $\tau \in \mathfrak{t}$, the centralizer $C_{E_8}(\tau)$ has identity component $T$ by [DG70, XIII.6.1(d), XIV.3.18] and in fact equals $T$ by [GG16, Proposition 9.2]. Since $\mathfrak{t}$ misses $\mathrm{Lie}(G)$, the annihilator of $\tau$ in $\mathrm{Lie}(G)$ is 0 as claimed. Furthermore, $G_\tau(k) = T(k) \cap G(k)$, i.e. the elements of $T(k)$ that commute with the involution, so $G_\tau(k) \cong \mu_2(k)^8$. □

Corollary 4.3. *If $\mathrm{char}\, k \neq 2$, then $\mathrm{Spin}_{15}$ acts generically freely on $V$.*

*Proof.* Of course the Lie algebra does because this is true for $\mathrm{Lie}(\mathrm{Spin}_{16})$.

For the group, a generic stabilizer is $\mathrm{Spin}_{15}(k) \cap X$ where $X$ is a generic stabilizer in $\mathrm{Spin}_{16}(k)$. Now $X$ is finite and meets the center of $\mathrm{Spin}_{16}$ in the kernel of $\mathrm{Spin}_{16} \to \mathrm{HSpin}_{16}$, whereas $\mathrm{Spin}_{15}$ injects into $\mathrm{HSpin}_{16}$. Therefore, by Lemma 4.1 a generic conjugate of $X$ intersect $\mathrm{Spin}_{15}$ is trivial. □

Corollary 4.4. *If $\mathrm{char}\, k \neq 2$, then $\mathrm{Spin}_{16}$ acts generically freely on $V \oplus W$, where $V$ is a half-spin module and $W$ is the natural (16-dimensional) module.*

*Proof.* Now the generic stabilizer is already 0 for the Lie algebra on $V$ whence on $V \oplus W$.

In the group $\mathrm{Spin}_{16}$, a generic stabilizer is conjugate to $X^g \cap \mathrm{Spin}_{15}$ where $X$ is the finite stabilizer on $V$ and as in the proof of the previous corollary, this is generically trivial. □

## 5. Proof of Theorem 1.1 for $n \leqslant 20$ and characteristic 2

To complete the proof of Theorem 1.1, it remains to prove, in case $\mathrm{char}\, k = 2$, that the following representations $G \to \mathrm{GL}(V)$ are generically free:

(i) $G = \mathrm{Spin}_{15}$, $\mathrm{Spin}_{17}$, $\mathrm{Spin}_{19}$ and $V$ is a spin representation;
(ii) $G = \mathrm{Spin}_{18}$ and $V$ is a half-spin representation;
(iii) $G = \mathrm{Spin}_{16}$ or $\mathrm{Spin}_{20}$ and $V$ is a direct sum of the vector representation and a half-spin representation;
(iv) $G = \mathrm{HSpin}_{20}$ and $V$ is a half-spin representation.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

Since we are in bad characteristic, the class of unipotent and nilpotent elements are more complicated. On the other hand, since we are in a fixed small characteristic and the dimensions of the modules and Lie algebras are relatively small, one can actually do some computations.

In particular, we check that in each case that there exists a $v \in V$ over the field of two elements such that $\mathrm{Lie}(G_v) = 0$. (This can be done easily in various computer algebra systems.) It follows that the same is true over any field of characteristic 2. Since the set of $w \in V$ where $\mathrm{Lie}(G_w) = 0$ is an open subvariety of $V$, this shows that $\mathrm{Lie}(G_w)$ is generically zero.

It remains to show that the group of $k$-points $G_v(k)$ of the stabilizer of a generic $v \in V$ is the trivial group.

First consider $G = \mathrm{Spin}_{16}$. By Lemma 4.1, it suffices to show that, for generic $w$ in a half-spin representation $W$, $G_w(k)$ is finite, which is true by the appendix. Alternatively, one can prove the finiteness of $G_w(k)$ by working in $\mathrm{Lie}(E_8) = \mathfrak{hspin}_{16} \oplus W$ and exhibiting a regular nilpotent of $\mathrm{Lie}(E_8)$ in $W$ whose stabilizer in $\mathfrak{hspin}_{16}$ is trivial. Since the set of $w$ where $(\mathrm{Spin}_{16})_w(k)$ is finite is open, the result follows.

Similarly, $\mathrm{Spin}_{15}$ acts generically freely on the spin representation.

As in the previous section, it suffices to show that for $G$ one of $\mathrm{HSpin}_{20}$ and $\mathrm{Spin}_{18}$ and $V$ a half-spin representation, $G_v(k) = 1$ for generic $v \in V$.

We first consider involutions. We recall that an involution $g \in \mathrm{SO}_{2n} = \mathrm{SO}(W)$ (in characteristic 2) is essentially determined by the number $r$ of nontrivial Jordan blocks of $g$ (equivalently $r = \dim(g-1)W$) and whether the subspace $(g-1)W$ is totally singular or not with $r$ even (and $r \leqslant n$); see [AS76, LS12] or see [FGS16, §§ 5, 6] for a quick elementary treatment. If $r < n$ or $(g-1)V$ is not totally singular, there is one class for each possible pair of invariants. If $r = n$ (and so $n$ is even) and $(g-1)V$ is totally singular, then there are two such classes interchanged by a graph automorphism of order 2.

LEMMA 5.1. *Suppose* $\mathrm{char}\, k = 2$. *Let* $G = \mathrm{Spin}_{2n}, n > 4$ *and let* $W$ *be a half-spin representation. If* $g \in G$ *is an involution other than a long root element, then* $\dim W^g \leqslant (5/8) \dim W$.

*Proof.* By passing to closures, we may assume that $r \leqslant 4$. Thus, $g \in \mathrm{Spin}_8$. By applying triality, we may assume that $g$ has precisely two nontrivial Jordan blocks, i.e. $r = 2$, for otherwise $g$ has a four-dimensional fixed space on each of the three eight-dimensional modules. There are two such conjugacy classes of involutions. One of them is the class of long root elements. The other is not invariant under the triality automorphism and it follows that $g$ has a six-dimensional fixed space on one representation and four-dimensional fixed spaces on the other two eight-dimensional representations.

If $2n = 10$, then $W$ restricted to $\mathrm{Spin}_8$ is a direct sum of two distinct half-spin representations, whence $\dim W^g \leqslant 10$ and the result follows. For $2n > 10$, the result follows by induction, since $W$ is a direct sum of the two half-spin representations of $\mathrm{Spin}_{2n-2}$. □

LEMMA 5.2. *Suppose* $\mathrm{char}\, k = 2$. *Let* $G = \mathrm{Spin}_{18}$ *or* $\mathrm{HSpin}_{20}$ *with* $V$ *a half-spin representation of dimension* 256 *or* 512, *respectively. Then* $G_v(k) = 1$ *for generic* $v \in V$.

*Proof.* It suffices to show that $\dim V^g + \dim g^G < \dim V$ for every noncentral $g \in G$ with $g$ of prime order.

The proof for semisimple elements is essentially identical to the case of odd characteristic (except that we need not consider involutions). Alternatively, since we know the result in characteristic zero, it follows that generic stabilizers have no nontrivial semisimple elements as in the proof of [GG15, Lemma 10.3].

Thus, it suffices to consider $g$ of order 2. If $g$ is not a long root element, then $\dim V^g \leqslant \frac{5}{8} \dim V$. On the other hand, $\dim g^G \leqslant 99$ for $n = 10$ and 79 for $n = 9$ by [AS76, LS12] or [FGS16]; in either case $\dim g^G < \frac{3}{8} \dim V$.

The remaining case to consider is when $g$ is a long root element. Then $\dim V^g = \frac{3}{4} \dim V$ while $\dim g^G = 34$ or 30, respectively, and again the inequality holds. □

## 6. Proof of Corollary 1.3

For $n$ not divisible by 4, the (half-)spin representation $\mathrm{Spin}_n$ is generically free by Theorem 1.1, so by, e.g., [Mer13, Theorem 3.13] we have

$$\mathrm{ed}(\mathrm{Spin}_n) \leqslant \dim V - \dim \mathrm{Spin}_n.$$

This gives the upper bound on $\mathrm{ed}(\mathrm{Spin}_n)$ for $n$ not divisible by 4. For $n = 16$, we use the same calculation with $V$ the direct sum of the vector representation of $\mathrm{Spin}_{16}$ and a half-spin representation. For $n \geqslant 20$ and divisible by 4, Theorem 1.1 gives that $\mathrm{ed}(\mathrm{HSpin}_n)$ is at most the value claimed; with this in hand, the argument in [CM14, Theorem 2.2] (referring now to [Lot13] instead of [BRV10] for the stacky essential dimension inequality) establishes the upper bound on $\mathrm{ed}(\mathrm{Spin}_n)$ for $n \geqslant 20$ and divisible by 4.

It is trivially true that $\mathrm{ed}_2(\mathrm{Spin}_n) \leqslant \mathrm{ed}(\mathrm{Spin}_n)$. Finally, that $\mathrm{ed}_2(\mathrm{Spin}_n)$ is at least the expression on the right-hand side of the display was proved in [BRV10, Theorem 3-3(a)] for $n$ not divisible by 4 and in [Mer09, Theorem 4.9] for $n$ divisible by 4; the lower bound on $\mathrm{ed}_2(\mathrm{HSpin}_n)$ is from [BRV10, Remarks 3–10]. □

## 7. $\mathrm{Spin}_n$ for $6 \leqslant n \leqslant 12$ and characteristic 2

Suppose now that $6 \leqslant n \leqslant 12$ and $\mathrm{char}\, k = 2$. Let us now calculate the stabilizer in $\mathrm{Spin}_n$ of a generic vector $v$ in a (half-)spin representation, which will justify those entries in Table 1. For $n = 6$, $\mathrm{Spin}_6 \cong \mathrm{SL}_4$ and the representation is the natural representation. For $n = 8$, the half-spin representation is indistinguishable from the vector representation $\mathrm{Spin}_8 \to \mathrm{SO}_8$ and again the claim is clear.

For the remaining $n$, we verify that the $k$-points $(\mathrm{Spin}_n)_v(k)$ of the generic stabilizer are as claimed, i.e. that the claimed group scheme is the reduced subgroup scheme of $(\mathrm{Spin}_n)_v$. The cases $n = 9, 11, 12$ are treated in [GLMS97, Lemma 2.11] and the case $n = 10$ is treated in [Lie87, p. 496].

For $n = 7$, view $\mathrm{Spin}_7$ as the stabilizer of an anisotropic vector in the vector representation of $\mathrm{Spin}_8$; it contains a copy of $G_2$. As a $G_2$ module, the half-spin representation of $\mathrm{Spin}_8$ is self-dual and has composition factors of dimensions one, six, one, so $G_2$ fixes a vector in $V$. As $G_2$ is a maximal closed connected subgroup of $\mathrm{Spin}_7$, it is the identity component of the reduced subgroup of $(\mathrm{Spin}_7)_v$.

We have verified that the reduced subgroup scheme of $(\mathrm{Spin}_n)_v$ agrees with the corresponding entry, call it $S$, in Table 1. We now proceed as in § 5 and find a $w$ such that $\dim(\mathfrak{spin}_n)_w = \dim S$, which shows that $(\mathrm{Spin}_n)_v$ is smooth, completing the proof of Table 1 for $n \leqslant 12$.

## 8. $\mathrm{Spin}_{13}$ and $\mathrm{Spin}_{14}$ and characteristic $\neq 2$

In this section, we determine the stabilizer in $\mathrm{Spin}_{14}$ and $\mathrm{Spin}_{13}$ of a generic vector in the (half-)spin representation $V$ of dimension 64. We assume that $\mathrm{char}\, k \neq 2$ and $k$ is algebraically closed.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

Let $C_0$ denote the trace zero subspace of an octonion algebra with quadratic norm $N$. We may view the natural representation of $\mathrm{SO}_{14}$ as a sum $C_0 \oplus C_0$ endowed with the quadratic form $N \oplus -N$. This gives an inclusion $G_2 \times G_2 \subset \mathrm{SO}_{14}$ that lifts to an inclusion $G_2 \times G_2 \subset \mathrm{Spin}_{14}$. There is an element of order 4 in $\mathrm{SO}_{14}$ such that conjugation by it interchanges the two copies of $G_2$, the element of order 2 in the orthogonal group with this property has determinant $-1$, so the normalizer of $G_2 \times G_2$ in $\mathrm{SO}_{14}(k)$ is isomorphic to $((G_2 \times G_2) \rtimes \mu_4)(k)$ and in $\mathrm{Spin}_{14}$ it is $((G_2 \times G_2) \rtimes \mu_8)(k)$.

Viewing $V$ as an internal Chevalley module for $\mathrm{Spin}_{14}$ (arising from the embedding of $\mathrm{Spin}_{14}$ in $E_8$), it follows that $\mathrm{Spin}_{14}$ has an open orbit in $\mathbb{P}(V)$, see for example [ABS90, Theorem 2f]. Moreover, the unique $(G_2 \times G_2)$-fixed line $kv$ in $V$ belongs to this open orbit, see [Pop80, Ros99a, p. 225, Proposition 11] or [Gar09, § 21]. That is, for $H$ the reduced subgroup scheme of $(\mathrm{Spin}_{14})_v$, $H^\circ \supseteq G_2 \times G_2$. By dimension count this is an equality. A computation analogous to that in the preceding paragraph shows that the idealizer of $\mathrm{Lie}(G_2 \times G_2)$ in $\mathfrak{so}_{14}$ is $\mathrm{Lie}(G_2 \times G_2)$ itself, hence $\mathrm{Lie}((\mathrm{Spin}_{14})_v) = \mathrm{Lie}(H^\circ)$, i.e. $(\mathrm{Spin}_{14})_v$ is smooth. It follows from the construction above that the stabilizer of $kv$ in $\mathrm{Spin}_{14}$ is all of $(G_2 \times G_2) \rtimes \mu_8$ (as a group scheme). The element of order 2 in $\mu_8$ is in the center of $\mathrm{Spin}_{14}$ and acts as $-1$ on $V$, so the stabilizer of $v$ is $G_2 \times G_2$ as claimed in Table 1.

Now fix a vector $(c, c') \in C_0 \oplus C_0$ so that $N(c)$, $N(c')$ and $N(c) - N(c')$ are all nonzero. The stabilizer of $(c, c')$ in $\mathrm{Spin}_{14}$ is a copy of $\mathrm{Spin}_{13}$, and the stabilizer of $v$ in $\mathrm{Spin}_{13}$ is its intersection with $G_2 \times G_2$, i.e. the product $(G_2)_c \times (G_2)_{c'}$. Each term in the product is a copy of $\mathrm{SL}_3$ (see, for example, [KMRT98, p. 507, Exercise 6]), as claimed in Table 1. (On the level of Lie algebras and under the additional hypothesis that $\mathrm{char}\, k = 0$, this was shown by Kac and Vinberg in [GV78, § 3.2].)

## 9. $\mathbf{Spin_{13}}$ and $\mathbf{Spin_{14}}$ and characteristic 2

We will calculate the stabilizer in $\mathrm{Spin}_n$ of a generic vector in an irreducible (half-)spin representation for $n = 13, 14$ over a field $k$ of characteristic 2.

PROPOSITION 9.1. *The stabilizer in* $\mathrm{Spin}_{14}$ *(over a field $k$ of characteristic 2) of a generic vector in a half-spin representation is the group scheme* $(G_2 \times G_2) \rtimes \mathbb{Z}/2$.

We use the following pushout construction. Let $X$, $V_1$, $V_2$ be vector spaces endowed with quadratic forms $q_X$, $q_1$, $q_2$ such that $q_X$, $q_1$ and $q_2$ are nonsingular and there exist isometric embeddings $f_i : (X, q_X) \hookrightarrow (V_i, q_i)$. There is a natural quadratic form $q_V$ on the pushout $V := (V_1 \oplus V_2)/(f_1 - f_2)(X)$; if we write $V_i \cong V_i' \perp f_i(X)$, then $(V, q_V)$ is isomorphic to $V_1' \perp V_2' \perp X$ and $f_1$ and $f_2$ define the same embedding $(X, q_X) \hookrightarrow (V, q_V)$.

Now pick a subspace $R \subset X$. Applying the same construction where the role of $V_i$ is played by the subspace $f_i(R)^\perp$ and the pushout is $(f_1(R)^\perp \oplus f_2(R)^\perp)/(f_1 - f_2)(R)$, one obtains $R^\perp \subset V$. In the case $\mathrm{char}\, k = 2$, $\dim X = 2$, and $R$ is an anisotropic line, this gives a homomorphism of algebraic groups $B_{\ell_1} \times B_{\ell_2} \to B_{\ell_1 + \ell_2}$ where $2\ell_i + 2 = \dim V_i$. We apply this construction where $V_1$ and $V_2$ are copies of an octonion algebra $C$, $X$ is a quadratic étale subalgebra and $R$ is the span of the identity element of $C$.

*Proof of Proposition 9.1.* The seven-dimensional Weyl module of the split $G_2$ gives an embedding $G_2 \hookrightarrow \mathrm{SO}_7$. Combining this with the construction in the previous paragraph gives maps

$$G_2 \times G_2 \to \mathrm{SO}_7 \times \mathrm{SO}_7 \to \mathrm{SO}_{13} \to \mathrm{SO}_{14}$$

which lift to maps where every SO is replaced by Spin.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

Put $V$ for a half-spin representation of $\mathrm{Spin}_{14}$. It restricts to the spin representation of $\mathrm{Spin}_{13}$. Calculating the restriction of the weights of $V$ to $\mathrm{Spin}_7 \times \mathrm{Spin}_7$ using the explicit description of the embedding, we see that $V$ is the tensor product of the eight-dimensional spin representations of $\mathrm{Spin}_7$. By triality, the restriction of one of the spin representations to $G_2$ is the action of $G_2$ on the octonions $C$, which is a uniserial module with one-dimensional socle $S$ (spanned by the identity element in $C$) and seven-dimensional radical, the Weyl module of trace zero octonions. The restriction of $V = C \otimes C$ to the first copy of $G_2$ is eight copies of $C$, so has an eight-dimensional fixed space $S \otimes C$. As $(S \otimes C)^{1 \times G_2} = S \otimes S$, we find that $S \otimes S$ is the unique line in $V$ stabilized by $G_2 \times G_2$.

We now argue that the $\mathrm{Spin}_{14}$ orbit of $S \otimes S$ is open in $\mathbb{P}(V)$. To see this, by [Roh93], it suffices to verify that $G_2 \times G_2$ is not contained in the Levi subgroup of a parabolic subgroup of $\mathrm{Spin}_{14}$. This is easily verified; the most interesting case is where the Levi has type $A_6$, and $G_2 \times G_2$ cannot be contained in such because the restriction of $V$ to $A_6$ has composition factors of dimension 1, 7, 21 and 35. We conclude that every nonzero $v \in S \otimes S$ is a generic vector in $V$ and $(\mathrm{Spin}_{14})_v$ has dimension 28.

If one constructs on a computer the representation $V$ of the Lie algebra $\mathfrak{spin}_{14}$ over a finite field $F$ of characteristic 2, then it is a matter of linear algebra to calculate the dimension of the stabilizer $(\mathfrak{spin}_{14})_x$ of any given vector $x \in V$. One finds for some $x$ that the stabilizer has dimension 28, which is the minimum possible, so by semicontinuity of dimension $\dim((\mathfrak{spin}_{14})_v) = 28 = \dim(G_2 \times G_2)$. That is, $(\mathrm{Spin}_{14})_v$ is smooth with identity component $G_2 \times G_2$. Consequently we may compute $(\mathrm{Spin}_{14})_v$ by determining its $K$-points for $K$ an algebraic closure of $k$.

There is an element $\tau$ of order 2 in $\mathrm{SO}_{14}(K)$ that interchanges the two copies of $\mathrm{SO}_7(K)$, hence of $G_2(K)$. As the centralizer of $(G_2 \times G_2)(K)$ in $\mathrm{SO}_{14}(K)$ is trivial (as can be seen from the composition series for $k^{14}$ as a representation of $G_2 \times G_2$) and $\mathrm{Aut}(G_2 \times G_2) = (G_2 \times G_2) \rtimes \langle \tau \rangle$, it follows that $(G_2 \times G_2)(K) \rtimes \mathbb{Z}/2$ is the normalizer of $G_2$ in $\mathrm{SO}_{14}(K)$.

As $\tau$ normalizes $(G_2 \times G_2)(K)$, it leaves the fixed subspace $S \otimes S \otimes K = Kv$ invariant, and we find a homomorphism $\chi\colon \mathbb{Z}/2 \to \mathbb{G}_\mathrm{m}$ given by $\tau v = \chi(\tau) v$ which must be trivial because $\mathrm{char}\, K = 2$. That is, the normalizer of $G_2 \times G_2$, which contains the stabilizer of $v$, actually equals the stabilizer of $v$. □

The above proof, which is somewhat longer than some alternatives, was chosen because of the details it provides on the embedding of $G_2 \times G_2$ in $\mathrm{Spin}_{14}$.

Proposition 9.2. *The stabilizer in* $\mathrm{Spin}_{13}$ *(over a field of characteristic 2) of a generic vector in the spin representation is the group scheme* $(\mathrm{SL}_2 \times \mathrm{SL}_2) \rtimes \mathbb{Z}/2$.

*Proof.* We imitate the argument used in § 8. View $\mathrm{Spin}_{13}$ as $(\mathrm{Spin}_{14})_y$ for an anisotropic $y$ in the 14-dimensional vector representation of $\mathrm{Spin}_{14}$. That representation, as a representation of $\mathrm{Spin}_{13}$, has socle $ky$ and radical $y^\perp$. Let $v$ be a generic element of the spin representation $V$ of $\mathrm{Spin}_{13}$. Our task is to determine the group

$$(\mathrm{Spin}_{13})_v = (\mathrm{Spin}_{14})_y \cap (\mathrm{Spin}_{14})_v. \tag{9.3}$$

The stabilizer $(\mathrm{Spin}_{14})_v$ described above is contained in a copy $(\mathrm{Spin}_{14})_e$ of $\mathrm{Spin}_{13}$ where $ke$ is the radical of the 13-dimensional quadratic form given by the pushout construction. As $v$ is generic, $y$ and $e$ are in general position, so tracing through the pushout construction we see that the intersection (9.3) contains the product of 2 copies of the stabilizer in $G_2$ of a generic octonion $z$. The quadratic étale subalgebra of $C$ generated by $z$ has normalizer $\mathrm{SL}_3 \rtimes \mathbb{Z}/2$ in $G_2$,



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

hence the stabilizer of $z$ is $\mathrm{SL}_3$. We conclude that, for $K$ an algebraic closure of $k$, the group of $K$-points of $(\mathrm{Spin}_{13})_v$ equals that of the claimed group, hence the stabilizer has dimension 16. Calculating with a computer as in the proof for $\mathrm{Spin}_{14}$, we find that $\dim(\mathfrak{spin}_{13})_v \leqslant 16$, and therefore the stabilizer of $v$ is smooth as claimed. □

Acknowledgements

We thank Alexander Merkurjev and Zinovy Reichstein for helpful comments, and for posing the questions answered in Proposition 9.1 and Theorem 1.1.

# Appendix. Generic stabilizers associated with a peculiar half-spin representation

Alexander Premet

## A.1 The main theorem

Throughout this appendix we work over an algebraically closed field $k$ of characteristic two. Let $G = \mathrm{HSpin}_{16}(k)$ and let $V$ be the natural (half-spin) $G$-module. The theorem stated below describes the generic stabilizers for the actions of $G$ and $\mathfrak{g} = \mathrm{Lie}(G)$ on $V$.

Theorem A.1. *The following are true.*

(i) *There exists a nonempty Zariski-open subset $U$ in $V$ such that for every $x \in U$ the stabilizer $G_x$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.*

(ii) *For any $x \in U$ the stabilizer $\mathfrak{g}_x$ is a four-dimensional toral subalgebra of $\mathfrak{g}$.*

(iii) *If $x, x' \in U$, then the stabilizers $G_x$ and $G_{x'}$ and the infinitesimal stabilizers $\mathfrak{g}_x$ and $\mathfrak{g}_{x'}$ are $G$-conjugate.*

(iv) *The scheme-theoretic stabilizer of any $x \in U$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \times (\mu_2)^4$.*

A more precise description of $G_x$ and $\mathfrak{g}_x$ with $x \in U$ is given in § A.5. It should be mentioned here that our Theorem A.1 can also be deduced from more general invariant-theoretic results recently announced by Eric Rains.

## A.2 Preliminary remarks and recollections

Let $\widetilde{G}$ be a simple algebraic group of type $\mathrm{E}_8$ over $k$ and $\tilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$. The Lie algebra $\tilde{\mathfrak{g}}$ is simple and carries an $(\mathrm{Ad}\, G)$-equivariant $[p]$th power map $x \mapsto x^{[p]}$. Since $p = 2$, Jacobson's formula for $[p]$th powers is surprisingly simple: we have that

$$(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x, y] \quad (\forall\, x, y \in \tilde{\mathfrak{g}}).$$

Let $T$ be a maximal torus of $\widetilde{G}$ and $\mathfrak{t} = \mathrm{Lie}(T)$. Write $\tilde{\Phi}$ for the root system of $\widetilde{G}$ with respect to $T$. In what follows we will make essential use of Bourbaki's description of roots in $\tilde{\Phi}$; see [Bou02, Planche VII]. More precisely, let $\mathbf{E}$ be an eight-dimensional Euclidean space over $\mathbb{R}$ with orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_8\}$. Then $\tilde{\Phi} = \tilde{\Phi}_0 \sqcup \tilde{\Phi}_1$ where

$$\tilde{\Phi}_0 = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant 8\}$$

and

$$\tilde{\Phi}_1 = \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu(i)} \varepsilon_i \,\middle|\, \sum_{i=1}^{8} \nu(i) \in 2\mathbb{Z} \right\}.$$

The roots $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, $\alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2$, $\alpha_5 = \varepsilon_4 - \varepsilon_3$, $\alpha_6 = \varepsilon_5 - \varepsilon_4$, $\alpha_7 = \varepsilon_6 - \varepsilon_5$, $\alpha_8 = \varepsilon_7 - \varepsilon_6$ form a basis of simple roots in $\tilde{\Phi}$ which we denote by $\tilde{\Pi}$. Let $(\cdot|\cdot)$ be the scalar product of $\mathbf{E}$. It is invariant under the action of the Weyl group $W(\tilde{\Phi}) \subset \mathrm{GL}(\mathbf{E})$.

Given $\alpha \in \tilde{\Phi}$ we denote by $U_\alpha$ and $e_\alpha$ the unipotent root subgroup of $\widetilde{G}$ and a root vector in $\mathrm{Lie}(U_\alpha)$. Let $V$ be the $k$-span of all $e_\alpha$ with $\alpha \in \tilde{\Phi}_1$ and write $G$ for the subgroup of $\widetilde{G}$ generated by $T$ and all $U_\alpha$ with $\alpha \in \tilde{\Phi}_0$. It is well known (and straightforward to see) that the algebraic $k$-group $G$ is isomorphic to $\mathrm{HSpin}_{16}(k)$ and the $G$-stable subspace $V$ of $\tilde{\mathfrak{g}}$ is isomorphic to the natural (half-spin) $G$-module: one can choose a Borel subgroup $B$ of $G$ in such a way that the fixed-point space $V^{R_u(B)}$ is spanned by $e_{-\alpha_1}$. We write $W$ for the subgroup of $W(\tilde{\Phi})$ generated all orthogonal reflections $s_\alpha$ with $\alpha \in \tilde{\Phi}_0$. Clearly, $W \cong N_G(T)/T$ is the Weyl group of $G$ relative to $T$. Since $G$ has type $\mathrm{D}_8$ the group $W$ is a semidirect product of its subgroup $W_0 \cong \mathfrak{S}_8$ acting by permutations of the set $\{\varepsilon_1, \dots, \varepsilon_8\}$ and its abelian normal subgroup $A \cong (\mathbb{Z}/2\mathbb{Z})^7$ consisting of all maps $\varepsilon_i \mapsto (\pm 1)_i \varepsilon_i$ with $\prod_{i=1}^8 (\pm 1)_i = 1$; see [Bou02, Planche IV].

We may (and will) assume further that the $e_\alpha$ are obtained by base change from a Chevalley $\mathbb{Z}$-form, $\tilde{\mathfrak{g}}_\mathbb{Z}$, of a complex Lie algebra of type $\mathrm{E}_8$. Since the group $\widetilde{G}$ is a simply connected the nonzero elements $h_\alpha := [e_\alpha, e_{-\alpha}] \in \mathfrak{t}$ with $\alpha \in \tilde{\Phi}$ span $\mathfrak{t}$. They have the property that $[h_\alpha, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha} = 0$ and $h_\alpha = h_{-\alpha}$ for all $\alpha \in \tilde{\Phi}$. It is well known that $e_\alpha^{[2]} = 0$ and $h_\alpha^{[2]} = h_\alpha$ for all $\alpha \in \tilde{\Phi}$. The set $\{h_\alpha \mid \alpha \in \tilde{\Pi}\}$ is a $k$-basis of $\mathfrak{t}$. Since $\tilde{\mathfrak{g}}$ is a simple Lie algebra, for every nonzero $t \in \mathfrak{t}$ there is a simple root $\beta \in \tilde{\Pi}$ such that $(\mathrm{d}\beta)_e(t) \neq 0$. This implies that $\mathfrak{t}$ admits a nondegenerate $W(\tilde{\Phi})$-invariant symplectic bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle h_\alpha, h_\beta \rangle = (\alpha|\beta) \mod 2$ for all $\alpha, \beta \in \tilde{\Phi}$.

### A.3 Orthogonal half-spin roots and Hadamard–Sylvester matrices

Following the Wikipedia webpage on Hadamard matrices we define the matrices $H_{2^k}$ of order $2^k$, where $k \in \mathbb{Z}_{\geqslant 0}$, by setting $H_1 = [1]$ and

$$H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} = H_2 \otimes H_{2^k}$$

for $k \geqslant 0$. These Hadamard matrices were first introduced by Sylvester in 1867 and they have the property that $H_{2^k} \cdot H_{2^k}^\top = 2^k \cdot I_{2^k}$ for all $k$. We are mostly interested in

$$H_8 = H_2 \otimes H_2 \otimes H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

To each row $r_i = (r_{i1}, \dots, r_{i8})$ of $H_8$ we assign the root $\gamma_i = \frac{1}{2}(r_{i1}\varepsilon_1 + \cdots + r_{i8}\varepsilon_8)$. This way we obtain 16 distinct roots $\pm\gamma_1, \dots, \pm\gamma_8$ in $\tilde{\Phi}_1$ with the property that $(\gamma_i|\gamma_j) = 0$ for all $i \neq j$. As $\pm\gamma_i \pm \gamma_j \notin \tilde{\Phi}$ for $i \neq j$, the semisimple regular subgroup $S$ of $\widetilde{G}$ generated by $T$ and all $U_{\pm\gamma_i}$ is connected and has type $\mathrm{A}_1^8$. It is immediate from the Bruhat decomposition in $\widetilde{G}$ that $G \cap S = N_G(T) \cap N_S(T)$.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

Using the explicit form of the simple roots $\alpha_1, \ldots, \alpha_8$ it is routine to determine the matrix $M := \big[(\gamma_i|\alpha_j)\big]_{1\leqslant i,j\leqslant 8}$. It has the following form:

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

It is then straightforward to check that $M$ is row-equivalent over the integers to a block-triangular matrix $M' = \left[\begin{smallmatrix} M_1 & M_2 \\ O_4 & 2M_3 \end{smallmatrix}\right]$ with $M_1, M_2, M_3 \in \mathrm{Mat}_4(\mathbb{Z})$ and $\det(M_1) = \det(M_3) = 1$. From this it follows that $\gamma_1, \ldots, \gamma_8$ span $\mathbf{E}$ over $\mathbb{R}$ and $h_{\gamma_1}, \ldots, h_{\gamma_8}$ span a maximal (four-dimensional) totally isotropic subspace of the symplectic space $\mathfrak{t}$. We call it $\mathfrak{t}_0$.

### A.4 A dominant morphism

Put $\Gamma = \{\gamma_1, \ldots, \gamma_8\}$ and let $\mathfrak{r}$ denote the subspace of $V$ spanned by $e_\gamma$ with $\gamma \in \pm\Gamma$. If $x = \sum_{i=1}^{8}(\lambda_i e_{\gamma_i} + \mu_i e_{-\gamma_i}) \in \mathfrak{r}$, then Jacobson's formula shows that $x^{[2]} = \sum_{i=1}^{8}(\lambda_i\mu_i)h_{\gamma_i} \in \mathfrak{t}_0$. It follows that

$$x^{[2]^{k+1}} = \sum_{i=1}^{8}(\lambda_i\mu_i)^{2^k} h_{\gamma_i} \quad (\forall\, k \geqslant 0). \tag{A.1}$$

Our discussion at the end of § A.3 shows that $\mathfrak{t}_0$ has a basis $t_1, \ldots, t_4$ contained in the $\mathbb{F}_2$-span of $\{h_\gamma \mid \gamma \in \Gamma\}$. Since $h_\alpha^{[2]} = h_\alpha$ for all roots $\alpha$, we have that $t_i^{[2]} = t_i$ for $1 \leqslant i \leqslant 4$. In view of (A.1) this yields that the subset of $\mathfrak{r}$ consisting of all $x$ as above with the property that $\{x^{[2]^k} \mid 1 \leqslant k \leqslant 4\}$ spans $\mathfrak{t}_0$, $\lambda_i\mu_i \neq 0$ for all $i$, and $\lambda_i\mu_i \neq \lambda_j\mu_j$ for $i \neq j$ is nonempty and Zariski open in $\mathfrak{r}$. We call this subset $\mathfrak{r}^\circ$ and consider the morphism

$$\psi : G \times \mathfrak{r} \longrightarrow V, \quad (g, x) \mapsto (\mathrm{Ad}\ g) \cdot x.$$

Note that $\dim(G \times \mathfrak{r}) = 120 + 16 = 136$ and $\dim V = 128$. By the theorem on fiber dimensions of a morphism, in order to show that $\psi$ is dominant it suffices to find a point $(g, x) \in G \times \mathfrak{r}$ such that all components of $\psi^{-1}((\mathrm{Ad}\ g) \cdot x)$ containing $(g, x)$ have dimension at most eight.

We take $x \in \mathfrak{r}^\circ$ and $g = 1_{\tilde{G}}$. Clearly, $\psi^{-1}(x) \subset \{(g, y) \in G \times \mathfrak{r} \mid y \in (\mathrm{Ad}\ G) \cdot x\}$. If $(g, y) \in \psi^{-1}(x)$, then $y \in \mathfrak{r}$ and $(\mathrm{Ad}\ g)^{-1}$ maps the $k$-span, $\mathfrak{t}(x)$, of $\{x^{[2]^k} \mid 1 \leqslant k \leqslant 4\}$ onto the $k$-span, $\mathfrak{t}(y)$, of $\{y^{[2]^k} \mid 1 \leqslant k \leqslant 4\}$. As $y^{[2]} \in \mathfrak{t}_0$ and $\mathfrak{t}_0$ is a restricted subalgebra of $\mathfrak{t}$, this implies that $\mathfrak{t}(x) = \mathfrak{t}(y) = \mathfrak{t}_0$. It follows that Ad $g$ preserves the Lie subalgebra $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}_0)$ of $\mathfrak{g}$. The centralizer $\mathfrak{c}_{\tilde{\mathfrak{g}}}(\mathfrak{t}_0)$ is spanned by $\mathfrak{t}$ and all root vectors $e_\alpha$ such that $\langle h_\alpha, h_{\gamma_i}\rangle = (\mathrm{d}\alpha)_e(h_{\gamma_i}) = 0$ for $1 \leqslant i \leqslant 8$. As $\mathfrak{t}_0$ is a maximal totally isotropic subspace of the symplectic space $\mathfrak{t}$, our concluding remark in § A.3 shows that $\mathfrak{c}_{\tilde{\mathfrak{g}}}(\mathfrak{t}_0) = \mathrm{Lie}(S)$. Since $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{g} \cap \mathrm{Lie}(S) = \mathfrak{t}$ we obtain that $g \in N_G(T)$. But then $\psi^{-1}(x) \subseteq \{(g, (\mathrm{Ad}\ g)^{-1} \cdot x) \in G \times \mathfrak{r}^\circ \mid g \in N_G(T)\}$. Since $\dim N_G(T) = \dim T = 8$, *all* irreducible components of $\psi^{-1}(x)$ have dimension at most eight. We thus deduce that the morphism $\psi$ is dominant. As the set $G \times \mathfrak{r}^\circ$ is Zariski open in $G \times \mathfrak{r}$, the $G$-saturation of $\mathfrak{r}^\circ$ in $V$ contains a Zariski-open subset of $V$.

### A.5 Generic stabilizers

Let $x = \sum_{i=1}^{8}(\lambda_i e_{\gamma_i} + \mu_i e_{-\gamma_i}) \in \mathfrak{r}^\circ$. In view of our discussion in § A.4 we now need to determine the stabilizer $G_x$. If $g \in G_x$ then Ad $g$ fixes $\mathfrak{t}_0 = \mathrm{span}\{x^{[2]^i} \mid 1 \leqslant i \leqslant 4\}$ and hence preserves $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{t}$. This yields $G_x \subseteq N_G(T)$. Working over a field of characteristic two has some advantages: after reduction modulo two we are no longer affected by the ambiguity in the choice of a Chevalley basis in $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ and the torus $T$ has no elements of order two. It follows that $N_{\widetilde{G}}(T)$ contains a subgroup isomorphic to $W(\tilde{\Phi})$ which intersects trivially with $T$. In the notation of [Ste68, § 3] this group is generated by all elements $\omega_\alpha = w_\alpha(1)$ with $\alpha \in \tilde{\Phi}$. As a consequence, $W$ embeds into $N_G(T)$ in such a way that $N_G(T) = W \ltimes T$.

Our discussion in § A.3 implies that for any $\alpha \in \tilde{\Pi}$ the element $16\alpha \in \mathbb{Z}\tilde{\Phi}$ lies in the $\mathbb{Z}$-span of $\gamma_1, \ldots, \gamma_8$. Since $T$ has no elements of order two and $\widetilde{G}$ is a group of adjoint type, it follows that for any collection $(t_1, \ldots, t_8) \in (k^\times)^8$ there exists a unique element $h = h(t_1, \ldots, t_8) \in T$ with $\gamma_i(h) = t_i$ for all $1 \leqslant i \leqslant 8$. Conversely, any element of $T$ has this form. As a consequence, $\widetilde{G}_x \cap T = \{1_{\widetilde{G}}\}$. For $1 \leqslant i \leqslant 8$ we set $h_i := h(1, \ldots, \mu_i/\lambda_i, \ldots, 1)$, an element of $T$, where the entry $\mu_i/\lambda_i$ occupies the $i$th position. Since Ad $s_{\gamma_i}$ permutes $e_{\pm\gamma_i}$ and fixes $e_{\pm\gamma_j}$ with $j \neq i$, it is straightforward to check that $s_{\gamma_i} h_i \in \widetilde{G}_x$. If $w_0$ is the longest element of $W(\tilde{\Phi})$, then it acts on $\mathbb{Z}\tilde{\Phi}$ as $-\mathrm{Id}$ and hence lies in $A \subset W \hookrightarrow N_G(T)$. (The abelian normal subgroup $A$ of $W$ was introduced in § A.2.) Since $w_0 = \prod_{i=1}^{8} s_{\gamma_i}$ we now deduce that $n_0 := w_0(\prod_{i=1}^{8} h_i) \in G_x$.

Suppose $\widetilde{G}_x \cap N_{\widetilde{G}}(T)$ contains an element $n = wh$, where $w \in W(\tilde{\Phi})$ and $h = h(a_1, \ldots, a_8) \in T$, such that $w(\gamma_i) = \gamma_j$ for $i \neq j$. Then $n(e_{\gamma_i}) = a_i e_{\gamma_j}$ and $n(e_{-\gamma_i}) = a_i^{-1} e_{-\gamma_j}$ implying that $\lambda_j = \lambda_i a_i$ and $\mu_j = \mu_i a_i^{-1}$. But then $\lambda_j/\lambda_i = \mu_i/\mu_j$ forcing $\lambda_i\mu_i = \lambda_j\mu_j$ for $i \neq j$. Since $x \in \mathfrak{r}^\circ$ this is false. As $n_0 \in G_x$ and $w_0(\pm\gamma_i) = \mp\gamma_i$ for all $i$, this argument shows that $\widetilde{G}_x \cap N_{\widetilde{G}}(T) = \langle s_{\gamma_i} h_i \mid 1 \leqslant i \leqslant 8\rangle$ is isomorphic to an elementary abelian 2-group of order $2^8$.

Let $\mathcal{A}_{2^k} \cong (\mathbb{Z}/2\mathbb{Z})^{2^k}$ denote the direct product of $2^k$ copies of $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. The group operation in $\mathcal{A}_{2^k}$ is defined componentwise. We write $u \bullet v$ for the product of $u, v \in \mathcal{A}_{2^k}$ and denote by $\mathbf{1}_{2^k}$ the identity element of $\mathcal{A}_{2^k}$ (all components of $\mathbf{1}_{2^k}$ are equal to 1). The set of rows, $R_{2^k}$, of the Hadamard–Sylvester matrix $H_{2^k}$ may be regarded as a subset $\mathcal{A}_{2^k}$ and easy induction on $k$ shows that $\pm R_{2^k}$ is a subgroup of $\mathcal{A}_{2^k}$. In particular, $\pm R_8$ is a subgroup of $\mathcal{A}_8$. As mentioned in § A.2 the subgroup $W_0 \cong \mathfrak{S}_8$ of the Weyl group $W = W(\tilde{\Phi}_0)$ acts on $\mathcal{A}_8$ by permuting components whereas the normal subgroup $A \cong (\mathbb{Z}/2\mathbb{Z})^7$ of $W$ embeds into $\mathcal{A}_8$ and acts on it by translations.

If $n \in G_x$, then $n = wh \in N_G(T)$ and $w$ preserves $\pm R_8$ setwise. If $w = a\sigma$, where $\sigma \in W_0$ and $a \in A$, then our discussion in the previous paragraph shows that $w(u) = (a\sigma)(u) = \pm u$ for all $u \in \pm R_8$. Taking $u = \mathbf{1}_8$ we get $\sigma(\mathbf{1}_8) = \mathbf{1}_8$ and $\pm\mathbf{1}_8 = w(\mathbf{1}_8) = a \bullet \sigma(\mathbf{1}_8) = a \bullet \mathbf{1}_8 = a$. This yields $a = \pm\mathbf{1}_8$ implying that $w \in W_0$ preserves $\pm R_8$. Also, $G_x \cap A$ is a cyclic group of order 2 generated by $n_0$.

We now consider three commuting involutions

$$\sigma_1 = (1,5)(2,6)(3,7)(4,8), \quad \sigma_2 = (1,4)(2,3)(5,8)(6,7) \quad \text{and} \quad \sigma_3 = (1,2)(3,4)(5,6)(7,8)$$

in $W \cong \mathfrak{S}_8$. One can see by inspection that each of them maps every $r \in R_8$ to $\pm r$. Hence, $\sigma_i \in \langle s_{\gamma_i} \mid 1 \leqslant i \leqslant 8\rangle$. Since $s_{\gamma_i} h_i \in \widetilde{G}_x$ for $1 \leqslant i \leqslant 8$, each $\sigma_i$ admits a unique lift in $G_x \subset N_G(T)$ which will be denoted by $n_i$. The subgroup $\langle n_i \mid 0 \leqslant i \leqslant 3\rangle$ of $G_x$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

Next we show that any element $\sigma h \in G_x$ with $\sigma \in W_0 \cong \mathfrak{S}_8$ lies in the subgroup generated by the $n_i$. Since $\sigma$ maps $\mathbf{1}_8$ to $\pm\mathbf{1}_8$ and $n_0 \in G_x$ we may assume that $\sigma(\mathbf{1}_8) = \mathbf{1}_8$. Since $\sigma$ maps $(\mathbf{1}_4, -\mathbf{1}_4)$ to $\pm(\mathbf{1}_4, -\mathbf{1}_4)$ and $n_1 \in G_x$ we may also assume that $\sigma$ fixes $(\mathbf{1}_4, -\mathbf{1}_4)$. Since $\sigma$



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

maps $(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_2, -\mathbf{1}_2)$ to $\pm(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_2, -\mathbf{1}_2)$ and $n_2 \in G_x$ we may assume that $\sigma$ fixes $(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_2, -\mathbf{1}_2)$ as well. Finally, since $\sigma$ maps $(1,-1,1,-1,1,-1,1,-1)$ to $\pm(1,-1,1,-1,1,-1,1,-1)$ and $n_3 \in G_x$ we may assume that $\sigma$ fixes $(1,-1,1,-1,1,-1,1,-1)$. This entails that $\sigma(i) = i$ for $i \in \{1,2,3,4\}$. As $\sigma(r) = \pm r$ for all $r \in R_8$ the latter shows that $\sigma = \mathrm{id}$ proving statement (i) of Theorem A.1.

Since $\mathfrak{g}_x$ contains the spanning set $\{x^{[2]^i} \mid 1 \leqslant i \leqslant 4\}$ of $\mathfrak{t}_0$, our remarks in § A.4 show that $\mathfrak{g}_x \subset \mathfrak{t}$. Since $[t,x] = 0$ for every $t \in \mathfrak{g}_x$ it must be that $(\mathrm{d}\gamma)_e(t) = 0$ for all $\gamma \in \Gamma$. Since $(\mathrm{d}\gamma)_e(t) = \langle h_\gamma, t\rangle$ and $\mathfrak{t}_0$ is a maximal isotropic subspace of the symplectic space $\mathfrak{t}$, we obtain that $t \in \mathfrak{t}_0$. As a result, $\mathfrak{g}_x = \mathfrak{t}_0$ for every $x \in \mathfrak{r}^\circ$. Statement (ii) follows.

In proving statement (iii) we may assume that $x = \sum_{i=1}^{8}(\lambda_i e_{\gamma_i} + \mu_i e_{-\gamma_i})$ and $x' = \sum_{i=1}^{8}(\lambda_i' e_{\gamma_i} + \mu_i' e_{-\gamma_i})$ are two elements of $\mathfrak{r}^\circ$. Our discussion in the previous paragraph shows that $\mathfrak{g}_x = \mathfrak{g}_{x'} = \mathfrak{t}_0$. Let $h_i' := h(1,\ldots,\mu_i'/\lambda_i',\ldots,1)$, where the entry $\mu_i'/\lambda_i'$ occupies the $i$th position. There is a unique element $h = h(b_1,\ldots,b_8) \in T$ such that

$$h \cdot s_{\gamma_i} h_i \cdot h^{-1} = s_{\gamma_i} h_i' \quad (1 \leqslant i \leqslant 8).$$

(We need to take $b_i = \sqrt{(\lambda_i \mu_i')/(\lambda_i' \mu_i)} \in k$ for all $1 \leqslant i \leqslant 8$ which is possible since all $\lambda_i\mu_i$ and $\lambda_i'\mu_i'$ are nonzero.) Our earlier remarks in this section now show that $h \cdot G_x \cdot h^{-1} = G_{x'}$. This proves statement (iii).

*Remark.* We stress that for an element $x = \sum_{i=1}^{8}(\lambda_i e_{\gamma_i} + \mu_i e_{-\gamma_i})$ to be in $\mathfrak{r}^\circ$ it is necessary that $\lambda_i\mu_i \neq \lambda_j\mu_j$ for all $i \neq j$. If one removes this condition and only requires that the set $\{x^{[2]^i} \mid 1 \leqslant i \leqslant 4\} \subset \mathfrak{t}$ is linearly independent, then one obtains an *a priori* bigger Zariski-open subset, $\mathfrak{r}'$, in $\mathfrak{r}$ which still has the property that $G_x$ is a finite group and $\mathfrak{g}_x = \mathfrak{t}_0$ for every $x \in \mathfrak{r}'$. However, it is not immediately clear that the stabilizers in $G$ of any two elements in $\mathfrak{r}'$ are isomorphic. It would be interesting to investigate this situation in more detail.

### A.6 Scheme-theoretic stabilizers

Let $\widetilde{\mathbf{G}}$ be a reductive group scheme over $k$ with root system $\tilde{\Phi}$ with respect to a maximal torus $\mathbf{T} \subset \widetilde{\mathbf{G}}$ and let $\mathbf{G}$ be the regular group subscheme of $\widetilde{\mathbf{G}}$ with root system $\tilde{\Phi}_0$. We may assume that $\mathbf{T}(k) = T$, $\widetilde{\mathbf{G}}(k) = \widetilde{G}$ and $\mathbf{G}(k) = G$. In this situation, we wish to describe the scheme-theoretic stabilizer $\mathbf{G}_x$ of $x \in \mathfrak{r}^\circ$, an affine group subscheme of $\mathbf{G}$ defined over $k$.

Let $F$ be any commutative associative $k$-algebra with 1. The subscheme $N_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{G}$ is smooth and since $p = 2$ we have an isomorphism $N_{\mathbf{G}}(\mathbf{T}) = W \times \mathbf{T}$ of affine group schemes over $k$. Arguing as in § A.5 one observes that $\mathbf{G}_x(F)$ is contained in the group of $F$-points of $N_{\mathbf{G}}(\mathbf{T})$. The latter contains $G_x = \mathbf{G}_x(k)$. Replacing $k^\times$ by the multiplicative group of $F$ and arguing as in § A.5 one observes that the canonical projection $N_{\mathbf{G}}(\mathbf{T}) \twoheadrightarrow W$ sends $\mathbf{G}_x(F)$ into the subgroup of $W$ generated by $\sigma_i$ with $0 \leqslant i \leqslant 3$. Since $n_i \in G_x$ for all $0 \leqslant i \leqslant 3$ it follows that the group $\mathbf{G}_x(F)$ is generated by $G_x = (\mathbf{G}_x)_{\mathrm{red}}$ and the scheme-theoretic stabilizer $\mathbf{T}_x$. On the other hand, our concluding remarks in § A.3 imply that the root lattice $\mathbb{Z}\tilde{\Phi}$ contains free $\mathbb{Z}$-submodules $\Lambda_1$ and $\Lambda_2$ of rank 4 such that $\mathbb{Z}\tilde{\Phi} = \Lambda_1 \oplus \Lambda_2$ and $\mathbb{Z}\Gamma := \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_8 = \Lambda_1 \oplus 2\Lambda_2$. Since $\mathbf{T}(F) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\tilde{\Phi}, F^\times)$, we have a short exact sequence

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_2/2\Lambda_2, F^\times) \to \mathbf{T}(F) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, F^\times) \to 1$$

which shows that the groups $\mathbf{T}_x(F)$ and $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2/2\Lambda_2, F^\times)$ are isomorphic. Since $\Lambda_2/2\Lambda_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, F^\times) = \mu_2(F)$ we have $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2/2\Lambda_2, F^\times) \cong (\mu_2)^4(F)$. Hence, $\mathbf{T}_x \cong (\mu_2)^4$ as affine group schemes over $k$.



https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press

Since $\sigma_i(\gamma_j) = \pm\gamma_j$ for all $0 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 8$ we have that $(\sigma_i(\lambda) - \lambda|\gamma_j) \in 2\mathbb{Z}$ for all $\lambda \in \mathbb{Z}\tilde{\Phi}$. Since $(\lambda|\lambda') \in 2\mathbb{Z}$ for all $\lambda, \lambda' \in \mathbb{Z}\Gamma$, it follows that each $\sigma_i$ acts trivially on $\Lambda_2/2\Lambda_2 \cong \mathbb{Z}\tilde{\Phi}/\mathbb{Z}\Gamma$. We thus deduce that the group scheme $\mathbf{G}_x$ is commutative. In view of the above this implies that $\mathbf{G}_x \cong (\mathbf{G}_x)_{\mathrm{red}} \times \mathbf{T}_x$ as affine group schemes over $k$. This completes the proof of Theorem A.1. □

## References


AP71 E. M. Andreev and V. L. Popov, *Stationary subgroups of points of general position in the representation space of a semisimple Lie group*, Funct. Anal. Appl. **5** (1971), 265–271.

AS76 M. Aschbacher and G. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.

ABS90 H. Azad, M. Barry and G. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), 551–562.

BM12 S. Baek and A. Merkurjev, *Essential dimension of central simple algebras*, Acta Math. **209** (2012), 1–27.

BS66 A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups*, in *Algebraic groups and discontinuous subgroups (Proceedings of Symposium in Pure Mathematics, Boulder, CO, 1965)* (American Mathematical Society, Providence, RI, 1966), 26–32.

Bou02 N. Bourbaki, *Lie groups and Lie algebras* (Springer, Berlin, 2002).

BRV10 P. Brosnan, Z. Reichstein and A. Vistoli, *Essential dimension, spinor groups, and quadratic forms*, Ann. of Math. (2) **171** (2010), 533–544.

BR97 J. Buhler and Z. Reichstein, *On the essential dimension of a finite group*, Compositio Math. **106** (1997), 159–179.

CM14 V. Chernousov and A. S. Merkurjev, *Essential dimension of spinor and Clifford groups*, Algebra Number Theory **8** (2014), 457–472.

CS06 V. Chernousov and J.-P. Serre, *Lower bounds for essential dimensions via orthogonal representations*, J. Algebra **305** (2006), 1055–1070.

Che97 C. Chevalley, *The algebraic theory of spinors* (Springer, Berlin, 1997), reprint of the 1954 edition.

CM93 D. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras* (Van Nostrant Reinhold, New York, 1993).

DG70 M. Demazure and A. Grothendieck, *Schémas en groupes II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Lecture Notes in Mathematics, vol. 152 (Springer, Berlin, 1970).

Flo08 M. Florence, *On the essential dimension of cyclic p-groups*, Invent. Math. **171** (2008), 175–189.

FGS16 J. Fulman, R. Guralnick and D. Stanton, *Asymptotics of the number of involutions in finite classical groups*. Preprint (2016), arXiv:1602.03611.

Gar98 S. Garibaldi, *Isotropic trialitarian algebraic groups*, J. Algebra **210** (1998), 385–418.

Gar09 S. Garibaldi, *Cohomological invariants: exceptional groups and spin groups*, Memoirs American Mathematical Society, vol. 937 (American Mathematical Society, Providence, RI, 2009), with an appendix by Detlev W. Hoffmann.

GG15 S. Garibaldi and R. M. Guralnick, *Simple groups stabilizing polynomials*, Forum Math.: Pi **3** (2015), e3 (41 pages), doi:10.1017/fmp.2015.3.

GG16 S. Garibaldi and R. M. Guralnick, *Essential dimension of algebraic groups, including bad characteristic*, Arch. Math. **107** (2016), 101–119.

GV78 V. Gatti and E. Viniberghi, *Spinors of 13-dimensional space*, Adv. Math. **30** (1978), 137–155.

GR09 P. Gille and Z. Reichstein, *A lower bound on the essential dimension of a connected linear group*, Comment. Math. Helv. **84** (2009), 189–212.

Gue97 M. Guerreiro, *Exceptional representations of simple algebraic groups in prime characteristic*, PhD thesis, University of Manchester (1997), arXiv:1210.6919.

GLMS97 R. M. Guralnick, M. W. Liebeck, D. Macpherson and G. M. Seitz, *Modules for algebraic groups with finitely many orbits on subspaces*, J. Algebra **196** (1997), 211–250.

Igu70 J.-I. Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028.

KM03 N. Karpenko and A. Merkurjev, *Essential dimension of quadrics*, Invent. Math. **153** (2003), 361–372.

KM08 N. Karpenko and A. Merkurjev, *Essential dimension of finite p-groups*, Invent. Math. **172** (2008), 491–508.

KMRT98 M.-A. Knus, A. S. Merkurjev, M. Rost and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44 (American Mathematical Society, Providence, RI, 1998).

Lie87 M. W. Liebeck, *The affine permutation groups of rank* 3, Proc. Lond. Math. Soc. (3) **54** (1987), 477–516.

LS12 M. Liebeck and G. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys Monographs, vol. 180 (American Mathematical Society, Providence, RI, 2012).

Lot13 R. Lötscher, *A fiber dimension theorem for essential and canonical dimension*, Compositio Math. **149** (2013), 148–174.

LMMR13 R. Lötscher, M. MacDonald, A. Meyer and Z. Reichstein, *Essential dimension of algebraic tori*, J. Reine Angew. Math. **677** (2013), 1–13.

Mer09 A. Merkurjev, *Essential dimension, quadratic forms–algebra, arithmetic, and geometry*, Contemporary Mathematics, vol. 493, eds R. Baeza, W. K. Chan, D. W. Hoffmann and R. Schulze-Pillot (American Mathematical Society, Providence, RI, 2009), 299–325.

Mer10 A. Merkurjev, *Essential p-dimension of* $\mathrm{PGL}(p^2)$, J. Amer. Math. Soc. **23** (2010), 693–712.

Mer13 A. Merkurjev, *Essential dimension: A survey*, Transform. Groups **18** (2013), 415–481.

Mer16 A. Merkurjev, *Essential dimension*, in *Séminaire Bourbaki*, Astérisque, vol. 380 (Société Mathématique de France, 2016), 423–448.

Mer17 A. Merkurjev, *Invariants of algebraic groups and retract rationality of classifying spaces*, in *Algebraic groups: structure and actions*, Proceedings of Symposia in Pure Mathematics, vol. 94 (American Mathematical Society, Providence, RI, 2017). Preprint (2015), http://www.math.ucla.edu/∼merkurev/papers/retract-class-space.pdf.

Pop88 A. M. Popov, *Finite isotropy subgroups in general position of simple linear Lie groups*, Trans. Moscow Math. Soc. (1988), 205–249 (Russian original: Trudy Moskov. Mat. Obschch. **50** (1987), 209–248, 262).

Pop80 V. L. Popov, *Classification of spinors of dimension* 14, Trans. Moscow Math. Soc. **37** (1980), 181–232.

Rei10 Z. Reichstein, *Essential dimension*, in *Proceedings of the international congress of mathematicians 2010* (World Scientific, Singapore, 2010).

RY00 Z. Reichstein and B. Youssin, *Essential dimensions of algebraic groups and a resolution theorem for G-varieties*, Canad. J. Math. **52** (2000), 1018–1056, with an appendix by J. Kollár and E. Szabó.

Roh93 G. Röhrle, *On certain stabilizers in algebraic groups*, Comm. Algebra **21** (1993), 1631–1644.

Ros99a M. Rost, *On 14-dimensional quadratic forms, their spinors, and the difference of two octonion algebras*, Preprint (1999), https://www.math.uni-bielefeld.de/∼rost/spin-14.html.

Ros99b M. Rost, *On the Galois cohomology of* Spin(14), Preprint (1999), https://www.math.uni-bielefeld.de/~rost/spin-14.html.

Ste68 R. Steinberg, *Lectures on Chevalley groups* (Yale University, New Haven, CT, 1968).

SF88 H. Strade and R. Farnsteiner, *Modular Lie algebras and their representations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 116 (Marcel Dekker, New York, 1988).



Skip Garibaldi skip@member.ams.org
Center for Communications Research, San Diego, CA 92121, USA

Robert M. Guralnick guralnic@usc.edu
Department of Mathematics, University of Southern California,
Los Angeles, CA 90089-2532, USA

Alexander Premet sasha.premet@manchester.ac.uk
The University of Manchester, Manchester M13 9PL, UK




https://doi.org/10.1112/S0010437X16008162 Published online by Cambridge University Press