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# An optimal transportation approach to the decay of correlations for non-uniformly expanding maps – CORRIGENDUM

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## 1. *Introduction*

The article [**Klo20**] contains a significant error in Lemma 2.14, used in the core Theorem 4.1. We describe here how to fix the error, changing slightly an assumption in Theorem 4.1 while leaving all main results intact. A consolidated version of the article is available from https://arxiv.org/abs/1711.08052v3.

It is easy to find counter-examples to the original Lemma 2.14, e.g. for the full shift, it suffices to consider a generic potential depending on a single coordinate. I apologize to readers of the previous version for this embarrassing mistake and warmly thank Manuel Stadlbauer for pointing out this error to me, much more gently than was deserved.

## 2. *Weighting a coupling by a normalized potential*

We start by some additional definitions.

*Definition 1.* Given a coupling $\mathsf{P}$ and a normalized potential $A : \Omega \to \mathbb{R}$, we define $\mathsf{P}_A = (e^{A^t(\bar{x})}\, d\Pi^t_{x,y}(\bar{x}, \bar{y}))_{t,x,y}$ and we say that $\mathsf{P}_A$ has *$\omega$-decay rate $F$* when for some constant $C = C(A)$, for all $t$ and for all $x, y$,

$$\int \omega \circ d(x_t, y_t) e^{A^t(\bar{x})}\, d\Pi^t_{x,y}(\bar{x}, \bar{y}) \leq CF(t, \omega \circ d(x, y)).$$

Here, $\mathsf{P}$ is then said to have *stable $\omega$-decay rate $F$* when the above holds for all bounded, normalized $A$.

The role of the constant $C$ is to allow a factor depending on $A$, but the decay rates we will consider (exponential or polynomial) are all defined up to a constant anyway. Observe that since $A$ is assumed to be normalized,

$$\int e^{A^t(\bar{x})}\, d\Pi^t_{x,y}(\bar{x}, \bar{y}) = \int e^{A^t(\bar{x})}\, dm^t_x(\bar{x}) = 1 \quad \text{for all } t, x, y,$$

so that for each $(t, x, y)$, $e^{A^t(\bar{x})}\, d\Pi^t_{x,y}(\bar{x}, \bar{y})$ is a probability measure; however, Definition 1 really is an extension of the original definition of $\omega$-decay rate since $\mathsf{P}_A$ is not a coupling: its first marginal is the Markov chain $\mathsf{M}_A = (e^{A(x_1)}\, dm_x(x_1))_x$ but its second marginal is different, and might not even be a Markov chain (in $\mathsf{P}_A$, the weight in the pairing of a $\bar{x}$

with a $\bar{y}$ is given by $A^t(\bar{x})$, independently of $\bar{y}$). A sufficient condition to have stable decay is given in §4.

## 3. *Replacements for Lemmas 2.12, 2.14 and Theorem 4.1*

We shall use a slightly extended version of Lemma 2.12 (the added part is a direct consequence of Lemma 2.9).

LEMMA 2. *Let $\mathsf{P} = (\Pi^t_{x,y})_{x,y,t}$ be a coupling of $\mathsf{M}$, $t \in \mathbb{N}$ and $\mu, \nu \in \mathcal{P}(\Omega)$. If $\mathsf{P}$ is $\omega$-Hölder with constant $C$, then*

$$\mathrm{W}_\omega(\mathscr{L}_0^{*t}\mu, \mathscr{L}_0^{*t}\nu) \leq C\, \mathrm{W}_\omega(\mu, \nu).$$

*If $\mathsf{P}$ has $\omega$-decay rate $F$, then*

$$\mathrm{W}_\omega(\mathscr{L}_0^{*t}\mu, \mathscr{L}_0^{*t}\nu) \leq F(t, \mathrm{W}_\omega(\mu, \nu))$$

*(in other words, $\mathscr{L}_0^*$ also has decay rate $F$ in the metric $\mathrm{W}_\omega$).*

We replace Lemma 2.14 by the following lemma.

LEMMA 3. *For all flat normalized potentials $A \in \mathscr{C}^\omega(\Omega)$, there exists a constant $B > 0$ such that for all $t$ and $\Pi^t_{x,y}$-almost all $(\bar{x}, \bar{y})$,*

$$e^{A^t(\bar{x}) - B\omega \circ d(x,y)} \leq e^{A^t(\bar{y})} \leq e^{A^t(\bar{x}) + B\omega\circ d(x,y)}.$$

*In particular, there exists a constant $C > 0$ such that*

$$\frac{1}{C} e^{A^t(\bar{x})} \leq e^{A^t(\bar{y})} \leq C e^{A^t(\bar{x})}.$$

*Proof.* By flatness,

$$e^{A^t(\bar{y})} \leq e^{|A^t(\bar{y}) - A^t(\bar{x})|} e^{A^t(\bar{x})} \leq e^{B\omega\circ d(x,y)} e^{A^t(\bar{x})} \leq \underbrace{e^{B\omega(\operatorname{diam}\Omega)}}_{C} e^{A^t(\bar{x})}$$

and similarly $e^{A^t(\bar{y})} \geq e^{-B\omega\circ d(x,y)} e^{A^t(\bar{x})}$. □

Theorem 4.1 should then be replaced by the following statement, where the assumption on the decay of $\mathsf{P}$ is replaced by the decay of $\mathsf{P}_A$ (for the same $\omega$ and $F$).

THEOREM 4. *Let $\mathsf{M}$ be a transition kernel on a compact metric space $\Omega$, and let $\omega$ be a modulus of continuity. Let $A \in \mathscr{C}^\omega(\Omega)$ be a flat, normalized potential and set $\mathscr{L} = \mathscr{L}_{\mathsf{M},A}$.*

*Assume $\mathsf{M}$ admits a coupling $\mathsf{P}$ such that $\mathsf{P}_A$ has $\omega$-decay with decay function $F$ and corresponding half-life $\tau = \tau_{1/2} : (0, +\infty) \to \mathbb{N}$.*

*Then there exist constants $C > 0$ and $k \in \mathbb{N}$ such that for all $\mu, \nu \in \mathcal{P}(\Omega)$ with $\mathrm{W}_\omega(\mu, \nu) =: r$, it holds*

$$\mathrm{W}_\omega(\mathscr{L}^{*k\tau(r/k)}\mu, \mathscr{L}^{*k\tau(r/k)}\nu) \leq \tfrac{1}{2}\, \mathrm{W}_\omega(\mu, \nu)$$

*and*

$$\mathrm{W}_\omega(\mathscr{L}^{*t}\mu, \mathscr{L}^{*t}\nu) \leq C\, \mathrm{W}_\omega(\mu, \nu) \quad \textit{for all } t \in \mathbb{N}.$$

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*In particular:*

- *if $F$ is exponential, then $\tau(r)$ is bounded and so is $k\tau(r/k)$, so that $\mathscr{L}^*_{\mathsf{M},A}$ decays exponentially in the metric $\mathrm{W}_\omega$, and $\mathscr{L}_{\mathsf{M},A}$ has a spectral gap on $\mathscr{C}^\omega(\Omega)$;*
- *if $F$ is polynomial, then $\tau(r) \leq D/r^\alpha$ so that $k\tau(r/k) \leq D'/r^\alpha$ and $\mathscr{L}^*_{\mathsf{M},A}$ decays polynomially, with the same degree.*

We provide the complete proof, very similar to the original one.

*Proof.*

*Step 1.* Construct a transport plan between $\mathscr{L}^{*t}\delta_x$ and $\mathscr{L}^{*t}\delta_y$.

Here we need the normalization assumption to ensure these two measures are both of the same mass. Fix $t \in \mathbb{N}$, $x, y \in \Omega$ and observe that $\mathscr{L}^{*t}\delta_x = (e_t)_*(e^{A^t}\, dm^t_x)$, where $e_t : \Omega^t \to \Omega$ is the projection to the last coordinates. We seek an efficient transport plan between $\mathscr{L}^{*t}\delta_x$ and $\mathscr{L}^{*t}\delta_y$, and we will construct it as $(e_t, e_t)_*\Pi$, where $\Pi$ is a transport plan between $e^{A^t}\, dm^t_x$ and $e^{A^t}\, dm^t_y$. What we are given by the coupling $\mathsf{P}$ is a transport plan $\Pi^t_{x,y}$ between $m^t_x$ and $m^t_y$, and we will modify it to take into account the $e^{A^t}$ factors. Define a function

$$\begin{aligned} a : \Omega^t \times \Omega^t &\to \mathbb{R} \\ (\bar{x}, \bar{y}) &\mapsto \min(e^{A^t(\bar{x})}, e^{A^t(\bar{y})}) \end{aligned}$$

so that $a\, d\Pi^t_{x,y}$ is a positive measure whose marginals are less than $e^{A^t}\, dm^t_x$ and $e^{A^t}\, dm^t_y$, respectively. There must thus exist some positive measure $\Lambda$ on $\Omega^t \times \Omega^t$ such that

$$\Pi := a\, d\Pi^t_{x,y} + \Lambda$$

is a probability measure with marginals exactly $e^{A^t}\, dm^t_x$ and $e^{A^t}\, dm^t_y$. We want to bound above the $\omega$-cost of $\Pi$; the basic idea is that the first term will be small by the decay hypothesis, the second one will be small because $\Lambda$ has small mass.

*Step 2.* Bound from above the mass of $\Lambda$.

By Lemma 3, for $\Pi^t_{x,y}$ almost all $(\bar{x}, \bar{y})$ and for some constant $B$,

$$a(\bar{x}, \bar{y}) \geq e^{A^t(\bar{x})} e^{-B\omega\circ d(x,y)};$$

using that $A$ is normalized, it comes that the total mass of $a\, d\Pi^t_{x,y}$ is at least $e^{-B\omega\circ d(x,y)}$. Since $\Omega$ has finite diameter, up to enlarging $B$, this total mass can be bounded from below both by a constant $e^{-B} \in (0, 1)$ and by $1 - B\omega \circ d(x, y)$. The total mass of $\Lambda$ is therefore bounded above as follows:

$$\int \mathbf{1}\, d\Lambda \leq \min(B\omega \circ d(x, y), 1 - e^{-B}).$$

*Step 3.* Bound the cost of $\Pi$ for a modified metric.

We introduce a new modulus of continuity

$$\omega' = \min(K\omega, \omega(\operatorname{diam} \Omega)),$$

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where $K$ is a positive constant to be specified later on (independently of $x, y$). We have $\omega' \circ d(x, y) \geq \omega \circ d(x, y)$ for all $x, y \in \Omega$ and $\omega' \leq K\omega$, so that $\omega \circ d$ and $\omega' \circ d$ are Lipschitz-equivalent metrics on $\Omega$, and as a consequence, $\mathrm{W}_\omega$ and $\mathrm{W}_{\omega'}$ are Lipschitz-equivalent (with the same constants).

We decompose the cost as

$$\int \omega' \circ d(x_t, y_t)\, d\Pi(\bar{x}, \bar{y}) = \int \omega' \circ d(x_t, y_t) a(\bar{x}, \bar{y})\, d\Pi^t_{x,y}(\bar{x}, \bar{y}) + \int \omega' \circ d(x_t, y_t)\, d\Lambda(\bar{x}, \bar{y}).$$

For the first term, we get from $\omega$-decay of $\mathsf{P}_A$ (with $D$ only depending on $A$):

$$\begin{aligned}\int \omega' \circ d(x_t, y_t) a(\bar{x}, \bar{y})\, d\Pi^t_{x,y}(\bar{x}, \bar{y}) &\leq K \int \omega \circ d(x_t, y_t) e^{A^t(\bar{x})}\, d\Pi^t_{x,y}(\bar{x}, \bar{y}) \\ &\leq DK \cdot F(t, \omega \circ d(x, y)) \\ &\leq DK \cdot F(t, \omega' \circ d(x, y)).\end{aligned}$$

For the second term, we distinguish two cases. If $\omega \circ d(x, y) \geq \omega(\operatorname{diam} \Omega)/K$, then $\omega' \circ d(x, y) = \omega(\operatorname{diam} \Omega) = \omega'(\operatorname{diam} \Omega)$ and we bound the mass of $\Lambda$ by $1 - e^{-B}$, so that

$$\begin{aligned}\int \omega' \circ d(x_t, y_t)\, d\Lambda(\bar{x}, \bar{y}) &\leq (1 - e^{-B})\omega'(\operatorname{diam} \Omega) \\ &\leq (1 - e^{-B})\omega' \circ d(x, y).\end{aligned}$$

If $\omega \circ d(x, y) \leq \omega(\operatorname{diam} \Omega)/K$, then $\omega' \circ d(x, y) = K\omega \circ d(x, y)$ and we bound the mass of $\Lambda$ by $B\omega \circ d(x, y)$:

$$\begin{aligned}\int \omega' \circ d(x_t, y_t)\, d\Lambda(\bar{x}, \bar{y}) &\leq B\omega \circ d(x, y)\omega'(\operatorname{diam} \Omega) \\ &\leq \frac{B\omega(\operatorname{diam} \Omega)}{K}\omega' \circ d(x, y).\end{aligned}$$

Choosing $K$ large enough to ensure $B\omega(\operatorname{diam} \Omega)/K \leq 1 - e^{-B}$, we get in both cases

$$\int \omega' \circ d(x_t, y_t)\, d\Pi(\bar{x}, \bar{y}) \leq DK \cdot F(t, \omega' \circ d(x, y)) + (1 - e^{-B})\omega'(d(x, y)). \tag{1}$$

*Step 4.* $\mathrm{W}_\omega(\mathscr{L}^{*t}\mu, \mathscr{L}^{*t}\nu) \leq C\, \mathrm{W}_\omega(\mu, \nu)$ for all $t \in \mathbb{N}$, for all $\mu, \nu \in \mathcal{P}(\Omega)$.

Since $F(t, r) \lesssim r$, the previous step implies in particular that $\Pi$, as a restricted coupling at time $t$, is $\omega'$-Hölder; but $\omega \leq \omega' \leq K\omega$ on $[0, \operatorname{diam} \Omega]$ so that $\Pi$ is also $\omega$-Hölder. Then the claim follows from Lemma 2.

*Step 5.* There exist $\theta_1 \in (0, 1)$ and $k_1 \in \mathbb{N}$ such that for all $r$, all $x, y \in \Omega$ such that $\omega' \circ d(x, y) \geq r$ and all $t \geq k_1\tau(r/2^{k_1})$,

$$\mathrm{W}_{\omega'}(\mathscr{L}^{*t}\delta_x, \mathscr{L}^{*t}\delta_y) \leq \theta_1\omega' \circ d(x, y).$$

We choose any $\theta_1 \in (1 - e^{-B}, 1)$ and $k_1$ large enough to ensure $DK/2^{k_1} + (1 - e^{-B}) \leq \theta_1$, and then apply equation (1) (note that $k_1\tau(r/2^{k_1}) \geq \tau(r) + \tau(r/2) + \cdots + \tau(r/2^{k_1})$).

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*Step 6.* There exist $\theta \in (0, 1)$ and $k_2 \in \mathbb{N}$ such that for all $r$, all $\mu, \nu \in \mathcal{P}(\Omega)$ such that $\mathrm{W}_{\omega'}(\mu, \nu) = r$ and all $t \geq k_2\tau(r/k_2)$,

$$\mathrm{W}_{\omega'}(\mathscr{L}^{*t}\mu, \mathscr{L}^{*t}\nu) \leq \theta\, \mathrm{W}_{\omega'}(\mu, \nu).$$

Choose any $\theta \in (\theta_1, 1)$ and let $\eta > 0$ be small enough to ensure $\theta_1 + C\eta \leq \theta$, where $C$ is the constant of Step 4. Let $\mu, \nu$ be any two probability measures and let $\Pi \in \Gamma(\mu, \nu)$ be optimal for $\mathrm{W}_{\omega'}(\mu, \nu) =: r$. Define $s := \eta r$ and $E := \{(x, y) \mid \omega' \circ d(x, y) \geq s\}$. For all $t \geq k_1\tau(s/2^{k_1})$, using Lemma 2.9, we obtain

$$\begin{aligned}
\mathrm{W}_{\omega'}(\mathscr{L}^{*t}\mu, \mathscr{L}^{*t}\nu) &\leq \int \mathrm{W}_{\omega'}(\mathscr{L}^{*t}\delta_x, \mathscr{L}^{*t}\delta_y)\, d\Pi(x, y)\\
&\leq \int_E \mathrm{W}_{\omega'}(\mathscr{L}^{*t}\delta_x, \mathscr{L}^{*t}\delta_y)\, d\Pi(x, y)\\
&\quad + \int_{\Omega\times\Omega\setminus E} \mathrm{W}_{\omega'}(\mathscr{L}^{*t}\delta_x, \mathscr{L}^{*t}\delta_y)\, d\Pi(x, y)\\
&\leq \theta_1 \int_E \omega' \circ d(x, y)\, d\Pi(x, y) + C \int_{\Omega\times\Omega\setminus E} \omega' \circ d(x, y)\, d\Pi(x, y)\\
&\leq \theta_1\, \mathrm{W}_{\omega'}(\mu, \nu) + C\eta r\\
&\leq \theta\, \mathrm{W}_{\omega'}(\mu, \nu).
\end{aligned}$$

It suffices to choose $k_2 \geq 2^{k_1}/\eta$.

*Step 7.* Conclude.

We deduce that the $\theta$ decay time $\tau_\theta^{\omega'}(r)$ of $\mathscr{L}^*$ with respect to $\mathrm{W}_{\omega'}$ is at most $k_2\tau(r/k_2)$. Then for all $n \in \mathbb{N}$,

$$\tau_{\theta^n}^{\omega'}(r) \leq k_2\tau(r/k_2) + k_2\tau(\theta r/k_2) + \cdots + k\tau(\theta^{n-1}r/k_2)$$

and taking $n$ large enough to ensure $\theta^n \leq 1/2K$, we get

$$\tau_{1/2K}^{\omega'}(r) \leq k_2 n\tau(\theta^{n-1}r/k_2) \leq k\tau(r/k) \quad \text{for some } k.$$

Now since $\mathrm{W}_\omega \leq \mathrm{W}_{\omega'} \leq K\, \mathrm{W}_\omega$, the decay time for $\mathscr{L}^*$ with respect to $\mathrm{W}_\omega$ satisfies $\tau_{1/2}^{\omega} \leq \tau_{1/2K}^{\omega'}$, and we are done. □

## 4. *A criterion for stable ω-decay*

Last, we give a criterion that can be used to check the new hypothesis in Theorem 4. We assume that P is itself Markovian, that is, is given by $(\Pi_{x,y}^t)_{x,y} = (\pi_{x,y})_{x,y} \circ \cdots \circ (\pi_{x,y})_{x,y}$ for some $\pi_{x,y} \in \Gamma(m_x, m_y)$.

*Definition 5.* We say that a Markov transition kernel $(\pi_{x,y})_{x,y}$ on $\Omega^2$ is *non-dilating and contracting with positive probability* when there exists $\lambda, p \in (0, 1)$ such that for all $x, y \in \Omega$:

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- for $\pi_{x,y}$-almost all $(x_1, y_1)$, $d(x_1, y_1) \leq d(x, y)$;
- there exists a set $E_{x,y} \subset \Omega^2$ such that for all $(x_1, y_1) \in E_{x,y}$, $d(x_1, y_1) \leq \lambda d(x, y)$ and $\pi_{x,y}(E_{x,y}) \geq p$.

LEMMA 6. *Let $(\pi_{x,y})_{x,y}$ be a Markovian coupling of* $\mathsf{M}$ *and $A$ be a bounded potential normalized with respect to* $\mathsf{M}$. *If $(\pi_{x,y})_{x,y}$ is non-dilating and contracting with positive probability, then $(e^{A(x_1)}\, d\pi_{x,y}(x_1, y_1))_{x,y}$ is non-dilating and contracting with positive probability.*

*Proof.* Both points are essentially obvious: $e^{A(x_1)}\, d\pi_{x,y}(x_1, y_1)$ is absolutely continuous with respect to $\pi_{x,y}$ and $\int_{E_{x,y}} e^{A(x_1)}\, d\pi_{x,y} \geq p \min(e^A) > 0$. For the claim to make sense though, one has to observe that $(e^A \pi_{x,y})_{x,y}$ is a Markov transition kernel, which follows from the normalization hypothesis:

$$\int \mathbf{1} e^{A(x_1)}\, d\pi_{x,y}(x_1, y_1) = \int \mathbf{1} e^{A(x_1)}\, dm_x(x_1) = \mathbf{1}. \qquad \square$$

LEMMA 7. *If $(\pi_{x,y})_{x,y}$ is a Markov transition kernel on $\Omega^2$ which is non-dilating and contracting with positive probability, then for some $\lambda \in (0, 1)$, all $(x, y) \in \Omega^2$ and all $t \in \mathbb{N}$,*

$$\int d(x_t, y_t)\, d\pi^t_{x,y}(\bar{x}, \bar{y}) \leq \lambda^t d(x, y).$$

Here we denoted by $\pi^t_{x,y}$ the iterates of the Markov transition kernel, so that when $(\pi_{x,y})_{x,y}$ comes from the coupling $\mathsf{P}$, $\pi^t_{x,y} = \Pi^t_{x,y}$.

*Proof.* It suffices to consider the case $t = 1$, then conclude by induction. However, this follows from the definition: denoting by $\lambda_1$ the contraction factor of $\pi_{x,y}$ on $E_{x,y}$,

$$\begin{aligned}\int d(x_1, y_1)\, d\pi_{x,y}(x_1, y_1) &= \int_{E_{x,y}} d(x_1, y_1)\, d\pi_{x,y}(x_1, y_1) \\ &\quad + \int_{\Omega^2 \setminus E_{x,y}} d(x_1, y_1)\, d\pi_{x,y}(x_1, y_1) \\ &\leq p\lambda_1 d(x, y) + (1 - p) d(x, y)\end{aligned}$$

then we take $\lambda = p\lambda_1 + (1 - p)$. $\square$

COROLLARY 8. *Assume* $\mathsf{P}$ *is Markovian with a transition kernel that is non-dilating and contracting with positive probability. Then for all bounded normalized potentials $A$,* $\mathsf{P}_A$ *has exponential decay with respect to $\omega_{\alpha+\beta \log}$ for all $\alpha \in (0, 1)$ and all $\beta \in \mathbb{R}$; and* $\mathsf{P}_A$ *has polynomial decay of degree $\beta$ with respect to $\omega_{\beta \log}$ for all $\beta > 0$.*

*Proof.* Let $A$ be a bounded, normalized potential. Then for all modulus of continuity $\omega$, some $\lambda \in (0, 1)$ and all $x, y, t$:

$$\begin{aligned}\int \omega \circ d(x_t, y_t)\, d(e^{A^t} \Pi^t_{x,y})(\bar{x}, \bar{y}) &\leq \omega\bigg(\int d(x_t, y_t)\, d(e^{A^t} \Pi^t_{x,y})(\bar{x}, \bar{y})\bigg) \\ &\leq \omega(\lambda^t d(x, y)).\end{aligned}$$

It only remains to observe that for all $\alpha' < \alpha$, $\omega_{\alpha+\beta\log}(\lambda^t r) \lesssim \lambda^{\alpha' t}\omega_{\alpha+\beta\log}(r)$ and that

$$\omega_{\beta\log}(\lambda^t r) = \frac{\omega_{\beta\log}(r)}{(1 + t\cdot\omega_{\beta\log}(r)^{1/\beta}\log(1/\lambda))^\beta},$$

which provides exactly the polynomial decay of degree $\beta$. □

To apply this machinery in practice, we can replace Lemma 5.3 by the following lemma.

LEMMA 9. *Let* $\mathsf{M}$ *be a weakly contracting* 1*-to-*$k$ *transition kernel and* $\mathsf{P}$ *be the natural coupling. For all bounded normalized potentials* $A$*,* $\mathsf{P}_A$ *has exponential decay with respect to* $\omega_{\alpha+\beta\log}$ *for all* $\alpha \in (0, 1)$ *and all* $\beta \in \mathbb{R}$*, and* $\mathsf{P}_A$ *has polynomial decay of degree* $\beta$ *with respect to* $\omega_{\beta\log}$ *for all* $\beta > 0$*.*

*Proof.* The natural coupling is Markovian, with transition kernel

$$\pi_{x,y} = \sum_j \frac{1}{k}\delta_{(x^{\eta(j)}, y^{\sigma(j)})},$$

which is non-dilating and contracting with positive probability thanks to the hypothesis that $\mathsf{M}$ is weakly contracting. Corollary 8 provides the conclusion. □

Let us in particular consider the one use of the original Theorem 4.1 I know of in the literature, Example 3.13 in [**CMRS21**]. They consider a full shift (with a weakly regular potential); the Markov chain to be considered jumps randomly uniformly from the current point $x$ to one of its antecedents by the shift map and is thus a contracting 1-to-$k$ transition kernel, so that Lemma 9 applies, providing the necessary hypothesis for Theorem 4.

Other statements and proofs in [**Klo20**] can be left as they are.

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