Equicontinuous Delone Dynamical Systems

Dedicated to Robert V. Moody on the occasion of his 70th birthday

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Abstract. We characterize equicontinuous Delone dynamical systems as those coming from Delone sets with strongly almost periodic Dirac combs. Within the class of systems with finite local complexity, the only equicontinuous systems are then shown to be the crystallographic ones. On the other hand, within the class without finite local complexity, we exhibit examples of equicontinuous minimal Delone dynamical systems that are not crystallographic. Our results solve the problem posed by Lagarias as to whether a Delone set whose Dirac comb is strongly almost periodic must be crystallographic.

Introduction

The study of disordered systems is one of the most prominent issues in mathematics and physics today. A special focus concerns aperiodic order *i.e.,* a special class of highly ordered structures (see *e.g.,* the monographs and conference proceedings [6, 20, 34, 41, 45]). This topic is particularly relevant for two reasons: On the one hand, the actual discovery of physical substances [19, 46], later called quasicrystals, exhibiting this form of order has triggered enormous research activities in the experimental and theoretical description of low complexity systems. On the other hand, aperiodic order has come up in disguise in various contexts in purely conceptually motivated studies.

As for such conceptual studies of almost periodic order (in disguise), we mention the work of Hedlund and Morse on Sturmian systems and on complexity of non periodic sequences [38, 39]. Geometric analogues have recently been studied by Lagarias [24,25] and Lagarias and Pleasants [27,28]. Very loosely speaking, the corresponding results show that in terms of suitable complexity notions there is a gap between the ordered world and the disordered world. Another approach to (dis)order in a spirit of Fourier analysis has been given in Meyer's work on harmonious sets [33]. There, a basic aim is to find and study a class of sets allowing for a Fourier type expansion. Meyer's considerations have been taken up by Moody [35,36]. The corresponding results have become a cornerstone in the study of diffraction aspects of aperiodic order initiated by Hof [18].

Of course, there are connections between the geometry and complexity based approach to aperiodic order and the Fourier analytic viewpoint. For example, pure

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pointedness on the Fourier side implies zero entropy [5]. A crucial link is provided by concepts of almost periodicity. On the one side almost periodicity is by its very definition a geometric concept about how a system "repeats itself". On the other side, it is an important ingredient in all Fourier analytic considerations concerning pure point measures. Almost periodicity has always played a role in the theory, see *e.g.,* Solomyak [48] and Queffelec [43]. Still, it seems that only recently with the work of Baake and Moody [7] and subsequent work, *e.g.,* [16, 17, 30, 32, 37, 49], that the role of almost periodicity in diffraction theory has begun to be properly ascertained.

A most basic question in this context is whether it is possible to characterize order by suitably strong forms of almost periodicity. An affirmative answer to one version of this question is given in the so-called Córdoba Theorem [13] (see [26] for a generalization as well). Another version of this question was posed by Lagarias in [26]. More precisely (with notions to be explained later), [26, Problem 4.4] asks whether a Delone set whose associated Dirac comb is strongly almost periodic is necessarily completely periodic.

The main aim of this paper is to answer this question. More specifically, we will show the following two points:

- For Delone sets with finite local complexity, complete periodicity is indeed equivalent to strong almost periodicity of the associated Dirac comb (Corollary 5.6).
- There exist Delone sets (without finite local complexity) that are strongly almost periodic but not completely periodic (Corollary 6.1).

So, in some sense, we show that the answer to the problem is both yes and no.

In order to avoid confusion let us mention that our notation differs from the notation of [26] in the following way. In line with [15] and recent work on almost periodicity such as [7, 30, 32, 37] we use the term strongly almost periodic measure for measures called uniformly almost periodic in [26]. We use the term crystallographic or completely periodic for what is called ideal crystal in [26].

The result given in the second point above is treated in Section 6 by providing a class of models based on the Kronecker flow on the two dimensional torus (Corollary 6.1).

As for the result given in the first point, it is quite clear that complete periodicity implies both finite local complexity and strong almost periodicity of the associated Dirac comb. Thus, the main work is to show the converse. Under the stronger assumption of Meyer property (instead of only finite local complexity) the result is already known. It can be found in the recent work [50] and could also rather directly be inferred from the earlier [4] (see [30] as well for results in a very similar spirit).

As mentioned already the starting point for our investigation is the question whether order can be characterized by suitable forms of almost periodicity. To shed further light on this question it might be worthwhile to compare our results with corresponding results for an even stronger notion of almost periodicity viz norm almost periodicity (see, *e.g.,* [7] for a discussion of this notion in the context of aperiodic order). Within the class of unweighted Dirac combs, the completely periodic ones are actually the only ones that are norm almost periodic (as can easily be seen). In this sense norm almost periodicity cannot be used to characterize non-trivial forms of order. Our main result essentially shows that strong almost periodicity together

with finite local complexity can also not be used for this purpose.

However, in this context our results can still be understood to illuminate the difference between norm almost periodicity and strong almost periodicity. First, they say that the strongly almost periodic Dirac combs with finite local complexity are exactly the norm almost periodic Dirac combs and that secondly there exist strongly almost periodic Dirac combs, which are not norm almost periodic. This should be compared with results of [7] showing that within the class of weighted Dirac combs coming from Meyer sets, strong almost periodicity actually implies norm almost periodicity without forcing complete periodicity.

Our approach to the first point relies on three ingredients. The first ingredient is the theory of equicontinuous systems discussed in Section 2. We say that a Delone set is equicontinuous if its associated dynamical system is equicontinuous. Equicontinuity of a dynamical system is equivalent to the system being a rotation on a compact abelian group (Ellis theorem, see Theorem 2.4). The second ingredient is the results of [30] discussed in Section 3 giving that a measure dynamical system is a rotation on a compact abelian group if and only if the measures are strongly almost periodic (Theorem 3.3). Combining these two pieces for Delone dynamical systems, we find that strong almost periodicity of the associate Dirac comb is equivalent to equicontinuity of the system (Theorem 4.2). Now, the third ingredient is a variant of a recent reasoning of Barge and Olimb [10] presented in Section 5. It gives that equicontinuity of a Delone dynamical system with finite local complexity is in fact equivalent to complete periodicity (Theorem 5.5). Put together, these considerations easily give the desired result.

We present both the theory of equicontinuous systems as well as the considerations on strongly almost periodic Delone dynamical systems in somewhat more detail than needed for the actual answer to the question of Lagarias. The reason is that we believe that these considerations may be of importance for future study as well.

Note added in proof. After this work was submitted we learned about recent investigations of S. Favorov entitled "Bohr and Besicovitch almost periodic discrete sets and quasicrystals", to appear in Proc. Amer. Math. Soc. (see arxiv:1011.4036 as well) giving similar results as ours based in the theory of almost periodic functions.

1 Preliminaries on Dynamical Systems

We deal with compact dynamical systems whose acting group is locally compact abelian. An important special case are dynamical systems consisting of measures on the group. The necessary notation is introduced in this section.

The space of continuous functions on a topological space X is denoted by $C(X)$; the subspace of continuous functions with compact support is denoted by $C_c(X)$, and the space of continuous bounded functions is denoted by $C_b(X)$. We will deal with spaces X that are locally compact σ-compact Hausdorff spaces. Then X carries the Borel σ-algebra generated by all closed subsets of X and by the Riesz–Markov representation theorem (see *e.g.,* [42]); the set $\mathcal{M}(X)$ of all complex regular Borel measures on X can be identified with the dual space $C_c(X)^*$ of complex valued, linear functionals on $C_c(X)$ that are continuous with respect to a suitable topology (see,

e.g., [42, Ch. 6.5] for details). We then write

$$\int_X f \mathrm{d}\mu = \mu(f)$$

for $f \in C_c(X)$. The space $\mathcal{M}(X)$ carries the *vague topology*, which is the weakest topology that makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(X)$, continuous.

The group operation of an locally compact abelian group will mostly be written additively as +. The neutral element will be denoted by *e*. Now, let G be a σ-compact locally compact abelian group. Whenever G acts on the compact space X by a continuous action

$$\alpha \colon G \times X \longrightarrow X, \quad (t, \omega) \longmapsto \alpha_t \omega,$$

where $G \times X$ carries the product topology, the pair (X, G) is called a *topological dynamical system* over G. We will mostly suppress the action α in our notation and write $t \cdot \omega := \alpha_t \omega$. Note that we use the notation with the dot to denote the action of the group, while the addition on (abelian) groups is written additively. A dynamical system (X, G) is called *minimal* if, for all $\omega \in X$, the G-orbit $\{t \cdot \omega : t \in G\}$ is dense in X.

Definition 1.1 (Factor)   Let two topological dynamical systems (X, G) and (Y, G) under the action of G be given. Then (Y, G) is called a *factor* of (X, G), with factor map ϱ, if $\varrho \colon X \to Y$ is a continuous surjection with $\varrho(t \cdot \omega) = t \cdot (\varrho(\omega))$ for all $\omega \in X$ and $t \in G$.

We will be concerned with special dynamical systems in which X is a compact group. In order to simplify the notation, we introduce the following notation for these systems.

Definition 1.2 (Rotation on a compact abelian group)   A dynamical system (X, G) is called a *rotation* on a compact abelian group if X is a compact abelian group and the action of G on X is induced by a homomorphism $\iota \colon G \to X$ such that $t \cdot x = \iota(t) + x$ for all $t \in G$ and $x \in X$.

2   Equicontinuous Systems and Proximality

In this section we recall some aspects of the theory of equicontinuous systems and proximality. The importance of the proximality relation for the dynamical systems defined by tilings or Delone sets seems only to have recently been emphasized. In [8] it was shown that topological closure of the proximality relation is a necessary and sufficient condition for (strong) Pisot substitution tilings to have pure point dynamical spectrum. This was generalized in [9]. Related to proximality are the notions of asymptotic tilings and augmentation that have led to finer topological invariants for substitution tilings [11]. Whereas the above mentioned work concerns the case in which the proximality relation is non-trivial, we consider the opposite case here, namely when this relation is trivial.

We refer the reader to Auslander's book [1] for more detailed information. The theory is cast within the framework of uniform structures instead of more conventional (but more restrictive) metric topologies.

A uniform structure $\mathcal{U}$ (see [23] for details) is a family of subsets of $X \times X$ satisfying:

(1) the diagonal $\Delta = \{(x, x) : x \in X\}$ is contained in all $U \subset \mathcal{U}$;
(2) with U any superset $V \supset U$ is contained in $\mathcal{U}$;
(3) with U and V also $U \cap V$ belong to $\mathcal{U}$;
(4) with U also $\{(y, x) : (x, y) \in U\}$ belongs to $\mathcal{U}$;
(5) for all $U \in \mathcal{U}$ exists $V \in \mathcal{U}$ such that, if $(x, y), (y, z) \in V$, then $(x, z) \in U$.

A uniform structure $\mathcal{U}$ generates a unique toplogy such that for each $x \in X$ the sets $V_x := \{y : (x, y) \in V\}$ (for $V \in \mathcal{U}$) are a neighborhood basis for x. A base of a uniformity structure $\mathcal{U}$ is any subfamily $\mathcal{U}' \subset \mathcal{U}$ such that all members of $\mathcal{U}$ are supersets of some menber of $\mathcal{U}'$.

Uniform structures have already been used in tiling theory. In fact, Schlottmann develops a basic theory of points sets (with finite local complexity) on locally compact abelian groups in the framework of uniform structures in [44] (see recent work of Müller and Richard [40] for related material). For arbitrary point sets a discussion can be found in [2] (see [4] as well). Note also that the construction of a cut and project scheme out of a suitable autocorrelation measure in [7] is based on uniform structures.

The reader who is not familiar with uniform structures may at first consider the case of a metric space whose uniformity structure has a base given by the ϵ-diagonals $\Delta_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. A map $f \colon X \to X$ is uniformly continuous (with respect to a given uniformity structure $\mathcal{U}$ of X) if for all $V \in \mathcal{U}$, we find $U \in \mathcal{U}$ such that $(f \times f)(U) \subset V$. A family of maps $\mathcal{F}$ from X to itself is called equicontinuous if for all $V \in \mathcal{U}$, we find $U \in \mathcal{U}$ such that for all $f \in \mathcal{F} \colon (f \times f)(U) \subset V$. These notions reduce to the conventional ones in a metric space provided one uses the fundamental system given by the ϵ-diagonals. On a compact space all uniformity structures coincide.

Consider a minimal dynamical system (X, G) where X is a compact Hausdorff space and G is a locally compact abelian group acting by α on X. If the action is free, then X can be seen as a compactification of G: it is the completion of one orbit and this orbit is a copy of G. One might ask when is X a group compactification; that is, when does X carry a group structure such that the orbit is a subgroup isomorphic to G, or, in other words, when is (X, G) a rotation on a compact abelian group? The question of when this happens is to the equicontinuity of the action.

Definition 2.1 The dynamical system (X, G, α) is called *equicontinuous* if the family of homeomorphisms $\{\alpha_t\}_{t \in G}$ is equicontinuous.

This reduces to the usual definition if (X, d) is a metric space:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in X \colon d(x, y) < \delta \implies \forall t \in G \colon d\big(\alpha_t(x), \alpha_t(y)\big) < \epsilon.$$

An equicontinuous metrizable system always admits an invariant metric that induces

the same topology. Indeed one can just take

$$\overline{d}(x, y) := \sup_{t \in G} d(t \cdot x, t \cdot y).$$

Likewise, any compact metrizable abelian group $\mathbb{T}$ admits a left invariant metric. Whenever d is a metric, $\overline{d}(x, y) := \sup_{t \in \mathbb{T}} d(x + t, y + t)$ is a metric on $\mathbb{T}$ that induces the same topology and is G-invariant.

We next provide a useful necessary condition for a system to be equicontinuous.

Definition 2.2 (Proximality)   Consider a G action on a compact Hausdorff space X. Two points $x, y \in X$ are proximal if for all $U \in \mathcal{U}$, there exists a $t \in G$ such that $(t \cdot x, t \cdot y)$ belongs to U.

In the context of compact metric spaces (X, d) this translates into saying that x, y are proximal whenever

$$\inf_{t \in G} d(t \cdot x, t \cdot y) = 0.$$

Corollary 2.3   *If (X, G) is equicontinuous, then the proximal relation is trivial.*

Moreover, a system with non-trivial proximality relation cannot carry an invariant metric that is compatible with the topology.

Remark   Within the class of model sets the phenomenon of proximality is also studied under the name of singular elements of the hull. That this is actually the same seems to be folklore. It can be infered together with much stronger statements from [9].

An equicontinuous dynamical system need not be minimal, but a transitive equicontinuous dynamical system is always minimal. So, in the context of Delone and tiling dynamical systems, equicontinuous systems are always minimal. The following theorem due to Ellis is of great importance for our results. Recall that we suppose that (X, G) is minimal.

Theorem 2.4   *(X, G) is conjugate to a minimal rotation on a compact abelian group if and only if it is equicontinuous.*

If (X, G) is equicontinuous, the group structure on X arises as follows: given any point $x_0 \in X$ the operation $t_1 \cdot x_0 + t_2 \cdot x_0 := (t_1 + t_2) \cdot x_0$ extends to an addition in X so that X becomes a group with x_0 as neutral element. Conversely any rotation on a compact abelian group is obviously equicontinuous.

To present the following characterization of equicontinuity in the context of metrizable dynamical systems we need some further notation. Let (X, G) be a dynamical system and d a metric on X. Then the ϵ-ball around $x \in X$ is denoted by $B_\epsilon(x)$. The elements of $\mathcal{R}(x, \epsilon) := \{t \in G : t \cdot x \in B_\epsilon(x)\}$ are called *return vectors* to $B_\epsilon(x)$. Now, (X, G) is called *uniformly almost periodic* if, for any $\epsilon > 0$, the joint set of return vectors to ϵ-balls, given by

$$A = \bigcap_{x \in X} \mathcal{R}(x, \epsilon),$$

is relatively dense (*i.e.,* there exists a compact K with $A + K = G$).

Theorem 2.5 ([1]) *(X, G) is equicontinuous if an only if it is uniformly almost periodic.*

3 Almost Periodic Measures on Locally Compact Abelian Groups

We will be concerned with dynamical systems built from measures. We will show that equicontinuity of such a system is equivalent to almost periodicity of the underlying measure (Theorem 3.3). This provides a characterization of equicontinuity in this framework. The considerations of this section can be understood as a (slight) reformulation of results obtained in [30]. Alternatively, they could, at least in part, be based on Theorem 2.5. A thorough study of measure dynamical systems in the framework of aperiodic order and diffraction can be found in [2, 3, 29] to which we refer the reader for further details.

Let G be a locally compact σ-compact abelian group. A measure $\nu \in \mathcal{M}(G)$ is *translation bounded* if for some (and then all) open non empty relatively compact set V in G there exists a $C = C_V \geq 0$ with $|\nu|(t + V) \leq C$ for every $t \in G$. Here, $|\nu|$ is the total variation measure of ν. The set of all translation bounded measures is denoted by $\mathcal{M}^\infty(G)$. As a subset of $\mathcal{M}(G)$, it carries the vague topology. There is an obvious action of G on $\mathcal{M}^\infty(G)$, given by

$$G \times \mathcal{M}^\infty(G) \longrightarrow \mathcal{M}^\infty(G), \quad (t, \nu) \mapsto \alpha_t \nu \quad \text{with} \quad (\alpha_t \nu)(\varphi) := \nu(\delta_{-t} * \varphi)$$

for $\varphi \in C_c(G)$. Here, δ_t denotes the unit point measure at $t \in G$, and the convolution $\omega * \varphi$ between $\varphi \in C_c(G)$ and $\omega \in \mathcal{M}^\infty(G)$ is defined by

$$\omega * \varphi(s) := \int \varphi(s - u) d\omega(u).$$

It is not hard to see that this action is continuous when restricted to a compact subset of $\mathcal{M}^\infty(G)$ (see, *e.g.,* [2]).

Definition 3.1 (X, G) is called a dynamical system on the translation bounded measures on G (TMDS) if X is a compact α-invariant subset of $\mathcal{M}^\infty(G)$.

Every translation bounded measure ν gives rise to a (TMDS) $(X(\nu), G)$, where

$$X(\nu) := \overline{\{\alpha_t \nu : t \in G\}}.$$

In fact, if ν satisfies the inequality $|\nu|(t + V) \leq C$ for some nonempty open relatively compact V in G and all $t \in G$, this inequality will be true for all $\mu \in X(\nu)$, and this shows the desired compactness by [2, Theorem 2].

As usual $\varphi \in C_b(G)$ is called *almost periodic* (in the sense of Bohr) if, for every $\epsilon > 0$, the set of $t \in G$ with $\|\delta_t * \varphi - \varphi\|_\infty \leq \epsilon$ is relatively dense in G. By standard reasoning this is equivalent to $\{\delta_t * \varphi : t \in G\}$ being relatively compact in $C_b(G)$ (see *e.g.,* [51]).

Definition 3.2 A translation bounded measure ν is called strongly almost periodic if $\nu * \varphi$ is almost periodic (in the Bohr sense) for every $\varphi \in C_c(G)$.

To deal with almost periodic functions it is useful to introduce the strong topology. To do so, we recall that for $\nu \in \mathcal{M}^\infty(G)$ and $\varphi \in C_c(G)$, the convolution $\nu * \varphi$ belongs to $C_b(G)$. The space $C_b(G)$ is equipped with the supremum norm. The strong topology is the weakest topology on $\mathcal{M}^\infty(G)$ such that all maps

$$\mathcal{M}^\infty(G) \longrightarrow C_b(G), \quad \nu \longmapsto \nu * \varphi,$$

are continuous.

Theorem 3.3 *Let G be a locally compact σ-compact abelian group. Let $\nu \in \mathcal{M}^\infty(G)$ be given. The following assertions are equivalent:*

(i) *The measure ν is strongly almost periodic (i.e., $\nu * \varphi$ is Bohr almost periodic for every $\varphi \in C_c(G)$).*
(ii) *$\{\alpha_t \nu : t \in G\}$ is relatively compact in the strong topology.*
(iii) *The topological space $X(\nu)$ is a topological group with addition $\dotplus$ satisfying $\alpha_s \nu \dotplus \alpha_t \nu = \alpha_{s+t} \nu$ for all $s, t \in G$.*
(iv) *The dynamical system $(X(\nu), G)$ is a rotation on a compact abelian group.*
(v) *The hull $(X(\nu), G)$ is an equicontinuous dynamical system.*

Proof The equivalence of (i), (ii), and (iii) is shown in [30, Lemma 4.2]. The equivalence of (iii) and (iv) is immediate from the definition of rotation on a compact abelian group. The equivalence of (iv) and (v) is Theorem 2.4. ∎

Factors of equicontinuous dynamical systems are equicontinuous as well. In our context the following variant will be of use in Section 6.

Corollary 3.4 *Let ν be a translation bounded measure on G. Then, the following assertions are equivalent:*

(i) *The measure ν is strongly almost periodic.*
(ii) *There exists a rotation of a compact abelian group $(\mathbb{T}, G)$ together with a continuous map $\pi \colon \mathbb{T} \to \mathcal{M}^\infty(G)$ with $\pi(e) = \nu$ and $\pi(t \cdot \xi) = t \cdot \pi(\xi)$ for all $\xi \in \mathbb{T}$ and $t \in G$.*

In this case $\pi(\mathbb{T}) = X(\nu)$ and π is a factor map as well as a group homomorphism.

Proof In the situation of (ii) we must have that π is a factor map with $\pi(\mathbb{T}) = X(\nu)$ as $\mathbb{T}$ is compact and π is continuous. Now the equivalence of (i) and (ii) follows from [30, Theorem 5.1]. The fact, that π is a group homomorphism now follows from [30, Lemma 5.2]. ∎

Remark The almost periodic measure dynamical systems considered above are of interest in the study of diffraction. In fact, they exhibit pure point diffraction and pure point dynamical spectrum with continuous eigenfunctions. Details can be found in [30] (see [4, 29, 32] for related arguments on diffraction as well).

4 Application to Delone Dynamical Systems

The previous sections dealt with first general dynamical systems and then measure dynamical systems. In this section we specialize even further and consider Delone dynamical systems. These systems are made of suitable uniformly discrete subsets of the underlying group. Such subsets can be considered as translation bounded measures in a canonical way. We will introduce the necessary notation to deal with these systems and to state Lagarias' question. We will then go on to characterize what it means for such a system to be equicontinuous in Theorem 7.1. This provides a main step in our dealing with the Lagarias question. Our strategy to view Delone sets as measures has the advantage of allowing us to treat colored Delone sets with no extra effort. This is discussed at the end of the section.

Let G be a locally compact abelian group. We will deal with subsets $\mathcal{L}$ of G. A subset $\mathcal{L}$ of G is called *uniformly discrete* if there exists an open neighborhood U of the identity in G such that $(x + U) \cap (y + U) = \varnothing$ for all $x, y \in L$ with $x \neq y$. A subset $\mathcal{L}$ of G is called *relatively dense* if there exists a compact neighborhood K of the identity of G such that

$$G = \bigcup_{x \in \mathcal{L}} (x + K).$$

A subset $\mathcal{L}$ of G is called a *Delone set* if it is both uniformly discrete and relatively dense. Any uniformly discrete set $\mathcal{L}$ (and hence any Delone set) in G can naturally be identified with the translation bounded measure

$$\delta_L := \sum_{x \in \mathcal{L}} \delta_x,$$

where δ_x denotes the unit point may at $x \in G$. The measure δ_L is then referred to as the *Dirac comb of L*. The *hull* of $X(\delta_{\mathcal{L}})$ in the vague topology then consists of measures of the form δ_M with $M \subset G$ uniformly discrete. In the sequel we will identify such measures with the underlying sets and then also write $X(\mathcal{L})$ instead of $X(\delta_{\mathcal{L}})$. We call $(X(\mathcal{L}), G)$ a *Delone dynamical* system if $\mathcal{L}$ is a Delone set. As discussed, *e.g.*, in [2], the topology on X inherited in this way from the vague topology on the measures can also be obtained from a uniform structure on the set $\mathcal{P}$ of all uniformly discrete point sets on G. Namely, for $K \subset G$ compact and V a neighbourhood of e in G, we set

$$U_{K,V} := \left\{ (\mathcal{L}_1, \mathcal{L}_2) \in \mathcal{P} \times \mathcal{P} : \mathcal{L}_1 \cap K \subset \mathcal{L}_2 + V \text{ and } \mathcal{L}_2 \cap K \subset \mathcal{L}_1 + V \right\}.$$

Then, the set $\mathcal{U}$ consisting of all $U_{K,V}$ with $K \subset G$ compact and V a neighborhood of $e \in G$, will provide a uniform structure. The induced topology is called the *local rubber topology* in [2] and is shown to agree with the topology induced by the vague topology on measures (see [12] as well).

Besides the hull of $\mathcal{L}$ we will also need the *canonical transversal* given by

$$\Xi(\mathcal{L}) = \{\mathcal{L}' \in X : e \in \mathcal{L}\}.$$

This set can easily be seen to be the closure of the set $\{\mathcal{L} - x : x \in \mathcal{L}\}$. For this reason it is also sometimes denoted as the *discrete hull* of $\mathcal{L}$. Of course, the discrete hull of $\mathcal{L}$ is a subset of the hull X and as such inherits a topology.

Whenever $\mathcal{L}$ is a subset of G, a set of the form $(-x + \mathcal{L}) \cap K$ with $x \in L$ and K compact is called a *patch of* $\mathcal{L}$. A set $\mathcal{L}$ in G is said to have *finite local complexity* (FLC) if, for any compact K in G, the set

$$\left\{ (-x + \mathcal{L}) \cap K : x \in \mathcal{L} \right\}$$

is finite. This just means that there are only finitely many patches for fixed "size" K. An alternative characterization of finite local complexity is that $\mathcal{L} - \mathcal{L}$ is locally finite [44].

The relevance of sets with finite local complexity in our setting comes from the following feature of the topology.

Lemma 4.1   *Let G be a locally compact abelian group. Let $\mathcal{L}$ be a Delone set with finite local complexity in G, $X = X(\mathcal{L})$ its hull, and $\Xi(\mathcal{L})$ its discrete hull. The unique uniform structure on the discrete hull compatible with the relative topology has a fundamental system given by the subsets*

$$U_C := \left\{ (\mathcal{L}'', \mathcal{L}') \in \Xi \times \Xi : \mathcal{L}'' \cap C = \mathcal{L}' \cap C \right\}$$

for $C \subset G$ compact. In particular, if $(\mathcal{L}_\iota)$ is a net in Ξ and $\mathcal{L}'$ belongs to X, then the following assertions are equivalent:

(i) $\mathcal{L}_\iota \to \mathcal{L}'$.
(ii)   *For any compact C in G there exists an ι_C with $\mathcal{L}_\iota \cap C = \mathcal{L}' \cap C$ for all $\iota > \iota_C$.*

Proof   This is well known and appears, at least implicitly, in *e.g.,* [4, 7, 44]. It can be seen as the statement that the local rubber topology introduced above agrees with the so called local topology in the case of finite local complexity. We sketch a proof for completeness: Let a compact C in G be given. By the definition of Ξ, we can assume without loss of generality that C contains the point e. Then, by finite local complexity, there are only finitely many possibilities for $\mathcal{L}' \cap C$ (for $\mathcal{L}' \in \Xi$). This implies that for any sufficiently small neighbourhood V of $e \in G$, the inclusions

$$\mathcal{L}_1 \cap C \subset \mathcal{L}_2 + V \quad \text{and} \quad \mathcal{L}_2 \cap C \subset \mathcal{L}_1 + V$$

can only hold if $\mathcal{L}_1 \cap C = \mathcal{L}_2 \cap C$. This shows that the U_C, $C \subset G$ compact, indeed form a basis of the uniformity. Now, the last statement characterizing convergence is immediate. ∎

An *occurrence of the patch* $(-x + \mathcal{L}) \cap K$ in a Delone set $\mathcal{L}$ is an element of

$$\left\{ y \in \mathcal{L} : (-x + \mathcal{L}) \cap K \subset (-y + \mathcal{L}) \right\}.$$

A Delone set $\mathcal{L}$ is called *repetitive* if for any patch the set of occurrences is relatively dense. For FLC Delone sets this condition is equivalent to minimality of the associated system $(X(\mathcal{L}), G)$, but for non-FLC this is no longer the case.

For any uniformly discrete subset $\mathcal{L}$ of G and any $\varphi \in C_c(G)$, we can define the function

$$f_{\mathcal{L}, \varphi} \colon G \longrightarrow \mathbb{C}, \quad f_{\mathcal{L}, \varphi}(t) = \sum_{x \in \mathcal{L}} \varphi(t - x).$$

From these definitions and Theorem 3.3, we immediately obtain the following theorem.

Theorem 4.2 *Let $\mathcal{L}$ be a Delone set in the locally compact σ-compact abelian group G. Then the following assertions are equivalent:*

(i) *The function $f_{\mathcal{L},\varphi}$ is Bohr-almost periodic for any $\varphi \in C_c(G)$.*

(ii) *The measure $\delta_{\mathcal{L}}$ is strongly almost periodic.*

(iii) *The hull $(X(\mathcal{L}), G)$ is a compact abelian group with neutral element $\mathcal{L}$ and group addition satisfying $(t + \mathcal{L}) + (s + \mathcal{L}) = (s + t + \mathcal{L})$ for all $t, s \in G$.*

(iv) *The dynamical system $(X(\mathcal{L}), G)$ is equicontinuous.*

We finish this section by discussing how the above above considerations can be carried over to colored subsets of groups.

Fix a finite set $\mathcal{C}$. We think of elements of $\mathcal{C}$ as colors. A *colored set* (with colors from $\mathcal{C}$) on the locally compact abelian group G is a pair $(\mathcal{L}, c)$ consisting of a subset $\mathcal{L}$ of G and a map $c\colon \mathcal{L} \to \mathcal{C}$. The set $\mathcal{L}$ is called the *support* of the colored set. A colored set is called *uniformly discrete, relatively dense,* and *Delone* respectively if and only if its support has the corresponding property. A *colored patch* (with colors from $\mathcal{C}$) in a colored set $(\mathcal{L}, c)$ is a pair (P, c_P) consisting of a patch P in $\mathcal{L}$ and the restriction c_P of c to P. The notion of finite local complexity then carries directly over to colored sets. As there are only finitely many colors, a colored set $(\mathcal{L}, c)$ has finite local complexity if and only if $\mathcal{L}$ has finite local complexity. A colored set whose support is uniformly discrete can be identified with a translation bounded measure on the following way. Chose an injective function $f\colon \mathcal{C} \to \mathbb{C}$. Then we can associate with $(\mathcal{L}, c)$ the measure

$$\delta_{(\mathcal{L},c)} := \sum_{x \in \mathcal{L}} f(c(x))\delta_x.$$

This allows one to transfer results from measures to colored Delone sets in the same way as they were transferred from measures to Delone sets. In particular, the direct analogue of Theorem 4.2 for colored Delone sets can be shown in this way. We refrain from an explicit statement.

5 An Affirmative Answer to the Question of Lagarias in the Case of Finite Local Complexity

For tilings of Euclidean space, Barge and Olimb have given a beautiful argument showing that in any FLC repetitive tiling system that is not completely periodic there exist two distinct tilings that are proximal, and hence their dynamical systems are not equicontinuous. We will present (a variant of) their reasoning for Delone sets on rather general locally compact abelian groups.

In short, the argument of Barge and Olimb contains two steps: First they show that failure of complete periodicity and repetitivity imply that no finite patch of a tiling can uniquely determine this tiling. We reproduce this step literally in Lemma 5.2. From this it is concluded that there exist distinct tilings that agree on arbitrarily large patches and hence are proximal. We will follow a somewhat different argumentation that applies to Delone sets on compactly generated locally compact abelian

groups (Lemma 5.3 and Corollary 5.4). This will allow us to give a characteriza-
tion of equicontinuous Delone dynamical systems that have finite local complexity
(Theorem 5.5) and yield an affirmative answer to a question of Lagarias for Delone
dynamical systems with finite local complexity (Corollary 5.6).

Definition 5.1   Let G be a locally compact abelian group and $\mathcal{L}$ a Delone set in G.
The elements of $P := \{t \in G : t + \mathcal{L} = \mathcal{L}\}$ are called periods of $\mathcal{L}$. The Delone set
$\mathcal{L}$ is called completely periodic or crystallographic if the set of its periods is relatively
dense in G.

We say that a patch of a Delone set *forces the whole Delone set* if any Delone set in
its hull that contains the patch at the same place is equal to the original Delone set.

Lemma 5.2   *Let $\mathcal{L}$ be a repetitive Delone set in a locally compact abelian group G,*
which is not completely periodic. Then it is not forced by any of its patches.

Proof   Assume the contrary. Then there exists a patch forcing the Delone set $\mathcal{L}$. This
means that all occurrences of this patch are periods of the Delone sets. As the Delone
set is repetitive, this means that the set of its periods is relatively dense. This is a
contradiction (as $\mathcal{L}$ is not completely periodic).   ∎

So far, we have not needed any restrictions on the geometry of the underling
group. For the next lemma we will need a further restriction viz the group G needs
to be compactly generated. This means that there exists a compact neighborhood W
of the identity such that the smallest subgroup of G containing W is in fact G. This
condition is clearly met for $G = \mathbb{R}^n$. Obviously, any compactly generated group is
σ-compact.

Lemma 5.3   *Let G be a locally compact, compactly generated group. Let $\mathcal{L}$ be a Delone*
set in G. Then there exists a compact set C in G with the following property. For all
$\mathcal{L}_1, \mathcal{L}_2$ in $X(\mathcal{L})$ with $e \in \mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_1 \neq \mathcal{L}_2$ there exists an $v \in \mathcal{L}_1 \cap \mathcal{L}_2$ with

$$(-v + \mathcal{L}_1) \cap C \neq (-v + \mathcal{L}_2) \cap C.$$

Proof   By the main structure theorem on locally compact, compactly generated abe-
lian groups, the group G has the form $G = \mathbb{R}^n \times \mathbb{Z}^f \times H$ with a compact group H
and non-negative integers n and f. For $r \geq 0$ let B_r and U_r be the closed and open
balls respectively with radius r around the origin in $\mathbb{R}^n \times \mathbb{Z}^f$ (with respect to the usual
Euclidean distance). As $\mathcal{L}$ is a Delone set, there exists an $R > 0$ such that

$$\mathcal{L}' \cap \big((t + B_R) \times H\big) \neq \varnothing$$

for any $t \in \mathbb{R}^n \times \mathbb{Z}^f$ and $\mathcal{L}' \in X(\mathcal{L})$.

To $\mathcal{L}_1 \neq \mathcal{L}_2$ in $X(\mathcal{L})$ with $e \in \mathcal{L}_1 \cap \mathcal{L}_2$ we can now define

$$r_M := \inf\big\{ r \geq 0 : \mathcal{L}_1 \cap (B_r \times H) \neq \mathcal{L}_2 \cap (B_r \times H) \big\}.$$

From $\mathcal{L}_1 \neq \mathcal{L}_2$ we obtain $r_M < \infty$.

We distinguish two cases:

Case 1: $r_M = 0$: Then $\mathcal{L}_1 \cap (\{0\} \times H) \neq \mathcal{L}_2 \cap (\{0\} \times H)$. Here, 0 is the origin of $\mathbb{R}^n \times \mathbb{Z}^f$. Thus, any compact set C containing $\{0\} \times H$ will do in this case (with $v = e \in G$).

Case 2: $r_M > 0$: In particular, n or f are non-zero. Without loss of generality we can assume $r_M > 2R + \epsilon$ for some $\epsilon > 0$ that we determine later (as otherwise, any compact C containing $B_{2R+\epsilon} \times H$ will do (with $v = e \in G$)). Now, by definition of r_M we can find a $t = (s, h) \in (B_{r_M} \setminus U_{r_M}) \times H$ that belongs to only one of $\mathcal{L}_1$ and $\mathcal{L}_2$. Without loss of generality we assume that t belongs to $\mathcal{L}_1$ but not to $\mathcal{L}_2$. On the other hand, $(U_{r_M} \times H) \cap \mathcal{L}_1 = (U_{r_M} \times H) \cap \mathcal{L}_1 \cap \mathcal{L}_2$. So the idea is to find a point $v = (s', h') \in (U_{r_M} \times H) \cap \mathcal{L}_1$ such that the distance between s and s' is bounded by a constant c that depends only on R, n and f and then choose $C = B_c \times H$.

To see that this is possible suppose first that $f = 0$. Let $\hat{s}$ be the unit vector in the direction of s. The R ball around $x = (r_M - R - \epsilon)\hat{s}$ lies in U_{r_M}. By definition of R it contains a point s' such that $v = (s', h') \in \mathcal{L}_1$. Clearly $\|s - s'\| \leq 2R + \epsilon$, and so we can take any $\epsilon > 0$.

Now if $f \neq 0$, then $(r_M - R - \epsilon)\hat{s}$ might not lie in $\mathbb{R}^n \times \mathbb{Z}^f$. So we take the nearest point to it in $\mathbb{R}^n \times \mathbb{Z}^f$ such that its R-ball is contained in U_{r_M}. This requires ϵ to be larger than $\sqrt{f}$.

The above considerations show that we can choose

$$C = B_{2R + \sqrt{f} + 1} \times H.$$

This finishes the proof. ∎

Remark　Note that the preceding lemma assumes neither that the Delone set in question in repetitive nor that it has finite local complexity.

Corollary 5.4　*Let G be a compactly generated group and $\mathcal{L}$ a repetitive not completely periodic Delone set in G with finite local complexity. Then $(X(\mathcal{L}), G)$ is not equicontinuous.*

Proof　Assume the contrary, *i.e.,* assume that $(X(\mathcal{L}), G)$ is equicontinuous. Then the induced partial action of G on $\Xi(\mathcal{L})$ is equicontinuous.

By Lemma 4.1 this means that for any compact $C \subset G$ there exists a compact C' such that for all $\mathcal{L}_1, \mathcal{L}_2$ in $\Xi(\mathcal{L})$, $\mathcal{L}_1 \cap C' = \mathcal{L}_2 \cap C'$ implies

$$(-v + \mathcal{L}_1) \cap C = (-v + \mathcal{L}_2) \cap C$$

for all $v \in \mathcal{L}_1 \cap \mathcal{L}_2$. Thus, if we chose C as in the previous lemma, we obtain that $\mathcal{L}_1 = \mathcal{L}_2$ whenever $\mathcal{L}_1 \cap C' = \mathcal{L}_2 \cap C'$. In particular, the patch $P := \mathcal{L}_1 \cap C'$ forces the Delone set $\mathcal{L}$. By Lemma 5.2, we then obtain that $\mathcal{L}$ is completely periodic. This is a contradiction. ∎

As a consequence of this corollary, we obtain the following characterization of equicontinuous Delone dynamical systems with finite local complexity.

Theorem 5.5 *Let G be a compactly generated group and $\mathcal{L}$ a Delone set in G with finite local complexity. Then, the following assertions are equivalent:*

(i) $\mathcal{L}$ *is completely periodic.*
(ii) $(X(\mathcal{L}), G)$ *is equicontinuous.*

Proof The implication (i) $\Rightarrow$ (ii) is clear (and does not depend on FLC). It remains to show the implication (ii) $\Rightarrow$ (i). By Corollary 5.4, it suffices to show that $\mathcal{L}$ is repetitive. This, in turn, follows from general principles as each transitive equicontinuous system must be minimal. We include a short proof for the convenience of the reader. By general principles, there will exist a minimal component (Y, G) of $(X(\mathcal{L}), G)$. Let $\mathcal{L}'$ be an element of Y. Now, as $\mathcal{L}'$ belongs to the hull of $\mathcal{L}$, there exists a net (t_i) in G with $t_i + \mathcal{L}$ converging to $\mathcal{L}'$. By the equicontinuity assumption (ii), we infer that then $-t_i + \mathcal{L}'$ must converge to $\mathcal{L}$. This shows that $\mathcal{L}$ belongs to Y and hence is repetitive. ∎

Remarks (a) The theorem can also be phrased in terms of the proximality relation.

(b) The preceding theorem can be combined with Theorem 4.2 to give various further characterizations of complete periodicity of a Delone set with finite local complexity. We single out one of these in the next corollary.

The previous theorem allows us to give an affirmative answer to a question of Lagarias (under the additional assumption of finite local complexity).

Corollary 5.6 *Let G be a compactly generated group and $\mathcal{L}$ a Delone set in G with finite local complexity. Then, the following assertions are equivalent:*

(i) $\mathcal{L}$ *is completely periodic.*
(ii) $\delta_{\mathcal{L}}$ *is strongly almost periodic.*

Proof Combine the previous theorem with Theorem 4.2. ∎

Remark It is not difficult to see that Lemmas 5.2 and 5.3 remain true for colored Delone sets. This allows one to obtain an analogue of the preceding corollary for colored Delone sets. We leave the details to the reader.

6 A Negative Answer to the Question of Lagarias for Systems Without Finite Local Complexity

In this section we show that Lagarias' question does not have an affirmative answer if the assumption of finite local complexity is dropped. More precisely, we provide examples of strongly almost periodic Delone sets that are not crystallographic. They have a significance in solid state physics too; namely, they can be used to describe incommensurable crystal phases [22] as we will recall below. In fact, such a Delone set can be described as a perturbation of a completely periodic set by displacements that are themselves periodic but with a period that is incommensurate with the first one [21]. We provide examples on the real line only, but it is quite clear that this description can be generalized to arbitrary Euclidean spaces (and even to more general locally compact abelian groups).

Our examples arise as factors of the Kronecker flow on the two-dimensional torus. By Corollary 3.4 such a factor automatically yields strongly almost periodic measures. These measures will be given by Dirac combs of certain Delone sets. Thus, it suffices to show that the Delone sets in question are not crystallographic. This will be clear from the construction.

Let $\mathbb{T} : \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus and let

$$p\colon \mathbb{R}^2 \longrightarrow \mathbb{T}, \quad p((x, y)) := (x, y) + \mathbb{Z}^2$$

be the canonical projection. Fix an irrational $\theta \in (0, 1)$ and define

$$h = h_\theta\colon \mathbb{R} \longrightarrow \mathbb{R}^2, h(t) = (t, \theta t) \quad \text{and} \quad \iota = \iota_\theta\colon \mathbb{R} \longrightarrow \mathbb{T}, \iota = p \circ h_\theta.$$

Then, ι induces an action of $\mathbb{R}$ on $\mathbb{T}$ via $t \cdot \xi := \iota(t) + \xi$, and in this way $(\mathbb{T}, \mathbb{R})$ becomes a rotation on a compact abelian group, known as *Kronecker flow* (with slope θ). Now let $g\colon [0, 1] \to \mathbb{R}^2$ be a continuous curve with the properties:

(1) g is everywhere transversal to the direction defined by the line $\mathrm{im}(h)$. More precisely, there exists $\epsilon > 0$ such that for all $(x, y) \in \mathrm{im}(g)$ we have $(x, y) + h(t) \in \mathrm{im}(g)$ only if $t = 0$ or $|t| > \epsilon$.

(2) $g(1) - g(0) \in \mathbb{Z}^2$.

In particular, $\Gamma := \mathbb{Z}^2 + \mathrm{im}(g)$ is the image of a closed connected curve on the torus that is transversal to the Kronecker flow. We also suppose that

(3) Γ does not intersect itself.

For example, any continuous strictly monotone $g\colon [0, 1] \to [0, 1]$ with $g(0) = 1$ and $g(1) = 0$ will satisfy these assumptions (1), (2), and (3). Now, we define the set

$$\mathcal{L}_{\Gamma,\theta} = \big\{ t \in \mathbb{R} : (t, \theta t) \in \Gamma \big\} = h^{-1}(\Gamma).$$

Corollary 6.1 *Let g be non-linear and let $\theta \in (0, 1)$ be irrational. Then $\mathcal{L}_{\Gamma,\theta}$ is a Delone set with a strongly almost periodic Dirac comb that is not completely periodic.*

Proof The construction gives directly that $\mathcal{L}_{\Gamma,\theta}$ is a Delone set. Since Γ is a closed curve transversal to the Kronecker flow, the set $\mathcal{L}_{\Gamma,\theta}$ is relatively dense, and since Γ does not intersect itself, $\mathcal{L}_{\Gamma,\theta}$ is uniformly discrete.

We now show that complete periodicity of $\mathcal{L}_{\Gamma,\theta}$ implies that g is linear. Assume without loss of generality that 0 belongs to $\mathcal{L}_{\Gamma,\theta}$. Then the orbit P of 0 under the set of periods of $\mathcal{L}_{\Gamma,\theta}$ is just this set of periods and hence a group. Let S' be the image of P under the map ι. Then S' is a subgroup of $\mathbb{T}$. Furthermore, by construction, S' is contained in the compact image of $\mathrm{im}(g)$ under p. Denote the closure of S' by S. Then S is again is a subgroup and contained in the image of $\mathrm{im}(g)$ under p. Hence, S cannot be $\mathbb{T}$. On the other hand, as θ is irrational, the subgroup S of $\mathbb{T}$ must contain an accumulation point. Thus, S cannot be discrete. Thus, S must be a circle. The image of g under p must then contain this circle. As this image is connected it must then agree with this circle. Hence, g is linear.

It remains to show the claim on strong almost periodicity. Define the map $\widetilde{\pi}$ from $\mathbb{R}^2$ to the measures on $\mathbb{R}$ via

$$\widetilde{\pi}(x, y) := \sum_{t:(x,y)+h(t)\in\Gamma} \delta_t.$$

In particular, $\widetilde{\pi}(0, 0) = \delta_{\mathcal{L}_{\Gamma,\theta}}$. Then $\widetilde{\pi}$ can easily be seen to have the following four properties:

- $\widetilde{\pi}(x, y)$ is translation bounded (as Γ is transversal to the Kronecker flow);
- $\widetilde{\pi}$ is continuous (as Γ is a continuous closed curve);
- $\widetilde{\pi}((x, y)) = \widetilde{\pi}((x, y) + (n, m))$ for all $(x, y) \in \mathbb{R}^2$ and $(n, m) \in \mathbb{Z}^2$ (as our construction is invariant under shifts by $\mathbb{Z}^2$);
- $\widetilde{\pi}(h(t) + (x, y)) = \alpha_{-t}\widetilde{\pi}(x, y)$ (as follows from a direct computation).

Thus, $\widetilde{\pi}$ induces a continuous map $\pi\colon \mathbb{T} \to \mathcal{M}^\infty(\mathbb{R})$ with $\pi(t \cdot \xi) = \alpha_{-t}\pi(\xi)$ for all $\xi \in \mathbb{T}$ and $t \in \mathbb{R}$. By Corollary 3.4, the measure $\pi((0, 0) + \mathbb{Z}^2) = \delta_{\mathcal{L}_{\Gamma,\theta}}$ is then strongly almost periodic. In this way, we have constructed a Delone set $\mathcal{L}_{\Gamma,\theta}$ whose Dirac comb is strongly almost periodic.   ∎

As mentioned, the sets $\mathcal{L}_{\Gamma,\theta}$ arise as models of (one-dimensional) incommensurate crystal phases (with one modulating period). In fact, [21] shows how to obtain a description of such a crystal phase in terms of a Delone set of the type $\mathcal{L}_{\Gamma,\theta}$. Let us explain in our framework how to view $\mathcal{L}_{\Gamma,\theta}$ as a pertubation of a periodic set by a displacement function that is periodic but with incommensurate period.

Let $(m, n) \in g(1) - g(0)$, which is a point in $\mathbb{Z}^2$. For simplicity we exclude the case that $m = 0$ that could be handled in a similar way. Now consider the linear function $g_0(t) = m(t, \frac{n}{m}t) \in \mathbb{R}^2$ and the curve Γ_0 on the torus defined by it. It then follows that $\mathcal{L}_{\Gamma_0,\theta}$ is a periodic Delone set, but which has an irrational period when compared to the length of the cycle corresponding to the x-axis modulo $\mathbb{Z}^2$. This simply comes from the fact that θ is irrational. Note that $g(1) - g(0) = g_0(1) - g_0(0)$ and that we may suppose without loss of generality $g(0) - g_0(0) \in \mathrm{im}(h)$. Due to the transversality condition there must exist a continuous function $\kappa\colon [0, 1] \to \mathbb{R}$ satisfying $\kappa(1) = \kappa(0)$ such that $g(t) = g_0(t) + (\kappa(t), \theta\kappa(t))$. Extend κ to a 1-periodic function on $\mathbb{R}$. Then $t \in \mathcal{L}_{\Gamma_0,\theta}$ whenever $(t, \theta t) - g_0(t) \in \mathbb{Z}^2$ (her we view g_0 as a linear function on all of $\mathbb{R}$) and hence $t + \kappa(t) \in \mathcal{L}_{\Gamma,\theta}$. This defines a map $\mathcal{L}_{\Gamma_0,\theta} \to \mathcal{L}_{\Gamma,\theta}\colon t \mapsto t + \kappa(t)$, which is, in fact, a bijection. Thus $\mathcal{L}_{\Gamma,\theta}$ is a perturbation of $\mathcal{L}_{\Gamma_0,\theta}$ with displacement κ. Since κ has rational period when compared to the length of the cycle corresponding to the x-axis modulo $\mathbb{Z}^2$, its period is incommensurable with the period of $\mathcal{L}_{\Gamma_0,\theta}$.

To complete the above discussion we determine the dynamical system $(X, \mathbb{R})$ of $\mathcal{L}_{\Gamma,\theta}$, or, equivalently, $\delta_{\mathcal{L}_{\Gamma,\theta}}$. By Corollary 3.4 π is a group homomorphism onto its image, and this image is X. Let $K = \ker \pi$. Then $(X, \mathbb{R})$ is the rotation on $\mathbb{T}/K$ induced by the Kronecker flow.

Clearly $\pi((x, y) + (k_1, k_2)) = \pi((x, y))$ if and only if $(k_1, k_2) + \Gamma = \Gamma$; *i.e.*, K is the stabiliser of Γ. Given that K is a closed subgroup of $\mathbb{T}^2$, we can distinguish two possibilities depending on whether it is discrete or not. As Γ is connected and non intersecting, K must be a cyclic sub-group of the torus in the discrete case, or a circle in the other case.

If K is not discrete, it actually coincides with a translate of Γ. It must then intersect the line $\mathrm{im}(h)$ more then once. It follows that the restriction of π to the line (which corresponds to the orbit of ν) is not injective. This implies that ν (and $\mathcal{L}_{\Gamma,\theta}$) are completely periodic and so π maps its orbit to S^1. Since S^1 is closed, $X = S^1$ in this case. If K is discrete, then X is a two dimensional torus again, and consequently ν is not completely periodic.

Remarks   (a) Since the Kronecker flow is dense in $\mathbb{T}^2$, distinct subgroups K yield systems that are not topologically conjugate. The finite cyclic sub-groups of $\mathbb{T}^2$ thus classify the possible hulls of Delone dynamical systems up to conjugacy that are not completely periodic and arise as factors of the Kronecker flow in the above manner. Likewise, this classification is given for periodic Delone sets by the sub-groups of $\mathbb{T}^2$ that are circles.

(b) If one drops the requirement that g be continuous or allows for $g(1) - g(0) \neq \mathbb{Z}^2$, then the above analysis completely breaks down. In fact, the Sturmian sequences associated with θ are obtained if one takes $\mathrm{im}(g)$ to be the closed interval in the orthocomplement of $\mathrm{im}(h)$, which is obtained by projecting (orthogonally) the unit cube onto that orthocomplement. The resulting Delone set is not periodic and has FLC, and is thus not uniformly almost periodic!

7   A Further Look at Delone Sets in Euclidean Space

Delone sets of Euclidean spaces are of particular importance for the theory both from the point of view of geometry and of physics. Thus, we have a closer look at these in this section. The main advantage is that the space of all Delone sets of a given Euclidean space carries a well-known metric, and with the help of this metric we can formulate more directly what it means for such a Delone set to be equicontinuous.

Let ρ be the standard Euclidean metric on $\mathbb{R}^n$, *i.e.*,

$$\rho(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2}.$$

Denote the closed ball (with respect to this metric) around the origin with radius r by B_r and the open ball with U_r. Then a subset $\mathcal{L}$ of $\mathbb{R}^n$ is a uniformly discrete if there exists an $r > 0$ with

$$(x + U_r) \cap (y + U_r) = \varnothing$$

for all $x, y \in \mathcal{L}$ with $x \neq y$. The largest such r is denoted by $r_{\min}$ and called the *packing radius* of $\mathcal{L}$. A subset $\mathcal{L}$ of $\mathbb{R}^n$ is relatively dense if there exists an $r > 0$ with

$$\mathbb{R}^n = \bigcup_{x \in \mathcal{L}} (x + B_r).$$

The smallest such r is denoted by $r_{\max}$ and called the *covering radius* of $\mathcal{L}$.

The intersection of B_r with a set $\mathcal{L}$ is denoted by $B_r[\mathcal{L}]$ and called an *r-patch*. The Hausdorff distance on compact subsets of $\mathbb{R}^n$ is denoted by d_H. It is not hard to see

that

(7.1) $d(\mathcal{L}_1, \mathcal{L}_2) := \sup \left\{ r > 0 : d_H \big(B_r[\mathcal{L}_1] \cup \partial B_r, B_r[\mathcal{L}_2] \cup \partial B_r \big) \leq \frac{1}{r} \right\}$

then gives a metric on the set of all Delone sets and that the induced topology agrees with the topology induced by considering the Delone sets as measures [2,3,12,14,31].

We will be concerned with dynamical systems $(X, \mathbb{R}^n)$ with $X = X(\mathcal{L})$ arising as the hull of a Delone set $\mathcal{L}$. Recall from Section 4 that the canonical transversal or discrete hull of $\mathcal{L}$ denoted by $\Xi := \Xi(\mathcal{L})$ is defined by

$$\Xi := \{ \mathcal{L}' \subset X(\mathcal{L}) : 0 \in \mathcal{L}' \}.$$

The metric d allows us now to use some notions introduced in Section 2 for such dynamical systems $(X(\mathcal{L}), \mathbb{R}^n)$. This is discussed next. For a Delone set $\mathcal{L}$, the set of r return vectors is given by

$$\mathcal{R}(\mathcal{L}, r) = \left\{ a \in \mathbb{R}^n : d(\mathcal{L}, -a + \mathcal{L}) \leq \frac{1}{r} \right\}$$
$$= \left\{ a \in \mathbb{R}^n : d_H \big(B_r[\mathcal{L}] \cup \partial B_r, B_r[\mathcal{L} - a] \cup \partial B_r \big) \leq \frac{1}{r} \right\}.$$

(If the set $\mathcal{R}(\mathcal{L}, r)$ is is relatively dense for all $r > 0$, the set $\mathcal{L}$ is called *rubber repetitive.* For Delone sets with finite local complexity, of course, rubber repetitivity coincides with repetitivity. In general, rubber repetitive is equivalent to minimality of the dynamical system.) A Delone set $\mathcal{L}$ is now uniformly almost periodic if for all $r > 0$ the set

$$A := \bigcap_{\mathcal{L}' \in X(\mathcal{L})} \mathcal{R}(\mathcal{L}', r)$$

is relatively dense.

Theorem 7.1 *Let $\mathcal{L}$ be a Delone set in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i)   *$\mathcal{L}$ is uniformly almost periodic;*
(ii)   *for all $r > 0$ the set $\bigcap_{\mathcal{L}' \in \Xi(\mathcal{L})} \mathcal{R}(\mathcal{L}', r)$ is relatively dense;*
(iii)   *for all $r > 0$ the set $\bigcap_{x \in \mathcal{L}} \mathcal{R}(\mathcal{L} - x, r)$ is relatively dense;*
(iv)   *for all $\epsilon > 0$ the set*

$$\bigcap_{x \in \mathcal{L}} \big(\mathcal{L} - x + B_\epsilon(0) \big) \cap \bigcap_{x \in \mathcal{L}} \big(-\mathcal{L} + x + B_\epsilon(0) \big)$$

   is relatively dense;
(v)   *For all $\epsilon > 0$ there exists a relatively dense set A such that for all $a \in A$ there is a bijection $f_a : \mathcal{L} \to \mathcal{L}$ satisfying $|f_a(x) - (x + a)| \leq \epsilon$.*
(vi)   *The dynamical system $(X(\mathcal{L}), \mathbb{R}^n)$ is equicontinuous.*

Proof Denote the covering radius of $\mathcal{L}$ by $r_{\max}$ and the packing radius by $r_{\min}$.

The equivalence (i) $\Leftrightarrow$ (vi) is just Theorem 2.5.

(i) $\Leftrightarrow$ (ii)   Clearly it suffices to consider large r. If $|x| \leq r_{\max}$ and r is large, then $\mathcal{R}(\mathcal{L} - x, r) \subset \mathcal{R}(\mathcal{L}, r - r_{\max})$. It follows that $\bigcap_{\mathcal{L}' \in X(\mathcal{L})} \mathcal{R}(\mathcal{L}', r)$ is relatively dense (for all $r > 0$) if and only if $\bigcap_{\mathcal{L}' \in \Xi(\mathcal{L})} \mathcal{R}(\mathcal{L}', r)$ is relatively dense (for all $r > 0$).

(ii) $\Rightarrow$ (iii)   This is clear, since $\{\mathcal{L} - x : x \in \mathcal{L}\}$ is a subset of $\Xi(\mathcal{L})$.

(iii) $\Rightarrow$ (ii)   As the discrete hull $\Xi(\mathcal{L})$ is the closure of the set $\{\mathcal{L} - x : x \in \mathcal{L}\}$, we can find for any $r > 0$, $\mathcal{L}' \in \Xi(\mathcal{L})$ and $c > 1$ an $x \in \mathcal{L}$ with

$$d_H\big(B_{cr}[\mathcal{L} - x], B_{cr}[\mathcal{L}']\big) \leq \frac{1}{cr}.$$

Let $a \in \mathcal{R}(\mathcal{L} - x, r)$ and $|a| \leq (c - 1)r$. Then, for $\mathcal{L}'$ and x as above,

$$
\begin{aligned}
d_H\big(B_r[\mathcal{L}' - a], B_r[\mathcal{L}']\big) &\leq d_H\big(B_r[\mathcal{L}' - a], B_r[\mathcal{L} - x - a]\big) \\
&\quad + d_H\big(B_r[\mathcal{L} - x - a], B_r[\mathcal{L} - x]\big) \\
&\quad + d_H\big(B_r[\mathcal{L} - x], B_r[\mathcal{L}']\big) \\
&\leq \frac{3}{r}.
\end{aligned}
$$

Thus $a \in \mathcal{R}(\mathcal{L}', \frac{r}{3})$. It follows that $\bigcap_{x \in \mathcal{L}} \mathcal{R}(\mathcal{L} - x, r) \cap B_{(c-1)r}(0)$ is contained in $\bigcap_{\mathcal{L}' \in \Xi(\mathcal{L})} \mathcal{R}(\mathcal{L}', \frac{r}{3})$. Letting $c \to \infty$ allows us to conclude the desired statement.

(iii) $\Rightarrow$ (iv)   Let $a \in \mathcal{R}(\mathcal{L}, r)$, that is $d(\mathcal{L} - a, \mathcal{L}) \leq \frac{1}{r}$. From (7.1) we conclude that $a \in \mathcal{L} + B_{\frac{1}{r}}(0)$ and hence $\mathcal{R}(\mathcal{L}, r) \subset \mathcal{L} + B_{\frac{1}{r}}(0)$. In particular there exists an $\tilde{a} \in \mathcal{L}$ such that $|a - \tilde{a}| < \frac{1}{r}$. From $d(\mathcal{L} - a, \mathcal{L}) \leq \frac{1}{r}$ we then obtain

$$d(\mathcal{L} - \tilde{a} + a, \mathcal{L} - \tilde{a}) \leq |\tilde{a} - a| + d(\mathcal{L}, \mathcal{L} - a) \leq \frac{2}{r}.$$

Thus, $a \in \mathcal{R}(\mathcal{L}, r)$ implies $-a \in \mathcal{R}(\mathcal{L} - \tilde{a}, \frac{r}{2})$. Hence,

$$\bigcap_{x \in \mathcal{L}} \mathcal{R}(\mathcal{L} - x, r) \subset \bigcap_{x \in \mathcal{L}} \big(\mathcal{L} - x + B_{\frac{1}{r}}(0)\big) \cap \bigcap_{x \in \mathcal{L}} \big(-\mathcal{L} + x + B_{\frac{2}{r}}(0)\big)$$

holds, and the desired statement (iv) follows.

(iv) $\Rightarrow$ (v)   We show that

$$A := \bigcap_{x \in \mathcal{L}} \big(\mathcal{L} - x + B_\epsilon(0)\big) \cap \bigcap_{x \in \mathcal{L}} \big(-\mathcal{L} + x + B_\epsilon(0)\big)$$

has the desired properties. Let $a \in A$ be given. Then, for each $x \in \mathcal{L}$ there exists an $y \in \mathcal{L}$ such that $a + x \in y + B_\epsilon(0)$. If $\epsilon > 0$ is small enough, for instance smaller than $\frac{r_{\min}}{2}$, then this y is unique and this defines a function $f_a \colon \mathcal{L} \to \mathcal{L}$, $f_a(x) = y$ with $|f_a(x) - (a + x)| \leq \epsilon$. Clearly, for small enough $\epsilon > 0$, such as *e.g.*, $0 < \epsilon < \frac{r_{\min}}{3}$, the function f_a is injective. Since $A = -A$, we can also construct f_{-a} which is easily seen to be the inverse of f_a.

(v) $\Rightarrow$ (iii)   Let $\epsilon > 0$. Given any r-patch $P \subset \mathcal{L}$ we have

$$f_a(P) \subset \mathcal{L} \quad \text{and} \quad d_H\big((P + a) \cap \partial B_r(0), f_a(P) \cup \partial B_r(0)\big) \leq \epsilon.$$

It follows that, if P is an $r = \frac{1}{\epsilon}$-patch of $\mathcal{L}$ at x, then

$$d_H\big((P - x) \cap \partial B_r(0), (f_a(P) - x - a) \cup \partial B_r(0)\big) \leq \frac{1}{r}.$$

Now since f_a is bijective and ϵ-close to the translation by a, it is, perhaps up to an irrelevant difference near $\partial B_r(f_a(x))$, the r-patch of $\mathcal{L}$ at $f_a(x)$, so we have

$$d_H\big((f_a(P) - x - a) \cup \partial B_r(0), B_r[\mathcal{L} - x - a] \cup \partial B_r(0)\big) \leq \frac{1}{r}$$

and thus $a \in \mathcal{R}(\mathcal{L} - x, \frac{r}{2})$.   ∎

Remarks   (a) Note that the theorem above has some overlap with Theorem 4.2. Indeed, (v) of the above theorem easily gives the uniform almost periodicity of all functions of the form $\sum_{x \in \mathcal{L}} \varphi(\cdot - x)$ with $\varphi \in C_c(\mathbb{R}^n)$, and this is Theorem 4.2(i) in the more restrictive case $G = \mathbb{R}^n$.

(b) The theorem can be used to provide a weak version of the so-called Córdoba theorem [13]. This theorem states that for a uniformly discrete set $\mathcal{L}$ of the form $\mathcal{L} = \cup_{j=1}^m \mathcal{L}_m$, and $w_1, \ldots, w_m \in \mathbb{C}$ the Fourier transform (taken in the sense of tempered distributions) of the generalized Dirac comb $\sum_{j=1}^m \sum_{x \in \mathcal{L}_j} w_j \delta_x$ is a translation bounded pure point measure if and only if each $\mathcal{L}_j$ is a lattice. In our context, we can show that for $\mathcal{L}$ with finite local complexity the Fourier transform of $\delta_{\mathcal{L}}$ (taken as a tempered distribution) is a pure point measure if and only if $\mathcal{L}$ is completely periodic. Here, the "if" part is clear from the Poisson summation formula and the "only if" part follows from the theorem, as pure pointedness of the Fourier transform of a measure implies almost periodicity of the measure [15].

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