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Special Values of Class Group L-Functions for CM Fields

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Abstract. Let H be the Hilbert class field of a CM number field K with maximal totally real subfield F of degree n over $\mathbb{Q}$. We evaluate the second term in the Taylor expansion at $s = 0$ of the Galois-equivariant L-function $\Theta_{S_\infty}(s)$ associated to the unramified abelian characters of $\mathrm{Gal}(H/K)$. This is an identity in the group ring $\mathbb{C}[\mathrm{Gal}(H/K)]$ expressing $\Theta^{(n)}_{S_\infty}(0)$ as essentially a linear combination of logarithms of special values $\{\Psi(z_\sigma)\}$, where $\Psi\colon \mathbb{H}^n \to \mathbb{R}$ is a Hilbert modular function for a congruence subgroup of $SL_2(\mathcal{O}_F)$ and $\{z_\sigma : \sigma \in \mathrm{Gal}(H/K)\}$ are CM points on a universal Hilbert modular variety. We apply this result to express the relative class number h_H/h_K as a rational multiple of the determinant of an $(h_K - 1) \times (h_K - 1)$ matrix of logarithms of ratios of special values $\Psi(z_\sigma)$, thus giving rise to candidates for higher analogs of elliptic units. Finally, we obtain a product formula for $\Psi(z_\sigma)$ in terms of exponentials of special values of L-functions.

1 Introduction

Let K be a number field, h_K be the order of the ideal class group Cl_K, w_K be the order of the torsion subgroup μ_K of the unit group $\mathcal{O}_K^\times$, and R_K be the regulator. The Dirichlet analytic class number formula evaluates the leading term in the Taylor expansion of the Dedekind zeta function $\zeta_K(s)$ at $s = 0$,

$$\zeta_K(s) = a_r s^r + a_{r+1} s^{r+1} + O(s^{r+2}),$$

as $a_r = -h_K R_K / w_K$, where r is the rank of the finitely generated abelian group $\mathcal{O}_K^\times$. Stark's Main conjecture [St1, St2, St3, St4] and its integral refinements due to Stark [St4], Tate [T], Chinburg [Ch], Rubin [R], and Popescu [P], among others, provide a vast Galois-equivariant generalization of the analytic class number formula with fundamental consequences for number theory. Roughly, the conjectures predict a relationship between the leading term at $s = 0$ of the imprimitive Artin L-functions $L_{M/K,S}(\rho, s)$ associated to a Galois extension M/K of number fields with Galois group $G = \mathrm{Gal}(M/K)$, and a certain $\mathbb{Q}[G]$-module-invariant associated to the unit group $\mathcal{O}_K^\times$ of K. In the conjectures the Dedekind zeta function $\zeta_K(s)$ is replaced by the G-equivariant L-function

$$\Theta_{M/K,S}(s) = \sum_{\rho \in \widehat{G}} L_{M/K,S}(\rho, s) \cdot e_{\check{\rho}}$$

with values in the group ring $\mathbb{C}[G]$, the regulator R_K is replaced by a G-equivariant regulator with values in $\mathbb{C}[G]$, and the rank r of $\mathcal{O}_K^\times$ is replaced by the local rank function of the projective $\mathbb{Q}[G]$-module $\mathbb{Q}\mathcal{O}_{M,S}^\times$ of S-units in M.

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Also of fundamental importance in number theory is the *second* term of $\zeta_K(s)$ at $s = 0$. For example, the Kronecker first limit formula evaluates the second term of $\zeta_{\mathbb{Q}(\sqrt{-D})}(s)$ at $s = 0$ as essentially the logarithm of the absolute value of the Dedekind eta function. See C. L. Siegel's book [Si] for a proof and some remarkable applications. The Rubin–Stark conjectures, and the even more general equivariant Tamagawa number conjecture of Burns and Flach [BF], predict nothing about the second and higher terms of $\Theta_{M/L,S}(s)$ at $s = 0$.

Our primary goal in this paper is to evaluate the second term at $s = 0$ of a G-equivariant L-function associated to a certain group of unramified abelian characters. We now summarize our main result. Let K be an imaginary quadratic extension of a totally real number field F of degree n over $\mathbb{Q}$. Let $\chi \in \widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ be an irreducible character of $G = \mathrm{Gal}(H/K)$, where H is the Hilbert class field of K. Let S_∞ be the set of infinite primes of K. The Artin L-function of χ is defined by

$$L_{S_\infty}(s,\chi) = \prod_{\mathfrak{p}\notin S_\infty} \left(1 - \chi(\sigma_{\mathfrak{p}}) N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1,$$

where $\sigma_{\mathfrak{p}} \in G$ is the Frobenius automorphism associated to the (unramified) prime $\mathfrak{p}$ in H/K. For each $\chi \in \widehat{G}$, let

$$e_\chi = \frac{1}{|G|} \sum_{\sigma\in G} \chi(\sigma) \cdot \sigma^{-1}$$

be the associated idempotent in the group ring $\mathbb{C}[G]$. The S_∞, G-equivariant L-function $\Theta_{S_\infty} : \mathbb{C} \to \mathbb{C}[G]$ is defined by

$$\Theta_{S_\infty}(s) = \sum_{\chi\in\widehat{G}} L_{S_\infty}(s,\chi) \cdot e_{\chi^{-1}}.$$

The order of vanishing of $\Theta_{S_\infty}(s)$ at $s = 0$ is $n-1$. In fact, because $L_{S_\infty}^{(n-1)}(0,\chi) = 0$ for all $\chi \in \widehat{G}$, $\chi \neq \mathbf{1}_G$ non-trivial, the leading term of $\Theta_{S_\infty}(s)$ at $s = 0$ arises from $\zeta_K^{(n-1)}(0)$, which is given by the analytic class number formula. In this case the Rubin–Stark conjecture is proved and is of minimal interest. However, as we will see, the second term at $s = 0$ is of considerable interest.

In the following theorem we summarize the main result of this paper, which is an evaluation formula in $\mathbb{C}[G]$ for the second term of $\Theta_{S_\infty}(s)$ at $s = 0$ (for the precise statement, see Theorem 1.9).

Main Theorem *There exists a positive, real-analytic Hilbert modular function* $\Psi : \mathbb{H}^n \to \mathbb{R}$ *and CM points* $\{z_\sigma\}_{\sigma\in G}$ *on a universal Hilbert modular variety* $\mathfrak{X}_0$ *such that*

$$\Theta_{S_\infty}^{(n)}(0) = \frac{n!}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{\sigma\in G} \log\left\{\alpha_\sigma \Psi(z_\sigma)\right\} \cdot \sigma^{-1} \quad \textit{in} \quad \mathbb{C}[G].$$

Here, the $\{\alpha_\sigma\}_{\sigma\in G}$ *are positive constants which are explicitly determined.*

In the remaining part of the introduction we describe the results of this paper in more detail.

Let $\mathbb{H}$ be the complex upper half-plane and $z = x + iy = (z_1, \ldots, z_n) \in \mathbb{H}^n$. Let $\{\sigma_1, \ldots, \sigma_n\}$ be the n real embeddings of F, $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals in F, $(a, b) \in \mathfrak{a} \times \mathfrak{b}$, and define

$$N(a + bz) = \prod_{j=1}^{n} (\sigma_j(a) + \sigma_j(b)z_j).$$

Then the non-holomorphic Hilbert modular Eisenstein series associated to $(\mathfrak{a}, \mathfrak{b})$ is defined by

$$E(s, z; \mathfrak{a}, \mathfrak{b}) = \sum_{(a,b)\in\mathfrak{a}\times\mathfrak{b}/\mathcal{O}_F^\times}{}' \frac{N(y)^s}{|N(a + bz)|^{2s}}, \quad \mathrm{Re}(s) > 1,$$

where the sum is over a complete set of non-zero, non-associate representatives of $\mathfrak{a} \times \mathfrak{b}$, and $N(y) = N(y(z))$ is the product of the imaginary parts of the components of $z \in \mathbb{H}^n$ (recall that (a, b) and (a', b') are associate if there exists a unit ϵ in $\mathcal{O}_F^\times$ such that $(a, b) = (\epsilon a', \epsilon b')$).

Let d_F be the absolute value of the discriminant of F and $N_{F/\mathbb{Q}}(\mathfrak{a}) = |\mathcal{O}_F : \mathfrak{a}|$. We will compute the Fourier expansion of $E(s, z; \mathfrak{a}, \mathfrak{b})$ and use this to prove that the Eisenstein series has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.

Theorem 1.1 *The Eisenstein series $E(s, z; \mathfrak{a}, \mathfrak{b})$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue*

$$\mathrm{Res}_{s=1}\, E(s, z; \mathfrak{a}, \mathfrak{b}) = \frac{2^{n-1}\pi^n R_F}{d_F w_F N_{F/\mathbb{Q}}(\mathfrak{a}\mathfrak{b})}.$$

Furthermore, $E(s, z; \mathcal{O}_F, \mathcal{O}_F)$ satisfies the functional equation

$$G(1 - s)E(s, z; \mathcal{O}_F, \mathcal{O}_F) = G(2(1 - s))E(1 - s, z; \mathcal{O}_F, \mathcal{O}_F),$$

where $G(s)$ is the gamma factor

$$G(s) = d_F^{s/2}\left[\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right]^n.$$

We will use the Fourier expansion to compute the Taylor expansion of $E(s, z; \mathfrak{a}, \mathfrak{b})$ at $s = 0$,

$$E(s, z; \mathfrak{a}, \mathfrak{b}) = E_{n-1}s^{n-1} + E_n(z)s^n + O(s^{n+1}). \tag{1.1}$$

The number E_{n-1} is essentially the regulator R_F, and the function $E_n(z)$ is a multiple of a positive, real-analytic Hilbert modular function $\Psi\colon \mathbb{H}^n \to \mathbb{R}$. For $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-D})$, and $\mathfrak{a} = \mathfrak{b} = \mathbb{Z}$, this function is given by

$$\log \Psi(z) = -\log(2\pi) + \frac{\pi y}{6} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \sigma_{-1}(|n|)e^{-2\pi|n|y}e^{2\pi i n x},$$

where

$$\sigma_{-1}(n) = \sum_{\substack{d>0 \\ d|n}} d^{-1}.$$

We will combine (1.1) with classical methods of Siegel [Si] to study the modular and analytic properties of $\Psi(z)$. Let

$$GL_2(F) = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in F \text{ and } \det(M) \in F^{\times} \right\},$$

and define the subgroup of matrices with totally positive determinant which stabilize the lattice $(\mathfrak{a}, \mathfrak{b})$,

$$\Gamma(\mathfrak{a}, \mathfrak{b}) = \left\{ M \in GL_2(F) : \det(M) \in \mathcal{O}_F^{\times,+}, M \cdot (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{b}) \right\}.$$

Here, $\mathcal{O}_F^{\times,+}$ denotes the totally positive units of F. The group $\Gamma(\mathfrak{a}, \mathfrak{b})$ embeds as a discrete subgroup of $GL_2(\mathbb{R})^n$, which induces a discontinuous action of $\Gamma(\mathfrak{a}, \mathfrak{b})$ on $\mathbb{H}^n$. We will determine the factor of automorphy occurring in the transformation formula satisfied by $\Psi(z)$ under the action of the group $\Gamma(\mathfrak{a}, \mathfrak{b})$.

Theorem 1.2 *For all $M \in \Gamma(\mathfrak{a}, \mathfrak{b})$,*

$$\Psi(M(z)) = |N(\gamma z + \delta)|^{-2^n R_F / w_F} \Psi(z).$$

Recall that the Dedekind eta function $\eta\colon \mathbb{H} \to \mathbb{C}$ is a non-vanishing, complex-valued, holomorphic function such that $|\eta(z)|$ appears in the Kronecker first limit formula. An interesting question is whether there exists a naturally defined complex-valued function whose absolute value equals $\Psi(z)$. One possible approach would be to prove that there exists an analytic function $A\colon \mathbb{H}^n \to \mathbb{C}$ such that $A(z) = \log \Psi(z) + iB(z)$; that is, $\log \Psi(z)$ is the real part of a complex-valued, analytic function. It would follow that $|e^{A(z)}| = \Psi(z)$.

It is well known that in one complex variable a harmonic function is the real part of an analytic function. Let

$$\Delta_j = y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$$

be the Laplacian associated to each factor in the product $\mathbb{H}^n$. We will prove that $\log \Psi(z)$ vanishes under each operator Δ_j.

Theorem 1.3 $\Delta_j \log \Psi(z) = 0$ *for* $j = 1, \ldots, n$.

Even though $\log \Psi(z)$ vanishes under each operator Δ_j, in several complex variables this condition is not strong enough to insure that $\log \Psi(z)$ is the real part of an analytic function (see Section 6).

Proposition 1.4 $\log \Psi(z)$ *is not the real part of an analytic function.*

Because K is an imaginary quadratic extension of F, the $2n$ embeddings of K occur in complex conjugate pairs. Let $\Phi = \{\tau_1, \ldots, \tau_n\}$ be a CM type for K/F, which is a choice of one embedding for each complex conjugate pair. We will construct for each ideal class $C \in \mathrm{Cl}_K$ a CM point

$$\Phi(z_C) = (\tau_1(z_C), \ldots, \tau_n(z_C)) \in \mathbb{H}^n$$

on a Hilbert modular variety $X_0(\mathfrak{a}_C) := \mathbb{H}^n/\Gamma_0(\mathfrak{a}_C)$ arising from the $\mathcal{O}_F$-module decomposition $A_C = \mathfrak{a}_C\omega_1 + \mathcal{O}_F\omega_2$ of a fixed integral ideal $A_C \in C^{-1}$. Here, $\mathfrak{a}_C$ is an integral ideal in F, $\Gamma_0(\mathfrak{a}_C) := \Gamma(\mathfrak{a}_C, \mathcal{O}_F)$, $(\omega_1, \omega_2) \in \mathfrak{a}_C^{-1}\mathcal{O}_K \times \mathcal{O}_K$, and $z_C := \omega_2/\omega_1$. We also show that the CM points $\{\Phi(z_C) : C \in \mathrm{Cl}_K\}$ can be viewed as living on a certain universal Hilbert modular variety $\mathfrak{X}_0$ constructed from the universal covers of the varieties $X_0(\mathfrak{a}_C)$.

Let $\zeta_K(s, C)$ be the Dedekind zeta function of the ideal class $C \in \mathrm{Cl}_K$. We will establish the identity

$$\zeta_K(s, C) = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}(2^n N_{F/\mathbb{Q}}(\mathfrak{a}_C) d_F/\sqrt{d_K})^s E(s, \Phi(z_C); \mathfrak{a}_C, \mathcal{O}_F). \tag{1.2}$$

The Artin map yields an isomorphism $G \cong \mathrm{Cl}_K$, from which one obtains the identity

$$L_{S_\infty}(s, \chi) = \sum_{C \in \mathrm{Cl}_K} \chi(C)\zeta_K(s, C). \tag{1.3}$$

Furthermore, for $\chi \neq \mathbf{1}_G$ non-trivial the orthogonality relations for group characters yields

$$L_{S_\infty}^{(n)}(0, \chi) = \sum_{C \in \mathrm{Cl}_K} \chi(C) a_n(C). \tag{1.4}$$

We will combine equations (1.1) through (1.4) to evaluate $L_{S_\infty}^{(n)}(0, \chi)$.

Theorem 1.5 *Using the same notation as above,*

$$L_{S_\infty}^{(n)}(0, \mathbf{1}_G) = \frac{n!}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \log\Big\{ (2^n d_F/\sqrt{d_K})^{h_K E_{n-1}} \prod_{C \in \mathrm{Cl}_K} \epsilon(C^{-1}) \Big\},$$

and for $\chi \neq \mathbf{1}_G$,

$$L_{S_\infty}^{(n)}(0, \chi) = \frac{n!}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|} \sum_{C \in \mathrm{Cl}_K} \chi(C) \log \epsilon(C^{-1}),$$

where

$$\epsilon(C^{-1}) = \left(N(y(\Phi(z_C)))N_{F/\mathbb{Q}}(\mathfrak{a}_C)^{-1}\right)^{E_{n-1}} \Psi(\Phi(z_C)).$$

When $K = \mathbb{Q}(\sqrt{-D})$, one can use the Kronecker first limit formula to express the relative class number h_H/h_K as the volume of the fundamental domain for a special subgroup of the unit group $\mathcal{O}_H^\times$ spanned by elliptic modular units formed from ratios of special values of $\eta(z)$ (see [Si]). Analogously, we will combine Theorem 1.5 with the class field theory product for $\zeta_H(s)$ and the Frobenius determinant relation to obtain a formula for h_H/h_K as a multiple of the determinant of an $(h_K-1)\times(h_K-1)$ matrix whose entries are logarithms of ratios of special values of the Hilbert modular function $\Psi(z)$ at CM points.

Theorem 1.6 *Using the same notation as above,*

$$\frac{h_H}{h_K} = \frac{w_H}{w_K}\frac{R_K}{R_H}\frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^{h_K-1}} \det_{C,C'\neq 1} \log\left\{\frac{\epsilon(C(C')^{-1})}{\epsilon(C)}\right\}.$$

Theorem 1.6 gives rise to the following interesting question.

Question 1.7 When $n > 1$ are the ratios $\epsilon(C(C')^{-1})/\epsilon(C)$ units, or simply algebraic numbers, in the Hilbert class field H?

It is important to point out that other functions similar to $\log\Psi(z)$ have been studied. For example, Konno in [K] defined a zeta function $Z(Q,\mathfrak{a},\mathfrak{b};s)$ associated to a certain positive definite quadratic form Q, meromorphically continued $Z(Q,\mathfrak{a},\mathfrak{b};s)$ to $\mathrm{Re}(s) > 1/2$, and computed the constant term in the Laurent expansion of $Z(Q,\mathfrak{a},\mathfrak{b};s)$ at $s=1$ in terms of a function of the form $\log\Psi(z)$. He then used this to evaluate $L_{S_\infty}(1,\chi)$ and express h_H/h_K as a multiple of a product of linear combinations of values of $\log\Psi(z)$.

Our approach using Hilbert modular Eisenstein series is different and provides many advantages. First, it allows us to study the modular and analytic properties of $\log\Psi(z)$ using the invariance of $E(s,z;\mathfrak{a},\mathfrak{b})$ under $\Gamma(\mathfrak{a},\mathfrak{b})$, and that $E(s,z;\mathfrak{a},\mathfrak{b})$ is an eigenfunction for the operators Δ_j. Second, it leads to an arithmetic interpretation of the points at which $\Psi(z)$ is evaluated as CM points on a universal Hilbert modular variety. This is potentially interesting, given the recent work on CM values of Hilbert modular functions by Bruinier and Yang [BY1, BY2, Y]). Third, it provides a conceptually simple and direct way of evaluating $L_{S_\infty}^{(n)}(s,\chi)$ at $s=0$, which is the most natural value to consider in the context of the Rubin–Stark conjectures.

Asai [A] showed how to define the non-holomorphic Eisenstein series for an arbitrary number field K of class number 1. He computed the constant term in the Laurent expansion of the Eisenstein series at $s=1$ in terms of a function of the form $\log\Psi(z)$, and used the methods of Siegel [Si] to prove results similar to Theorems 1.2 and 1.3 for $\log\Psi(z)$. Asai also related $\log\Psi(z)$ to certain Grössencharakter L-functions.

The Rubin–Stark conjectures are actually stated for (S,T)-modified G-equivariant L-functions. See [P] for a description of the hypotheses that the two finite sets of primes S and T must satisfy. The (S_∞,T)-modified L-functions arising in the context of this paper can be described as follows. Let T be a non-empty, finite set of primes in K. Let T_H be the set of primes in H dividing primes in T. We require that T satisfy the following hypotheses.

(1) $T \cap S_\infty = \varnothing$.
(2) There are no non-trivial elements in μ_H which are congruent to 1 modulo all the primes $\mathfrak{p}$ in T_H.

Hypothesis (2) is satisfied if, for example, T contains at least two primes of different residual characteristic or a prime whose residue field is large compared to the size of μ_K.

For such a set T and $\chi \in \widehat{G}$, define the complex-analytic function

$$\zeta_T(s,\chi) = \prod_{\mathfrak{p}\in T}(1 - \chi(\sigma_{\mathfrak{p}})N_{K/\mathbb{Q}}(\mathfrak{p})^{1-s}).$$

We then define the (S_∞, T)-modified L-function associated to χ by

$$L_{S_\infty,T}(s,\chi) = \zeta_T(s,\chi)L_{S_\infty}(s,\chi).$$

Note that for any $\chi \in \widehat{G}$, $\zeta_T(0,\chi) \neq 0$, and thus $L_{S_\infty}(s,\chi)$ and $L_{S_\infty,T}(s,\chi)$ have the same order of vanishing at $s = 0$. Define $\zeta_T\colon \mathbb{C} \to \mathbb{C}[G]$ by

$$\zeta_T(s) = \sum_{\chi\in\widehat{G}} \zeta_T(s,\chi)\cdot e_{\chi^{-1}}.$$

Then the (S_∞, T)-modified G-equivariant L-function $\Theta_{S_\infty,T}\colon \mathbb{C} \to \mathbb{C}[G]$ is defined by

$$\Theta_{S_\infty,T}(s) := \zeta_T(s)\Theta_{S_\infty}(s) = \sum_{\chi\in\widehat{G}} L_{S_\infty,T}(s,\chi)\cdot e_{\chi^{-1}}.$$

Remark 1.8. The function $\Theta_{S_\infty,T}(s)$ satisfies the remarkable integrality property $\Theta_{S_\infty,T}(1-n) \in \mathbb{Z}[G]$ for all integers $n \geq 1$. This is a consequence of a more general integrality property for (S, T)-modified L-functions due independently to Deligne and Ribet [DR], P. Cassou–Noguès [CN], and D. Barsky [B].

We now state precisely our main result, which is an evaluation formula in $\mathbb{C}[G]$ for the second terms of $\Theta_{S_\infty}(s)$ and $\Theta_{S_\infty,T}(s)$ at $s = 0$.

Theorem 1.9 *Using the same notation as above,*

$$\Theta^{(n)}_{S_\infty}(0) = \frac{n!}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}\sum_{\sigma\in G}\log\big\{(2^n d_F/\sqrt{d_K})^{E_{n-1}}\epsilon(\sigma^{-1})\big\}\cdot\sigma^{-1} \quad \textit{in} \quad \mathbb{C}[G].$$

In particular,

$$\Theta^{(n)}_{S_\infty,T}(0) = \delta_T(0)\Theta^{(n)}_{S_\infty}(0) \quad \textit{in} \quad \mathbb{C}[G].$$

Stark's integral refinement of his conjecture concerns abelian extensions M/K and their associated imprimitive Artin L-functions $L_{M/K,S}(\rho, s)$ of order of vanishing *at most 1* at $s = 0$ (see [St4]). The conjecture predicts the existence of a special S-unit ϵ_M whose construction is closely related to the solution of Hilbert's twelfth problem regarding explicit generation of the abelian class fields of the base field K. Tate [T] showed that the non-trivial solution to Stark's integral conjecture would lead to an explicit generation of the abelian class fields of the base field K by exponentials of special values of L-functions.

We obtain from equation (11.4) the following product formula for the special values $\{\Psi(\Phi(z_C)) : C \in \mathrm{Cl}_K\}$ in terms of exponentials of special values of L-functions.

Theorem 1.10 *For each $C \in \mathrm{Cl}_K$,*

$$\Psi(\Phi(z_C)) = \left(2^n d_F/\sqrt{d_K} N(y(\Phi(z_C)))N_{F/\mathbb{Q}}(\mathfrak{a}_C)^{-1}\right)^{-E_{n-1}} \times \prod_{\chi \in \widehat{G}} \exp\left\{ \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{h_K n!} \chi(C) L_{S_\infty}^{(n)}(0,\chi)\right\}.$$

2 Proof of Theorem 1.1

In this section we compute the Fourier expansion of $E(s, z; \mathfrak{a}, \mathfrak{b})$ and use this to prove Theorem 1.1. We will need formulas for the Fourier coefficients of the function

$$f(z) = \sum_{a \in \mathfrak{a}} |N(z+a)|^{-2s}, \quad \mathrm{Re}(s) > 1.$$

Let $\mathfrak{a}^*$ be the dual lattice of $\mathfrak{a}$, T be the trace, and $\mathrm{vol}(P)$ be the volume of a fundamental parallelotope P for $\mathfrak{a}$. Because the function $f(z)$ is holomorphic on $\mathbb{H}^n$ and periodic with respect to $\mathfrak{a}$, it has a Fourier expansion

$$f(z) = \sum_{a \in \mathfrak{a}^*} h_a(y,s) e^{2\pi i T(ax)},$$

where the Fourier coefficients are given by the formula

$$h_a(y,s) = \frac{1}{\mathrm{vol}(P)} \int_P f(z) e^{-2\pi i T(ax)} dx.$$

Proposition 2.1 *Using the same notation as above,*

$$h_0(y,s) = \frac{N(y)^{1-2s}}{\sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})} \left[\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \right]^n,$$

and for $a \neq 0$,

$$h_a(y,s) = \frac{2^n N(y)^{\frac{1}{2}-s}}{\sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})} \left[\frac{\pi^s}{\Gamma(s)} \right]^n \prod_{j=1}^n K_{s-\frac{1}{2}}(2\pi|\sigma_j(a)|y_j) N_{F/\mathbb{Q}}((a))^{s-\frac{1}{2}},$$

where

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)} \cosh(\nu t) dt, \quad t > 0,$$

is the K-Bessel function.

Proof Let $d(\mathfrak{a})$ be the absolute value of the discriminant of $\mathfrak{a}$. Then $d(\mathfrak{a}) = d_F N_{F/\mathbb{Q}}(\mathfrak{a})^2$, so that $\mathrm{vol}(P) = \sqrt{d(\mathfrak{a})} = \sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})$. Using the definition of $f(z)$ and that P is a fundamental parallelotope for the lattice $\mathfrak{a}$, we find that

$$h_a(y,s) = \frac{N(y)^{1-2s}}{\sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})} \int_{\mathbb{R}^n} |N(1-ix)|^{-2s} e^{-2\pi i T(ayx)} dx.$$

Define the 1-dimensional integral

$$h(y,s) = \int_{\mathbb{R}} |1-it|^{-2s} e^{-ity} dt = \int_{\mathbb{R}} (1+t^2)^{-s} e^{-ity} dt, \quad \mathrm{Re}(s) > \frac{1}{2}.$$

Then using the definition of the trace we find that

$$h_a(y,s) = \frac{N(y)^{1-2s}}{\sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})} \prod_{j=1}^{n} h(2\pi\sigma_j(a) y_j; s).$$

Thus, to compute $h_a(y,s)$, it suffices to compute $h(y,s)$.

For $y = 0$ we obtain

$$h(0,s) = \int_{\mathbb{R}} (1+t^2)^{-s} dt = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}$$

(see [L2, p. 272]). It follows that the zeroth Fourier coefficient is

$$h_0(y,s) = \frac{N(y)^{1-2s}}{\sqrt{d_F} N_{F/\mathbb{Q}}(\mathfrak{a})} \left[\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \right]^n.$$

Suppose $y \neq 0$. Since $1 + t^2$ is even, we can write

$$h(y,s) = \int_{\mathbb{R}} \left(1+t^2\right)^{-s} e^{-ity} dt = 2\int_0^\infty (1+t^2)^{-s} \cos(ty) dt.$$

Further, we have the formula

$$\int_0^\infty (1+t^2)^{-s} \cos(ty) dt = \frac{1}{\sqrt{\pi}} \left(\frac{2}{|y|}\right)^{\frac{1}{2}-s} \cos\left(\pi\left(\frac{1}{2}-s\right)\right) \Gamma(1-s) K_{s-\frac{1}{2}}(|y|),$$

where

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)} \cosh(\nu t) dt, \quad t > 0,$$

is the K-Bessel function (see [GR], pg. 426). It follows that

$$h(y,s) = \frac{2}{\sqrt{\pi}} \left(\frac{2}{|y|}\right)^{\frac{1}{2}-s} \cos\left(\pi\left(\frac{1}{2}-s\right)\right) \Gamma(1-s) K_{s-\frac{1}{2}}(|y|),$$

from which we obtain

$$(2.1)\quad \prod_{j=1}^{n} h(2\pi\sigma_j(a)y_j;s) = 2^n\left[\pi^{s-1}\cos\left(\pi\left(\tfrac{1}{2}-s\right)\right)\Gamma(1-s)\right]^n$$
$$\prod_{j=1}^{n}\left(|\sigma_j(a)|y_j\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi|\sigma_j(a)|y_j\right).$$

From the identity $\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right) = \frac{\pi}{\cos(\pi s)}$, we determine the identity $\pi^{s-1}\cos\left(\pi\left(\frac{1}{2}-s\right)\right)\Gamma(1-s) = \frac{\pi^s}{\Gamma(s)}$. Similarly, we determine the identity

$$\prod_{j=1}^{n}(|\sigma_j(a)|y_j)^{s-\frac{1}{2}} = N(y)^{s-\frac{1}{2}}N_{F/\mathbb{Q}}((a))^{s-\frac{1}{2}}.$$

Substituting these identities in (2.1) yields

$$h_a(y,s) = \frac{2^n N(y)^{\frac{1}{2}-s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\pi^s}{\Gamma(s)}\right]^n\prod_{j=1}^{n}K_{s-\frac{1}{2}}(2\pi|\sigma_j(a)|y_j)N_{F/\mathbb{Q}}((a))^{s-\frac{1}{2}}. \qquad \blacksquare$$

We will need the following lemma, which can be proved in a manner analogous to equation (8.1).

Lemma 2.2 *Let* $[\mathfrak{a}]$ *be the ideal class of* F *containing* $\mathfrak{a}$. *Then*

$$\zeta_F(2s,[\mathfrak{a}^{-1}]) = N_{F/\mathbb{Q}}(\mathfrak{a})^{2s}\sum_{\alpha\in\mathfrak{a}/\mathcal{O}_F^\times}{}' |N(\alpha)|^{-2s},$$

where $\mathfrak{a}/\mathcal{O}_F^\times$ *denotes the sum over a collection of non-zero* α *in* $\mathfrak{a}$ *which are non-associate modulo* $\mathcal{O}_F^\times$.

We are now in a position to compute the Fourier expansion of $E(s,z;\mathfrak{a},\mathfrak{b})$. Using Lemma 2.2 and the change of summation

$$\sum_{\substack{(a,b)\in\mathfrak{a}\times\mathfrak{b}/\mathcal{O}_F^\times\\ b\neq 0}} = \sum_{a\in\mathfrak{a}/\mathcal{O}_F^\times}\sum_{b\in\mathfrak{b}}{}',$$

we compute

(2.2)
$$\begin{aligned}E(s,z;\mathfrak{a},\mathfrak{b}) &= \sum_{a\in\mathfrak{a}/\mathcal{O}_F^\times}{}' N(y)^s|N(a)|^{-2s} + \sum_{\substack{(a,b)\in\mathfrak{a}\times\mathfrak{b}/\mathcal{O}_F^\times\\ b\neq 0}} N(y)^s|N(a+bz)|^{-2s}\\
&= N(y)^sN_{F/\mathbb{Q}}(\mathfrak{a})^{-2s}\zeta_F(2s,[\mathfrak{a}^{-1}]) + \sum_{a\in\mathfrak{a}/\mathcal{O}_F^\times}\sum_{b\in\mathfrak{b}}{}' N(y)^s|N(a+bz)|^{-2s}\\
&= N(y)^sN_{F/\mathbb{Q}}(\mathfrak{a})^{-2s}\zeta_F(2s,[\mathfrak{a}^{-1}]) + \sum_{b\in\mathfrak{b}/\mathcal{O}_F^\times}{}'\sum_{a\in\mathfrak{a}} N(y)^s|N(a+bz)|^{-2s}.\end{aligned}$$

Write $z = x + iy$, so that $bz = bx + i(by)$. Letting $z \mapsto bz$ in $f(z)$ yields

$$\begin{aligned}\sum_{a\in\mathfrak{a}} |N(a+bz)|^{-2s} &= \sum_{a\in\mathfrak{a}^*} h_a(by,s)e^{2\pi i T(abx)} \\ &= h_0(by,s) + \sum_{a\in\mathfrak{a}^*}{}' h_a(by,s)e^{2\pi i T(abx)}.\end{aligned} \tag{2.3}$$

From the formula for $h_0(y,s)$ in Proposition 2.1,

$$h_0(by,s) = \frac{\big(N(y)N_{F/\mathbb{Q}}((b))\big)^{1-2s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right]^n.$$

Then substituting (2.3) into (2.2) yields

(2.4)

$$\begin{aligned}E(s,z;\mathfrak{a},\mathfrak{b}) &= N(y)^s N_{F/\mathbb{Q}}(\mathfrak{a})^{-2s}\zeta_F(2s,[\mathfrak{a}^{-1}]) \\ &\quad + \frac{N(y)^s|N(y)|^{1-2s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right]^n \sum_{b\in\mathfrak{b}/\mathcal{O}_F^\times}{}' N_{F/\mathbb{Q}}((b))^{1-2s} \\ &\quad + N(y)^s \sum_{b\in\mathfrak{b}/\mathcal{O}_F^\times}{}' \sum_{a\in\mathfrak{a}^*}{}' h_a(by,s)e^{2\pi i T(abx)} \\ &= N(y)^s N_{F/\mathbb{Q}}(\mathfrak{a})^{-2s}\zeta_F(2s,[\mathfrak{a}^{-1}]) \\ &\quad + \frac{N(y)^{1-s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right]^n N_{F/\mathbb{Q}}(\mathfrak{b})^{-(2s-1)}\zeta_F(2s-1,[\mathfrak{b}^{-1}]) \\ &\quad + N(y)^s \sum_{b\in\mathfrak{b}/\mathcal{O}_F^\times}{}' \sum_{a\in\mathfrak{a}^*}{}' h_a(by,s)e^{2\pi i T(abx)}.\end{aligned}$$

Using the definition of the trace we find that

$$h_a(by,s) = \frac{\big(N(y)N_{F/\mathbb{Q}}((b))\big)^{1-2s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\prod_{j=1}^n h(2\pi\sigma_j(ab)y_j;s).$$

Then combining terms with fixed ab yields

$$\sum_{b\in\mathfrak{b}/\mathcal{O}_F^\times}{}' \sum_{a\in\mathfrak{a}^*}{}' h_a(by,s)e^{2\pi i T(abx)} = \frac{N(y)^{1-2s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\sum_{\tilde{a}\in\mathfrak{a}^*}{}' \sigma_{\tilde{a}}(y,s)e^{2\pi i T(\tilde{a}x)}, \tag{2.5}$$

where the Fourier coefficients are given by the following sum of divisors:

$$\sigma_{\tilde{a}}(y,s) = \sum_{\substack{\tilde{a}=ab \\ a\in\mathfrak{a}^* \\ b\in\mathfrak{b}/\mathcal{O}_F^\times}} \prod_{j=1}^n h(2\pi\sigma_j(ab)y_j;s)N_{F/\mathbb{Q}}((b))^{1-2s}.$$

From the formula for $h_a(y,s)$, $a \neq 0$, in Proposition 2.1, we can express the Fourier coefficients as
(2.6)

$$\sigma_{\tilde{a}}(y,s) = 2^n N(y)^{s-\frac{1}{2}} \left[\frac{\pi^s}{\Gamma(s)}\right]^n \sum_{\substack{\tilde{a}=ab \\ a\in\mathfrak{a}^* \\ b\in\mathfrak{b}/\mathcal{O}_F^\times}} \left(\frac{N_{F/\mathbb{Q}}((a))}{N_{F/\mathbb{Q}}((b))}\right)^{s-\frac{1}{2}} \prod_{j=1}^n K_{s-\frac{1}{2}}(2\pi|\sigma_j(ab)|y_j).$$

Finally, by substituting (2.5) into (2.4) and using the formula (2.6), we obtain the Fourier expansion

$$\begin{aligned}(2.7)\qquad E(s,z;\mathfrak{a},\mathfrak{b}) &= N(y)^s N_{F/\mathbb{Q}}(\mathfrak{a})^{-2s}\zeta_F(2s,[\mathfrak{a}^{-1}]) \\ &+ \frac{N(y)^{1-s}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right]^n N_{F/\mathbb{Q}}(\mathfrak{b})^{-(2s-1)}\zeta_F(2s-1,[\mathfrak{b}^{-1}]) \\ &+ \frac{2^n N(y)^{\frac{1}{2}}}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\left[\frac{\pi^s}{\Gamma(s)}\right]^n \\ &\times \sum_{\tilde{a}\in\mathfrak{a}^*}{}' \sum_{\substack{\tilde{a}=ab \\ a\in\mathfrak{a}^* \\ b\in\mathfrak{b}/\mathcal{O}_F^\times}} \left(\frac{N_{F/\mathbb{Q}}((a))}{N_{F/\mathbb{Q}}((b))}\right)^{s-\frac{1}{2}} e^{2\pi i T(abx)} \prod_{j=1}^n K_{s-\frac{1}{2}}(2\pi|\sigma_j(ab)|y_j) \\ &= A(s)+B(s)+C(s).\end{aligned}$$

The expression (2.6) provides an analytic continuation of $\sigma_{\tilde{a}}(y,s)$ to an entire function on $\mathbb{C}$. Furthermore, by estimating $\sigma_{\tilde{a}}(y,s)$ on compact subsets of $\mathbb{C}$, one can show that the series $\sum_{\tilde{a}\in\mathfrak{a}^*}\sigma_{\tilde{a}}(y,s)e^{2\pi i T(\tilde{a}x)}$ converges uniformly on compact subsets and hence defines entire function on $\mathbb{C}$. Therefore, $C(s)$ is entire on $\mathbb{C}$.

The function $\zeta_F(s,C)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$. Therefore, $A(s)$ and $B(s)$ have meromorphic continuations to $\mathbb{C}$. We conclude that $E(s,z;\mathfrak{a},\mathfrak{b}) = A+B+C$ has a meromorphic continuation to $\mathbb{C}$.

We want to determine the poles of $E(s,z;\mathfrak{a},\mathfrak{b})$. The function $B(s)$ has a pole at $s=1$. To compute the residue, recall the Laurent expansion $\zeta_F(s,C) = \frac{\kappa}{s-1}+O(1)$ (see [L1, p. 254]), where the residue κ is given by $\kappa = \frac{2^n R_F}{w_F\sqrt{d_F}}$. Using the expansion $\zeta_F(2s-1,[\mathfrak{a}^{-1}]) = \frac{\kappa/2}{s-1}+O(1)$, and $\Gamma(1/2)=\sqrt{\pi}$, we find that the residue of the pole of $B(s)$ at $s=1$ is

$$\operatorname{Res}_{s=1} B(s) = \frac{\pi^n}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{ab})}\frac{\kappa}{2} = \frac{2^{n-1}\pi^n R_F}{d_F w_F N_{F/\mathbb{Q}}(\mathfrak{ab})}.$$

We claim that $A(s)$ and $B(s)$ have simple poles at $s=1/2$ with residues which cancel. Thus, since $C(s)$ is holomorphic at $s=1/2$, we can conclude that $E(s,z;\mathfrak{a},\mathfrak{b})$ has only the pole at $s=1$ coming from $B(s)$ with residue $\operatorname{Res}_{s=1} E(s,z;\mathfrak{a},\mathfrak{b}) = \operatorname{Res}_{s=1} B(s)$.

Using the expansion $\zeta_F(2s,[\mathfrak{a}^{-1}]) = \frac{\kappa/2}{s-\frac{1}{2}}+O(1)$, we obtain

$$A(s) = \frac{N(y)^{1/2}}{N_{F/\mathbb{Q}}(\mathfrak{a})}\frac{\kappa}{2}\frac{1}{s-\frac{1}{2}}+O(1) = \frac{N(y)^{1/2}2^{n-1}R_F}{\sqrt{d_F}w_F N_{F/\mathbb{Q}}(\mathfrak{a})}\frac{1}{s-\frac{1}{2}}+O(1).$$

Now, we know that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, so

$$\Gamma\left(s - \frac{1}{2}\right)^n = \frac{1}{(s-\frac{1}{2})^n} + O\left(s - \frac{1}{2}\right)^{1-n}. \tag{2.8}$$

The functional equation for $\zeta_F(s, C)$ is given by $G(s)\zeta_F(s, C) = G(1-s)\zeta_F(1-s, C)$, where $G(s)$ is the gamma factor $G(s) = d_F^{s/2}\left[\pi^{-s/2}\Gamma\left(s/2\right)\right]^n$ (see [L1, p. 254]). Using the functional equation, one can show that $\zeta_F(s, C)$ has a zero of order $n-1$ at $s = 0$ with leading term

$$\frac{\zeta_F^{(n-1)}(0, C)}{(n-1)!} = \lim_{s\to 0} s^{-(n-1)}\zeta_F(s, C) = -\frac{R_F}{w_F},$$

and therefore $\zeta_F(s, C) = -\frac{R_F}{w_F}s^{n-1} + O(s^n)$. In particular,

$$\zeta_F(2s-1, [\mathfrak{b}^{-1}]) = -\frac{2^{n-1}R_F}{w_F}\left(s - \frac{1}{2}\right)^{n-1} + O\left(s - \frac{1}{2}\right)^n, \tag{2.9}$$

so that multiplying (2.8) and (2.9) yields

$$B(s) = -\frac{N(y)^{1/2}2^{n-1}R_F}{\sqrt{d_F}w_F N_{F/\mathbb{Q}}(\mathfrak{a})}\frac{1}{s-\frac{1}{2}} + O(1).$$

We conclude that $A(s)$ and $B(s)$ have simple poles at $s = 1/2$ with residues which cancel.

Finally, assume that $\mathfrak{a} = \mathfrak{b} = \mathcal{O}_F$. By applying the functional equations

$$G(s)\zeta_F(s, C) = G(1-s)\zeta_F(1-s, C)$$

and $K_{-\nu}(z) = K_\nu(z)$ in (2.7), one can show that $E(s, z; \mathcal{O}_F, \mathcal{O}_F)$ satisfies the functional equation

$$G(1-s)E(s, z; \mathcal{O}_F, \mathcal{O}_F) = G(2(1-s))E(1-s, z; \mathcal{O}_F, \mathcal{O}_F).$$

This completes the proof of Theorem 1.1. ■

3 Taylor Expansion of $E(s, z; \mathfrak{a}, \mathfrak{b})$ at $s = 0$

We now use the Fourier expansion (2.7) to compute the first two terms in the Taylor expansion of $E(s, z; \mathfrak{a}, \mathfrak{b})$ at $s = 0$,

$$E(s, z; \mathfrak{a}, \mathfrak{b}) = E_{n-1}s^{n-1} + E_n(z)s^n + O(s^{n+1}).$$

We will compute the Taylor expansions of A, B, and C separately.

First, observe that

$$N(y)^s N_{F/\mathbb{Q}}(\mathfrak{a})^{-2s} = 1 + \log\left(N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2}\right)s + O(s^2),$$

and arguing as in Section 2 we determine that

$$\zeta_F(2s,[\mathfrak{a}^{-1}]) = -\frac{2^{n-1}R_F}{w_F}s^{n-1} + \frac{2^n\zeta_F^{(n)}(0,[\mathfrak{a}^{-1}])}{n!}s^n + O(s^{n+1}).$$

Then

$$A(s) = -2^{n-1}\frac{R_F}{w_F}s^{n-1} + \left(\frac{2^n\zeta_F^{(n)}(0,[\mathfrak{a}^{-1}])}{n!} - 2^{n-1}\frac{R_F}{w_F}\log\left(N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2}\right)\right)s^n + O(s^{n+1}).$$

Second, using the expansion

$$\left[\frac{1}{\Gamma(s)}\right]^n = s^n + O(s^{n+1}) \tag{3.1}$$

and $\Gamma(-1/2) = -2\sqrt{\pi}$, we find that

$$B(s) = \frac{(-1)^n 2^n \pi^n N(y) N_{F/\mathbb{Q}}(\mathfrak{b})}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\zeta_F(-1,[\mathfrak{b}^{-1}])s^n + O(s^{n+1}).$$

Third, using $K_{-\nu}(z) = K_\nu(z)$ and $K_{1/2}(z) = \sqrt{\pi/2z}\cdot e^{-z}$, we compute

$$\prod_{j=1}^n K_{-1/2}(2\pi|\sigma_j(ab)|y_j) = \frac{N(y)^{-1/2}}{2^n}N_{F/\mathbb{Q}}((ab))^{-1/2}e^{-2\pi S(aby)}, \tag{3.2}$$

where $S(aby) := \sum_{j=1}^n |\sigma_j(ab)|y_j$. Using (3.1) and (3.2), we find that

$$C(s) = \frac{1}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\sideset{}{'}\sum_{\tilde{a}\in\mathfrak{a}^*}\sum_{\substack{\tilde{a}=ab\\ a\in\mathfrak{a}^*\\ b\in\mathfrak{b}/\mathcal{O}_F^\times}}\frac{e^{-2\pi S(aby)}}{N_{F/\mathbb{Q}}((a))}e^{2\pi i T(abx)}s^n + O(s^{n+1}).$$

Finally, from the sum $A+B+C$, we conclude that $E_{n-1} = -2^{n-1}\frac{R_F}{w_F}$, and

$$E_n(z) = \log\left((N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2})^{E_{n-1}}\Psi(z)\right),$$

where

$$\log\Psi(z) = \frac{2^n\zeta_F^{(n)}(0,[\mathfrak{a}^{-1}])}{n!} + \frac{(-1)^n 2^n\pi^n N(y)N_{F/\mathbb{Q}}(\mathfrak{b})}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\zeta_F(-1,[\mathfrak{b}^{-1}]) + \frac{1}{\sqrt{d_F}N_{F/\mathbb{Q}}(\mathfrak{a})}\sideset{}{'}\sum_{\tilde{a}\in\mathfrak{a}^*}\sum_{\substack{\tilde{a}=ab\\ a\in\mathfrak{a}^*\\ b\in\mathfrak{b}/\mathcal{O}_F^\times}}\frac{e^{-2\pi S(aby)}}{N_{F/\mathbb{Q}}((a))}e^{2\pi i T(abx)}.$$

4 Proof of Theorem 1.2

The group $\Gamma(\mathfrak{a}, \mathfrak{b})$ embeds discretely in $GL_2(\mathbb{R})^n$,

$$M \to \left(\begin{pmatrix} \sigma_1(\alpha) & \sigma_1(\beta) \\ \sigma_1(\gamma) & \sigma_1(\delta) \end{pmatrix}, \dots, \begin{pmatrix} \sigma_n(\alpha) & \sigma_n(\beta) \\ \sigma_n(\gamma) & \sigma_n(\delta) \end{pmatrix} \right),$$

which induces a discontinuous action of $\Gamma(\mathfrak{a}, \mathfrak{b})$ on $\mathbb{H}^n$,

$$M(z) = \left(\frac{\sigma_1(\alpha)z_1 + \sigma_1(\beta)}{\sigma_1(\gamma)z_1 + \sigma_1(\delta)}, \dots, \frac{\sigma_n(\alpha)z_n + \sigma_n(\beta)}{\sigma_n(\gamma)z_n + \sigma_n(\delta)} \right).$$

From the definition of $\Gamma(\mathfrak{a}, \mathfrak{b})$ we have the invariance property

$$E(s, M(z); \mathfrak{a}, \mathfrak{b}) = E(s, z; \mathfrak{a}, \mathfrak{b})$$

for all $M \in \Gamma(\mathfrak{a}, \mathfrak{b})$. Then using the Taylor expansion

$$E(s, z; \mathfrak{a}, \mathfrak{b}) = E_{n-1}s^{n-1} + E_n(z)s^n + O(s^{n+1}),$$

we see that $E_n(M(z)) = E_n(z)$ for all $M \in \Gamma(\mathfrak{a}, \mathfrak{b})$.

Write

$$E_n(z) = \log \Psi(z) + E_{n-1} \log\Big(N(\mathrm{Im}(z)) N_{F/\mathbb{Q}}(\mathfrak{a})^{-2} \Big), \tag{4.1}$$

where we have set $N(y) = N(\mathrm{Im}(z))$. Then using (4.1) and the invariance of $E_n(z)$ under $\Gamma(\mathfrak{a}, \mathfrak{b})$ we compute

$$\begin{aligned} \log \Psi(M(z)) &= E_n(M(z)) - E_{n-1} \log\big(N(\mathrm{Im}(M(z))) N_{F/\mathbb{Q}}(\mathfrak{a})^{-2} \big) \\ &= E_n(z) - E_{n-1} \log\big(N(\mathrm{Im}(M(z))) N_{F/\mathbb{Q}}(\mathfrak{a})^{-2} \big) \\ &= \log \Psi(z) + E_{n-1} \log\big(N(\mathrm{Im}(z)) N_{F/\mathbb{Q}}(\mathfrak{a})^{-2} \big) \\ &\quad - E_{n-1} \log\big(N(\mathrm{Im}(M(z))) N_{F/\mathbb{Q}}(\mathfrak{a})^{-2} \big) \\ &= \log \Psi(z) + E_{n-1} \log\Big(\frac{N(\mathrm{Im}(z))}{N(\mathrm{Im}(M(z)))} \Big). \end{aligned}$$

Using $N(\mathrm{Im}(z)) = \prod_{j=1}^n \mathrm{Im}(z_j)$ and the identity

$$\mathrm{Im}\Big(\frac{az + b}{cz + d} \Big) = \frac{\mathrm{Im}(z)}{|cz + d|^2},$$

we compute

$$N(\mathrm{Im}(M(z))) = \prod_{j=1}^n \frac{\mathrm{Im}(z_j)}{|\sigma_j(\gamma)z_j + \sigma_j(\delta)|^2},$$

so that

$$\frac{N(\mathrm{Im}(z))}{N(\mathrm{Im}(M(z)))} = \prod_{j=1}^{n} |\sigma_j(\gamma)z_j + \sigma_j(\delta)|^{-2}.$$

This yields the transformation formula

$$\log \Psi(M(z)) = \log \Psi(z) + \log\Bigl(\prod_{j=1}^{n} |\sigma_j(\gamma)z_j + \sigma_j(\delta)|^{-2E_{n-1}}\Bigr). \tag{4.2}$$

Finally, exponentiate both sides of (4.2) and use the definition of the norm $N(\,\cdot\,)$, and $E_{n-1} = -2^{n-1}R_F/w_F$.

5 Proof of Theorem 1.3

Let σ be an embedding of F, $(a,b) \in \mathfrak{a} \times \mathfrak{b}$, and $z \in \mathbb{H}$. It can be shown by direct computation that

$$y^2\Delta(y^s|\sigma(a) + \sigma(b)z|^{-2s}) = s(s-1)y^s|\sigma(a) + \sigma(b)z|^{-2s}.$$

Using the relation

$$N(y)^s|N(a+bz)|^{-2s} = \prod_{j=1}^{n} y_j^s|\sigma_j(a) + \sigma_j(b)z_j|^{-2s},$$

it follows immediately that

$$\Delta_j E(s,z;\mathfrak{a},\mathfrak{b}) = s(s-1)E(s,z;\mathfrak{a},\mathfrak{b}). \tag{5.1}$$

Thus, $E(s,z;\mathfrak{a},\mathfrak{b})$ is an eigenfunction for the operators Δ_j with eigenvalue $s(s-1)$.

Substitute the Taylor expansion

$$E(s,z;\mathfrak{a},\mathfrak{b}) = E_{n-1}s^{n-1} + E_n(z)s^n + O(s^{n+1})$$

into the right-hand side of (5.1), expand, and equate coefficients to obtain the recurrence relation $\Delta_j E_k = E_{k-2} - E_{k-1}$, for $k = 0, 1, \ldots$, where $E_k = 0$ for $k = -2, -1, \ldots, n-2$.

From the definition of $E_n(z)$ we see that

$$\log \Psi(z) = E_n(z) - E_{n-1}\log\bigl(N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2}\bigr).$$

We claim that

$$\Delta_j\bigl(E_n(z) - E_{n-1}\log(N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2})\bigr) = 0,$$

and hence $\log \Psi(z)$ vanishes under the operators Δ_j. To see this, note that as a consequence of the recurrence relation, $\Delta_j E_n(z) = -E_{n-1}$, and a straightforward computation yields

$$\Delta_j \log\bigl(N(y)N_{F/\mathbb{Q}}(\mathfrak{a})^{-2}\bigr) = -1.$$

The claim follows from these two facts. ■

6 Proof of Proposition 1.4

Let $a, b \in \mathbb{C}^n$, $n > 1$. The set $L = \{a + bz : z \in \mathbb{C}\}$ is called a *complex line* in $\mathbb{C}^n$. Let Ω be a domain in $\mathbb{C}^n$. A C^2 function $f : \Omega \to \mathbb{C}$ is called *pluriharmonic* if for every complex line L the function $z \to f(a + bz)$ is harmonic on the set $\Omega_L = \{z \in \mathbb{C} : a + bz \in \Omega\}$.

If $z^0 \in \mathbb{C}^n$ and $r > 0$, the open polydisc centered at z^0 of radius r is defined by

$$D^n(z^0, r) = \{z \in \mathbb{C}^n : |z_j - z_j^0| < r,\ j = 1, \ldots, n.\}.$$

Pluriharmonic functions are characterized as follows, (see [Kr, p. 82]).

Proposition 6.1 *Let $D^n(z, r) \subset \mathbb{C}^n$ be a polydisc and $f : D^n(z, r) \to \mathbb{R}$ be C^2. Then f is pluriharmonic on $D^n(z, r)$ if and only if f is the real part of an analytic function on $D^n(z, r)$.*

Define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

One can establish the following useful criterion for pluriharmonicity.

Proposition 6.2 *A C^2 function f on Ω is pluriharmonic if and only if*

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} f = 0$$

for all $j, k = 1, \ldots, n$. In particular, the second condition holds if and only if f satisfies the Neumann equation $\partial\overline{\partial} f = 0$.

It can be shown that $\log \Psi(z)$ does not satisfy the second condition of Proposition 6.2, and thus $\log \Psi(z)$ is not the real part of an analytic function.

7 CM Points on Hilbert Modular Varieties

In this section we follow in part Moreno [M, Section 3.2]. Let $\mathfrak{D}_{K/F}$ be the relative different. Since K is a quadratic extension of F, $K = F(\sqrt{\alpha})$ for some $\alpha \in F$. By considering prime ideal factors, it can be shown that $(\sqrt{\alpha})\mathfrak{D}_{K/F}^{-1} = \tilde{\mathfrak{a}}\mathcal{O}_K$ for some ideal $\tilde{\mathfrak{a}}$ in F. The ideal class $[\tilde{\mathfrak{a}}]$ is independent of the choice of α.

The following lemma is due to Chevalley [C].

Lemma 7.1 *Let A be an integral ideal in K. Then the relative norm $\mathcal{N}_{K/F}(A)$ lies in the ideal class of the form $\mathfrak{a}[\tilde{\mathfrak{a}}]$, $\mathfrak{a}$ being an integral ideal in F, if and only if there exist $\omega_1 \in \mathfrak{a}^{-1}\mathcal{O}_K$ and $\omega_2 \in \mathcal{O}_K$ such that $A = \mathfrak{a}\omega_1 + \mathcal{O}_F\omega_2$.*

Choose a complete set of representatives $\{\mathfrak{a}_j\}_{j \in J}$ of ideal classes of F. Among the ideal classes $\{\mathfrak{a}_j[\tilde{\mathfrak{a}}]\}_{j \in J}$ of F, choose the sub-collection $\{\mathfrak{a}_i[\tilde{\mathfrak{a}}]\}_{i \in I}$ of ideal classes which contain the relative norm of an ideal in K. We may assume that the ideals $\{\mathfrak{a}_i\}_{i\in I}$ are integral.

Let C be an ideal class of K, and let A be an integral ideal in C. Then $\mathcal{N}_{K/F}(A) \in \mathfrak{a}_i[\tilde{\mathfrak{a}}]$ for some $i \in I$. It follows from Lemma 7.1 that there is a decomposition $A = \mathfrak{a}_i\omega_1 + \mathcal{O}_F\omega_2$, where $(\omega_1, \omega_2) \in \mathfrak{a}_i^{-1}\mathcal{O}_K \times \mathcal{O}_K$. Up to multiplication by a unit in F, we may assume that the imaginary parts of the components of $z_C := \omega_2/\omega_1$ under a given choice of n real embeddings of F are positive.

Because K is an imaginary quadratic extension of F, the $2n$ embeddings of K occur in complex conjugate pairs. Let $\Phi = \{\tau_1, \dots, \tau_n\}$ be a CM type for K/F, which is a choice of one embedding for each complex conjugate pair. Define the CM point

$$\Phi(z_C) := (\tau_1(z_C), \dots, \tau_n(z_C)) \in \mathbb{H}^n.$$

Let $\Gamma_0(\mathfrak{a}_i) := \Gamma(\mathfrak{a}_i, \mathcal{O}_F)$, and form the Hilbert modular variety

$$X_0(\mathfrak{a}_i) := \mathbb{H}^n/\Gamma_0(\mathfrak{a}_i).$$

There is an embedding

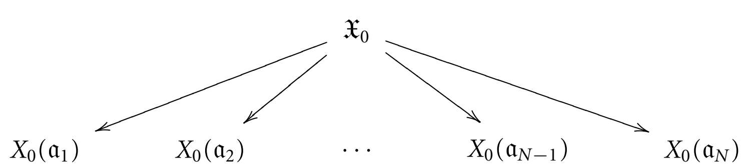
$$\left\{C \in \mathrm{Cl}_K \colon \mathcal{N}_{K/F}(C) = \mathfrak{a}_i[\tilde{\mathfrak{a}}]\right\} \hookrightarrow X_0(\mathfrak{a}_i)$$

defined by $C \mapsto \Phi(z_C) \bmod \Gamma_0(\mathfrak{a}_i)$.

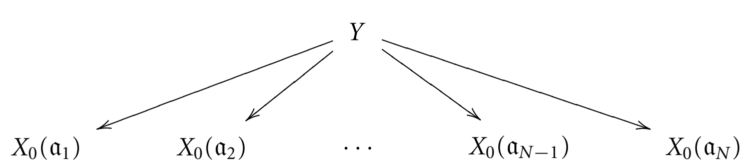
Proposition 7.2 *The map* $C \mapsto \Phi(z_C) \bmod \Gamma_0(\mathfrak{a}_i)$ *is well defined.*

Proof Suppose B is an integral ideal in K which is equivalent to A. Then by Lemma 7.1, $B = \mathfrak{a}_i\omega_1' + \mathcal{O}_F\omega_2'$, where $(\omega_1', \omega_2') \in \mathfrak{a}_i^{-1}\mathcal{O}_K \times \mathcal{O}_K$. Again, we may assume that the imaginary parts of the components of $z_C' = \omega_2'/\omega_1'$ are positive. It can be shown that there exists a matrix $M \in \Gamma_0(\mathfrak{a}_i)$ such that $M(\Phi(z_C)) = \Phi(z_C')$ (see [M, Lemma 1]). Thus, the map is well defined. ■

Let $\mathcal{A}$ be the ideal defined by taking the least common multiple of the ideals $\{\mathfrak{a}_i\}_{i\in I}$. Form the discrete subgroup $\Gamma_0(\mathcal{A}) < GL_2(\mathbb{R})^n$, and the corresponding Hilbert modular variety, $X_0(\mathcal{A}) := \mathbb{H}^n/\Gamma_0(\mathcal{A})$. Then $\mathfrak{X}_0 := X_0(\mathcal{A})$ is the minimal common cover of the varieties $X_0(\mathfrak{a}_i)$. That is, if $|I| = N$, there is the following covering diagram.



If Y is a cover which fits into the diagram



there is a cover $Y \longrightarrow \mathfrak{X}_0$. Finally, using the covers $\mathfrak{X}_0 \longrightarrow X_0(\mathfrak{a}_i)$, one obtains an embedding $\mathrm{Cl}_K \hookrightarrow \mathfrak{X}_0$ defined by $C \mapsto \Phi(z_C) \bmod \Gamma_0(\mathcal{A})$.

8 CM Values of Hilbert Modular Eisenstein Series

Let $C \in \mathrm{Cl}_K$ and fix an integral ideal $A_C \in C^{-1}$. Then as A runs over all integral ideals in C, the ideal $A \cdot A_C = (\alpha)$ runs over all principal ideals (α) with $(\alpha) \equiv 0 \bmod A_C$. It follows that

$$\zeta_K(s, C) = \sum_{A \in C}{}' N_{K/\mathbb{Q}}(A)^{-s} = \sum_{(\alpha) \subset A_C}{}' N_{K/\mathbb{Q}}(A_C^{-1}(\alpha))^{-s} = N_{K/\mathbb{Q}}(A_C)^s \sum_{\alpha \in A_C/\mathcal{O}_K^\times}{}' N_{K/\mathbb{Q}}((\alpha))^{-s}. \tag{8.1}$$

From (8.1) and a counting argument, we obtain

$$|\mathcal{O}_K^\times : \mathcal{O}_F^\times| \zeta_K(s, C) = N_{K/\mathbb{Q}}(A_C)^s |\mathcal{O}_K^\times : \mathcal{O}_F^\times| \sum_{\alpha \in A_C/\mathcal{O}_K^\times}{}' N_{K/\mathbb{Q}}((\alpha))^{-s} = N_{K/\mathbb{Q}}(A_C)^s \sum_{\alpha \in A_C/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((\alpha))^{-s}. \tag{8.2}$$

From Section 7 there is a decomposition $A_C = \mathfrak{a}_C\omega_1 + \mathcal{O}_F\omega_2$, where $(\omega_1, \omega_2) \in \mathfrak{a}_C^{-1}\mathcal{O}_K \times \mathcal{O}_K$ and $\mathfrak{a}_C \in \{\mathfrak{a}_i\}$. Therefore, we can express (8.2) as

$$|\mathcal{O}_K^\times : \mathcal{O}_F^\times| \zeta_K(s, C) = N_{K/\mathbb{Q}}(A_C)^s \sum_{(a,b) \in \mathfrak{a}_C \times \mathcal{O}_F/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((a\omega_1 + b\omega_2))^{-s} = N_{K/\mathbb{Q}}(A_C/\omega_1)^s \sum_{(a,b) \in \mathfrak{a}_C \times \mathcal{O}_F/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((a + bz_C))^{-s}. \tag{8.3}$$

We will need the following lemma.

Lemma 8.1 *Let Φ be a CM type for K/F. Then*

$$N_{K/\mathbb{Q}}((a + bz_C)) = |N(a + b\Phi(z_C))|^2.$$

Proof For an embedding τ of K let $\tilde{\tau}$ be its restriction to F. Then

$$N_{K/\mathbb{Q}}((a + bz_C)) = \prod_{j=1}^n \tau_j(a + bz_C)\overline{\tau}_j(a + bz_C) = \prod_{j=1}^n (\tilde{\tau}_j(a) + \tilde{\tau}_j(b)\tau_j(z_C))(\tilde{\tau}_j(a) + \tilde{\tau}_j(b)\overline{\tau}_j(z_C)) =$$

$$= \prod_{j=1}^{n} (\tilde{\tau}_j(a) + \tilde{\tau}_j(b)\tau_j(z_C))(\overline{\tilde{\tau}_j(a) + \tilde{\tau}_j(b)\tau_j(z_C)})$$

$$= \prod_{j=1}^{n} |\tilde{\tau}_j(a) + \tilde{\tau}_j(b)\tau_j(z_C)|^2$$

$$= |N(a + b\Phi(z_C))|^2. \qquad \blacksquare$$

Using Lemma 8.1 we can express (8.3) as

$$|\mathcal{O}_K^\times : \mathcal{O}_F^\times|\zeta_K(s,C) = N_{K/\mathbb{Q}}(A_C/\omega_1)^s \sideset{}{'}\sum_{(a,b)\in\mathfrak{a}_C\times\mathcal{O}_F/\mathcal{O}_F^\times} |N(a+b\Phi(z_C))|^{-2s}. \tag{8.4}$$

We will also need the following lemma, which follows from a calculation with determinants, (see also [M, p. 237]).

Lemma 8.2 *Using the same notation as above,*

$$N_{K/\mathbb{Q}}(A_C/\omega_1) = N(y(\Phi(z_C)))2^n N_{F/\mathbb{Q}}(\mathfrak{a}_C)d_F/\sqrt{d_K}.$$

Using Lemma 8.2 we can express (8.4) as

$$\begin{aligned}
&|\mathcal{O}_K^\times : \mathcal{O}_F^\times|\zeta_K(s,C)\\
&\quad = (2^n N_{F/\mathbb{Q}}(\mathfrak{a}_C)d_F/\sqrt{d_K})^s \sideset{}{'}\sum_{(a,b)\in\mathfrak{a}_C\times\mathcal{O}_F/\mathcal{O}_F^\times} N(y(\Phi(z_C)))^s|N(a+b\Phi(z_C))|^{-2s}\\
&\quad = (2^n N_{F/\mathbb{Q}}(\mathfrak{a}_C)d_F/\sqrt{d_K})^s E\big(s,\Phi(z_C);\mathfrak{a}_C,\mathcal{O}_F\big)\,.
\end{aligned}$$

We now obtain the identity

$$\zeta_K(s,C) = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}(2^n N_{F/\mathbb{Q}}(\mathfrak{a}_C)d_F/\sqrt{d_K})^s E(s,\Phi(z_C);\mathfrak{a}_C,\mathcal{O}_F). \tag{8.5}$$

Remark. It is important to observe that the above construction depends only on the inverse class C^{-1}. This can be seen by combining the argument used in section 7 to show that the map $C \mapsto \Phi(z_C) \bmod \Gamma_0(\mathfrak{a}_C)$ is well-defined with the invariance of the Eisenstein series $E(s,z;\mathfrak{a}_C,\mathcal{O}_F)$ with respect to the group $\Gamma_0(\mathfrak{a}_C)$.

9 Proof of Theorem 1.5

From (8.5) we obtain the identity

$$\zeta_K(s,C) = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}(2^n d_F/\sqrt{d_K})^s N_{F/\mathbb{Q}}(\mathfrak{a}_C)^s E(s,\Phi(z_C);\mathfrak{a}_C,\mathcal{O}_F).$$

It follows from (1.3) that
(9.1)

$$L_{S_\infty}(s,\chi) = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}(2^n d_F/\sqrt{d_K})^s \sum_{C\in \mathrm{Cl}_K} \chi(C) N_{F/\mathbb{Q}}(\mathfrak{a}_C)^s E(s, \Phi(z_C); \mathfrak{a}_C, \mathcal{O}_F).$$

Suppose that $\chi = \mathbf{1}_G$. Substitute the Taylor expansions

$$\left(2^n d_F/\sqrt{d_K}\right)^s = 1 + \log\left(2^n d_F/\sqrt{d_K}\right) s + O(s^2),$$

$$N_{F/\mathbb{Q}}(\mathfrak{a}_C)^s = 1 + \log\left(N_{F/\mathbb{Q}}(\mathfrak{a}_C)\right) s + O(s^2),$$

$$E(s, \Phi(z_C); \mathfrak{a}_C, \mathcal{O}_F) = E_{n-1}s^{n-1} + E_n(\Phi(z_C))s^n + O(s^{n+1})$$

into the RHS of (9.1) and differentiate both sides n times with respect to s to obtain

$$(9.2)\quad \frac{L_{S_\infty}^{(n)}(s,\mathbf{1}_G)}{n!} = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}\sum_{C\in \mathrm{Cl}_K}\Big(E_{n-1}\log\left(2^n d_F/\sqrt{d_K}\right) + E_{n-1}\log(N_{F/\mathbb{Q}}(\mathfrak{a}_C)) + E_n(\Phi(z_C))\Big) + O(s).$$

Using the definition of $E_n(\Phi(z_C))$, we compute

$$\begin{aligned}(9.3)\quad & E_{n-1}\log(N_{F/\mathbb{Q}}(\mathfrak{a}_C)) + E_n(\Phi(z_C)) \\ &= \log(N_{F/\mathbb{Q}}(\mathfrak{a}_C)^{E_{n-1}}) + \log((N(y(\Phi(z_C)))N_{F/\mathbb{Q}}(\mathfrak{a}_C)^{-2})^{E_{n-1}}\Psi(\Phi(z_C))) \\ &= \log\left((N(y(\Phi(z_C)))N_{F/\mathbb{Q}}(\mathfrak{a}_C)^{-1})^{E_{n-1}}\Psi(\Phi(z_C))\right).\end{aligned}$$

Substitute (9.3) in (9.2) and let $s \to 0$ to obtain the formula for $L_{S_\infty}^{(n)}(0,\mathbf{1}_G)$.

Suppose that $\chi \neq \mathbf{1}_G$. By the orthogonality relations for group characters, $\sum_{C\in\mathrm{Cl}_K}\chi(C) = 0$. Proceeding as above, we find that

$$(9.4)\quad \frac{L_{S_\infty}^{(n)}(s,\chi)}{n!} = \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}\sum_{C\in \mathrm{Cl}_K}\chi(C)\left\{E_{n-1}\log\left(N_{F/\mathbb{Q}}(\mathfrak{a}_C)\right) + E_n(\Phi(z_C))\right\} + O(s).$$

Again, substitute (9.3) into (9.4) and let $s \to 0$ to obtain the formula for $L_{S_\infty}^{(n)}(0,\chi)$. ■

10 Proof of Theorem 1.6

From class field theory there is the decomposition,

$$(10.1)\qquad \zeta_H(s) = \zeta_K(s)\prod_{\substack{\chi\in\widehat{G}\\ \chi\neq\mathbf{1}_G}} L_{S_\infty}(s,\chi).$$

(see [N, p. 524]). Using the functional equation for the Dedekind zeta function as in section 2 we compute

$$\lim_{s\to 0} s^{-(n-1)}\zeta_H(s) = -\frac{h_H R_H}{w_H} \quad \text{and} \quad \lim_{s\to 0} s^{-(n-1)}\zeta_K(s) = -\frac{h_K R_K}{w_K}.$$

Then equating leading terms of the Taylor expansions at $s = 0$ of both sides of (10.1) yields

$$-\frac{h_H R_H}{w_H} = -\frac{h_K R_K}{w_K} \prod_{\substack{\chi\in\widehat{G} \\ \chi\neq \mathbf{1}_G}} \frac{L^{(n)}_{S_\infty}(0,\chi)}{n!}. \tag{10.2}$$

Using Theorem 1.5 we can express (10.2) as

$$-\frac{h_H R_H}{w_H} = -\frac{h_K R_K}{w_K} \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^{h_K-1}} \prod_{\substack{\chi\in\widehat{G} \\ \chi\neq \mathbf{1}_G}} \sum_{C\in \mathrm{Cl}_K} \chi(C)\log \epsilon(C^{-1}),$$

or equivalently as,

$$\frac{h_H}{h_K} = \frac{w_H}{w_K}\frac{R_K}{R_H} \frac{1}{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|^{h_K-1}} \prod_{\substack{\chi\in\widehat{G} \\ \chi\neq \mathbf{1}_G}} \sum_{C\in \mathrm{Cl}_K} \chi(C)\log \epsilon(C^{-1}). \tag{10.3}$$

Let G be a finite abelian group and $\chi \in \widehat{G}$. Let f be a complex-valued function on G. The following formula is a consequence of the Frobenius determinant relation (see [L2, p. 283]):

$$\prod_{\substack{\chi\in\widehat{G} \\ \chi\neq \mathbf{1}_G}} \sum_{a\in G} \chi(a) f(a^{-1}) = \det_{a,b\neq 1}\left\{ f(ab^{-1}) - f(a)\right\}. \tag{10.4}$$

Define $f\colon \mathrm{Cl}_K \to \mathbb{R}$ by

$$f(C) = \log \epsilon(C), \quad C \in \mathrm{Cl}_K\,.$$

Using (10.4) we obtain the relation

$$\prod_{\substack{\chi\in\widehat{G} \\ \chi\neq \mathbf{1}_G}} \sum_{C\in \mathrm{Cl}_K} \chi(C)\log \epsilon(C^{-1}) = \det_{C,C'\neq 1} \log\left\{\frac{\epsilon(C(C')^{-1})}{\epsilon(C)}\right\}. \tag{10.5}$$

Substitute (10.5) into (10.3) to complete the proof. ■

11 Proof of Theorem 1.9

Define $f\colon \mathrm{Cl}_K \to \mathbb{R}$ by $f(C) = \log \epsilon(C^{-1})$, and $g\colon \widehat{G} \to \mathbb{C}$ by

$$g(\chi) = \sum_{C \in \mathrm{Cl}_K} \chi(C) f(C).$$

Then

$$\begin{aligned}\sum_{\chi \in \widehat{G}} \overline{\chi}(C') g(\chi) &= \sum_{\chi \in \widehat{G}} \overline{\chi}(C') \sum_{C \in \mathrm{Cl}_K} \chi(C) f(C) \\ &= \sum_{C \in \mathrm{Cl}_K} f(C) \sum_{\chi \in \widehat{G}} \overline{\chi}(C') \chi(C).\end{aligned}$$

By the orthogonality relations for group characters,

$$\sum_{\chi \in \widehat{G}} \overline{\chi}(C') \chi(C) = \begin{cases} h_K & \text{if } C = C', \\ 0 & \text{if } C \neq C'. \end{cases}$$

Thus, $\sum_{\chi \in \widehat{G}} \overline{\chi}(C') g(\chi) = h_K f(C')$, or equivalently,

$$f(C) = \frac{1}{h_K} \{ g\Big(\mathbf{1}_G\Big) + \sum_{\substack{\chi \in \widehat{G} \\ \chi \neq \mathbf{1}_G}} \overline{\chi}(C) g(\chi) \}. \tag{11.1}$$

From (11.1) and Theorem 1.5,

$$\log \epsilon(C^{-1}) - \frac{1}{h_K} \sum_{C \in \mathrm{Cl}_K} \log \epsilon(C^{-1}) = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{h_K n!} \sum_{\substack{\chi \in \widehat{G} \\ \chi \neq \mathbf{1}_G}} \chi(C) L_{S_\infty}^{(n)}(0, \chi). \tag{11.2}$$

Furthermore, by Theorem 1.5,

$$\frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{h_K n!} L_{S_\infty}^{(n)}(0, \mathbf{1}_G) = \log\Big(\Big(2^n d_F / \sqrt{d_K}\Big)^{E_{n-1}} \Big) + \frac{1}{h_K} \sum_{C \in \mathrm{Cl}_K} \log \epsilon(C^{-1}). \tag{11.3}$$

Add (11.3) to both sides of (11.2) to obtain

$$\log\Big(\Big(2^n d_F / \sqrt{d_K}\Big)^{E_{n-1}} \epsilon(C^{-1}) \Big) = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{h_K n!} \sum_{\chi \in \widehat{G}} \chi(C) L_{S_\infty}^{(n)}(0, \chi). \tag{11.4}$$

Finally, as a consequence of (11.4) and the isomorphism $G \cong \mathrm{Cl}_K$, we obtain the following equality in $\mathbb{C}[G]$,

$$\begin{aligned}
&\sum_{\sigma\in G} \log\left((2^n d_F/\sqrt{d_K})^{E_{n-1}} \epsilon(\sigma^{-1})\right) \cdot \sigma^{-1} \\
&\quad = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{h_K n!} \sum_{\sigma\in G}\sum_{\chi\in\widehat{G}} \chi(\sigma) L_{S_\infty}^{(n)}(0,\chi)\cdot\sigma^{-1} \\
&\quad = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{n!} \sum_{\chi\in\widehat{G}} L_{S_\infty}^{(n)}(0,\chi)\Big(\frac{1}{h_K}\sum_{\sigma\in G}\chi(\sigma)\cdot\sigma^{-1}\Big) \\
&\quad = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{n!} \sum_{\chi\in\widehat{G}} L_{S_\infty}^{(n)}(0,\chi)\cdot e_{\chi^{-1}} \\
&\quad = \frac{|\mathcal{O}_K^\times : \mathcal{O}_F^\times|}{n!} \Theta_{S_\infty}^{(n)}(0).
\end{aligned}$$

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