ON EXPLICIT BOUNDS IN LANDAU'S THEOREM

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1. The theorem of Landau in question may be stated in the form that if the function $F(Z)$ is regular for $|Z| < 1$ and does not take the values 0 and 1, while

$$F(Z) = a_0 + a_1 Z + \dots$$

is its Taylor expansion about $Z = 0$, then $|a_1|$ has a bound depending only on a_0. In fact $|a_1|$ has a bound depending only on $|a_0|$ and Hayman **(1)** gave the explicit bound

$$|a_1| \leqslant 2\,|a_0|\{|\log|a_0|| + 5\pi\}.$$

In a recent paper **(2)** I gave a simple method for obtaining explicit bounds in Schottky's Theorem and applied it also to improving the above bound to

$$|a_1| \leqslant 2\,|a_0|\{|\log|a_0|| + 7.77\}.$$

Since writing that paper I have observed that by relatively small modifications of the argument that bound can still be substantially improved.

2. It is well known that, for a given a_0, the maximum value of $|a_1|$ is attained for the function $F_0(Z)$ mapping $|Z| < 1$ onto the universal covering surface of the finite W-plane punctured at 0 and 1 and taking the value a_0 at $Z = 0$. Now $|Z| < 1$ is mapped conformally onto $\Re z > 0$ in such a way that if the mapping function is $Z = Z(z)$ and we set $F_0(Z(z)) = f(z)$, then for a suitable branch of $\log f(z)$ the mapping

$$w = \log f(z) \mp \pi i$$

(where $-$ or $+$ is chosen according as $\Im a_0 \geqslant 0$ or $\Im a_0 < 0$) carries the domain determined by the inequalities

$$-\pi < \Im z < \pi, \quad \Re z > 0, \quad |z - \tfrac{1}{2}\pi i| > \tfrac{1}{2}\pi, \quad |z + \tfrac{1}{2}\pi i| > \tfrac{1}{2}\pi$$

onto the strip

$$-\pi < \Im w < \pi$$

so that the boundary points $\pm\pi i$ correspond to themselves. Further, the boundary points of these domains at infinity in whose neighborhoods $\Re z$, $\Re w$ become large and positive correspond and the boundary point $z = 0$ corresponds to the point at infinity in whose neighborhood $\Re w$ becomes large and negative. We denote the point in the z-plane corresponding to $Z = 0$ by b. Moreover we set $\zeta = e^{-z}$, $\omega = e^{-w}$ and denote the corresponding mapping

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between these planes by $\zeta = \phi(\omega)$ or $\omega = \psi(\zeta)$. The function $\phi(\omega)$ is regular and univalent for $|\omega| < 1$ with $\phi'(0) = 1/16$.

Next we observe that, as was proved in (**2**, p. 80), in obtaining a bound of the form

$$|a_1| \leqslant 2\,|a_0|\{|\log |a_0|| + K\},$$

it is enough to confine ourselves to the situation $|a_0| \geqslant 1$, $|a_0 - 1| \geqslant 1$. Then we use distinct arguments according as $|a_0|$ is near 1 or bounded from 1.For $|a_0|$ near 1 we use the fact that under the mapping from the z-plane to the w-plane the half-plane $\Re z > \frac{1}{2}\pi$ is mapped into the w-plane slit along the half-infinite segments $\Im w = (2n+1)\pi$, $\Re w \leqslant 0$, n running through all integers. Comparing the inner radii of these domains with respect to b and its image with the derivative of the mapping function, namely $a_1/2a_0\Re b$, we get the bound (**2**, p. 81)

$$|a_1| \leqslant 2(|a_0|\,|a_0 - 1|)^{\frac{1}{2}} \log|2a_0 - 1 + 2\{a_0(a_0 - 1)\}^{\frac{1}{2}}|\ \Re b/(\Re b - \tfrac{1}{2}\pi).$$

Since the conditions $|a_0| \geqslant 1$, $|a_0 - 1| \geqslant 1$ imply $\Re b \geqslant \frac{1}{2}3^{\frac{1}{2}}\pi$ we have for $|a_0| = t$, $t \geqslant 1$

$$\begin{aligned} |a_1| &\leqslant (3 + 3^{\frac{1}{2}})\ |a_0|(1 + t^{-1})^{\frac{1}{2}} \log[2t + 1 + 2(t^2 + t)^{\frac{1}{2}}] \\ &\leqslant 2|a_0|\{\log|a_0| + \Lambda(t)\} \end{aligned}$$

where

$$\Lambda(t) = \tfrac{1}{2}(3 + 3^{\frac{1}{2}})(1 + t^{-1})^{\frac{1}{2}} \log[2t + 1 + 2(t^2 + t)^{\frac{1}{2}}] - \log t.$$

Unlike the function $L(t)$ used previously the function $\Lambda(t)$ is not monotone increasing. However direct calculation shows that on the range $t \geqslant 1$ it first decreases to a minimum and from then on increases. Thus, on an interval $1 \leqslant t \leqslant t_0$, $\Lambda(t)$ does not exceed the larger of $\Lambda(1)$ and $\Lambda(t_0)$. It proves advantageous to take the interval $1 \leqslant t \leqslant 1.84$. We readily find

$$\Lambda(1) < 5.90,\ \Lambda(1.84) < 5.94.$$

Thus for $1 \leqslant t \leqslant 1.84$, $\Lambda(t) \leqslant 5.94$ and for $1 \leqslant |a_0| \leqslant 1.84$ we have

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + 5.94\}.$$

Now we apply to the function $\phi(\omega)$ instead of the bound previously used (**2**, p. 81) the result due to Robinson (**3**, p. 444)

$$\left|\frac{d\zeta}{d\omega}\right| \geqslant 16|\zeta|^2\,\frac{1 - |\omega|^2}{|\omega|^2}.$$

Using the fact that

$$\phi'(-a_0^{-1}) = -\frac{2a_0^2\Re b}{a_1 e^b}$$

we get

$$|a_1| \leqslant \tfrac{1}{8}\,e^{\Re b}\,\Re b\,\frac{|a_0|^2}{|a_0|^2 - 1}.$$

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Moreover (**2**, p. 79)

$$e^{\mathfrak{R}b} \leqslant 16|a_0| + 8,$$

so

$$|a_1| \leqslant (2|a_0| + 1)\log(16|a_0| + 8)\,\frac{|a_0|^2}{|a_0|^2 - 1}.$$

Then for $|a_0| = t$, $t > 1$, we have

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + M(t)\},$$

where

$$M(t) = (t + \tfrac{1}{2})t\,(t^2 - 1)^{-1}\log(16t + 8) - \log t.$$

Direct calculation shows that $M(t)$ is decreasing for $t > 1$. Now $M(1.84) < 5.93$. Thus for $|a_0| \geqslant 1.84$ we have

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + 5.93\}.$$

Combining this with our previous estimate we have

THEOREM 1. *If $F(Z)$ is regular for $|Z| < 1$, does not take the values* 0 *and* 1 *and has Taylor expansion about $Z = 0$*

$$F(Z) = a_0 + a_1 Z + \ldots,$$

then

$$|a_1| \leqslant 2|a_0|\{|\log|a_0|| + 5.94\}.$$

As Hayman has remarked in his review of (**2**) (Mathematical Reviews, **16** (1955), 579) the value 5.94 cannot be replaced by 4.37.

REFERENCES

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