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UNIFORM HARMONIC APPROXIMATION WITH CONTINUOUS EXTENSION TO THE BOUNDARY

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1. Let G be a domain in the complex plane and F a nonempty subset of G such that F is the closure in G of its interior F^0. We will say $f \in C^1(F)$ if f is continuous on F and possesses continuous first partial derivatives in F^0 which extend continuously to F as finite-valued functions. Let $G^* - F$ be connected and locally connected, $f \in C^1(F)$ be harmonic in F^0, and E be a subset of $\partial F \cap \partial G$ (here G^* denotes the one-point compactification of G and the boundaries ∂F, ∂G are taken in the extended plane). Suppose there is a sequence $\langle h_n \rangle$ of functions harmonic in G such that

$$|f - h_n| \to 0, \left|\frac{\partial f}{\partial x} - \frac{\partial h_n}{\partial x}\right| \to 0, \quad \text{and} \quad \left|\frac{\partial f}{\partial y} - \frac{\partial h_n}{\partial y}\right| \to 0$$

uniformly on F as $n \to \infty$. We prove that if f extends continuously to $F \cup E$ then there is a sequence m_n of functions harmonic on Ω and continuous on $F \cup E$ such that $|f - m_n| \to 0$ uniformly on F. Our paper is motivated by a problem posed by Stray (cf. [**1**], p. 359) for harmonic functions. The analogous problem for analytic functions was solved in 1978 by Roth [**9**] and Stray [**11**]. In the analytic case, however, no assumptions are imposed on the partials of f. But the harmonic case, itself, must be dealt with separately since if $f \in C^1(F)$ and f is harmonic in the interior of F, f need not be the real part of a function continuous on F and analytic in the interior F^0. Finally we would like to make two additional comments. First, such approximations as given above, where the error $f - m_n$ can be continuously extended to certain subsets of the boundary, prove useful when constructing functions with prescribed boundary behavior. Secondly, our hypotheses are satisfied in specific instances. See, for example Shaginyan [**10**].

2. For a subset S of the extended plane $\mathbf{C}^*$ let S^0 be its interior, $\bar{S}$ its closure in $\mathbf{C}^*$ and $\partial S = \bar{S} - S^0$. By D^1 we mean $\partial/\partial x$ and by D^2 we mean $\partial/\partial y$. Our main tool will be the following Walsh-type fusion lemma.

LEMMA 1. *Let K_1 and K be compact subsets of $\mathbf{C}$ and let K_2 be a relatively closed subset of $\mathbf{C}$ such that $K_1 \cap K_2 = \emptyset$, $K_1 \cup K \cup K_2 \neq \mathbf{C}$, and $K^0 \neq \emptyset$.*

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Moreover, let D be an open disk such that

$$\bar{D} \subset \mathbf{C}\backslash(K_1 \cup K \cup K_2).$$

Then there exists a constant C_0 such that if u_1 and u_2 are essentially harmonic functions on $\mathbf{C}\backslash\bar{D}$ (for the relevant definitions, see the book by Gauthier-Hengartner [**3**] *) with*

$$||u_1 - u_2||_K < \epsilon, \quad \textit{and} \quad ||D^i(u_1 - u_2)||_K < \epsilon, \quad i = 1, 2,$$

then there exists an essentially harmonic function h on $\mathbf{C}\backslash\bar{D}$ with

$$||u_i - h||_{K \cup K_i} < C_0\epsilon, \quad \left|\left|\frac{\partial(u_i - h)}{\partial x}\right|\right|_{K \cup K_i} < C_0\epsilon, \quad \textit{and}$$

$$\left|\left|\frac{\partial(u_i - h)}{\partial y}\right|\right|_{K \cup K_i} < C_0\epsilon, \quad i = 1, 2.$$

Remark 1. C_0 depends on K_1, K_2, and K but is independent of u_1, u_2. All norms are sup norms.

Remark 2. In order to approximate u_i on $K \cup K_i$, $i = 1, 2$, only the assumption $||u_1 - u_2||_K < \epsilon$ is needed (cf. Lemma 2.2.16 [**3**]). But in order to simultaneously approximate u_i and its first partials on $K \cup K_i$, $i = 1, 2$, we need, in addition, a condition on the first partials of u_i on K. To see this consider the following example:

Let

$$K = K_1 = \{z \big| |z| \leqq 1\}, \quad K_2 = \{z \big| |z| \geqq 2\},$$

$$u_n(z) = \operatorname{Re} z^n, \quad \text{and} \quad v(z) \equiv 0.$$

Then $||u_n - v||_K \leqq 1$ but

$$\left|\left|\frac{\partial(u_n - v)}{\partial x}\right|\right|_K \geqq n.$$

So there is no essentially harmonic h such that $\partial h/\partial x$ approximates $\partial u_n/\partial x$ and $\partial v/\partial x$ on K for all n.

Proof. The proof is similar to that given for Lemma 2.2.16, [**3**] except that Gauthier and Hengartner do not simultaneously fuse the partial derivatives in their Lemma 2.2.16. Thus their proof needs some modifications.

We may assume without loss of generality that $u_2 = 0$. Let $u \equiv u_1$. Since $K^0 \neq \emptyset$ we may assume $0 < ||u||_K < \epsilon$, for otherwise the proof is trivial. Let Ω_1, Ω_2 be smoothly bounded open sets such that $K_1 \subset \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset \mathbf{C} - \bar{D}$ and $\bar{\Omega}_2 \cap K_2 = \emptyset$. Note that Ω_1, Ω_2 depend only on K_1, K_2, and D. Furthermore let V be a bounded open neighborhood of K such that

$$\|u\|_{\bar{V}} \leqq 2\|u\|_K, \quad \|D^i u\|_{\bar{V}} \leqq 2\|D^i u\|_K, \quad i = 1, 2,$$

$\partial V \in C^1$, and $V \subset \mathbf{C} - \bar{D}$. Introduce bounded open sets G_1, G_2 such that

$$K_1 \subset G_1 \subset \Omega_1,$$

$$\Omega_2, \subset G_2 \subset \mathbf{C} - (K_2 - V) - \bar{D},$$

$$(G_2 - G_1) \cap V = (\Omega_2 - \Omega_1) \cap V,$$

with

$$d(\partial G_1, \partial \Omega_1) \leqq (1/2) d(K_1, \partial \Omega_1),$$

$$d(\partial G_2, \partial \Omega_2) \leqq (1/2) d(K_2 - V, \partial \Omega_2),$$

$d \equiv$ distance. Further we choose both ∂G_1, $\partial G_2 \in C^1$ and u singularity free on ∂G_1, ∂G_2.

Let $H \in C^\infty(\mathbf{C})$ such that

$$H|_{\Omega_1} \equiv 1, \quad H|_{\mathbf{C} - \Omega_2} \equiv 0$$

and $0 \leqq H(x) \leqq 1$ on $\mathbf{C}$.

Let

$$\psi(z) = \begin{cases} H(z)u(z), & z \in G_2 \\ 0 \quad , & z \in \mathbf{C} - G_2. \end{cases}$$

Then ψ is C^∞ outside the singularities of u contained in G_2, and satisfies the following inequalities:

(1) $\|\psi - u\|_{K_1 \cup K} \leqq C\|u\|_K$

(2) $\|\psi\|_{K_2 \cup K} \leqq C\|u\|_K$

(3) $\|D^i(\psi - u)\|_{K_1 \cup K} \leqq C(\|u\|_K + \|D^i u\|_K)$

(4) $\|D^i \psi\|_{K_2 \cup K} \leqq C(\|u\|_K + \|D^i u\|_K)$,

where the constant C is independent of u.

Since ψ, however, is not necessarily essentially harmonic in $\mathbf{C} - \bar{D}$ it is not the desired function, but will serve as an auxiliary function.

We show (1) first. Now $\psi \equiv u$ on Ω_1 and $K_1 \subset \Omega_1$ imply

$$\|\psi - u\|_{K_1} = 0.$$

As for K we have

$$K = (K \cap G_2) \cup [K \cap (\mathbf{C} - G_2)]$$

and $\psi \equiv 0$ on $K \cap (\mathbf{C} - G_2)$. Hence

$$\|\psi - u\|_{K \cap (\mathbf{C} - G_2)} = \|u\|_{K \cap (\mathbf{C} - G_2)} \leqq \|u\|_K$$

and since $\psi \equiv Hu$ on $K \cap G_2$ we have

$$\|\psi - u\|_{K \cap G_2} = \|u(H - 1)\|_{K \cap G_2} \leqq 2\|u\|_K.$$

As for (2) we note that since $\psi \equiv 0$ on $\mathbf{C} - G_2$ it suffices to show (2) on $(K_2 \cup K) \cap G_2$. Now

$$\|\psi\|_{K \cap G_2} = \|Hu\|_{K \cap G_2} \leqq \|u\|_K.$$

Since $K_2 \cap G_2 = \emptyset$ we have (2).

In order to show (3) it suffices to show the inequality on K since on K_1, $\psi \equiv u$. As before

$$K = K \cap (\mathbf{C} - G_2) \cup (K \cap G_2)$$

and $\psi \equiv 0$ on $K \cap (\mathbf{C} - G_2)$ so

$$\|D^i(\psi - u)\|_{K \cap (\mathbf{C} - G_2)} = \|D^i u\|_{K \cap (\mathbf{C} - G_2)} \leqq \|D^i u\|_K$$

while on $K \cap G_2$, $\psi \equiv Hu$, so

$$\|D^i(\psi - u)\|_{K \cap G_2} = \|D^i(Hu) - D^i u\|_{K \cap G_2}$$
$$\leqq \|D^i(Hu) - D^i u\|_K.$$

Now H has compact support and so

$$\|D^i H\|_{\mathbf{C}} \leqq M < \infty.$$

Also $D^i(Hu) = uD^i H + HD^i u$ implies

$$\|D^i(Hu) - D^i u\|_K \leqq \|u\|_K M + \|(H - 1)D^i u\|_K$$
$$\leqq \|u\|_K M + \|D^i u\|_K,$$

and this proves (3).

Finally we need show (4) only on $K \cap G_2$, where we have

$$\|D^i \psi\|_{K \cap G_2} = \|D^i(Hu)\|_{K \cap G_2} \leqq \|D^i(Hu)\|_K$$
$$\leqq \|u\|_K M + \|D^i u\|_K.$$

We next proceed as in the proof of Theorem 2.2.9 [3] and let

$$W = G_1 \cup V \cup (\mathbf{C} - G_2),$$

S = set of singularities of u in G_1, and S_ϵ an ϵ-neighborhood of S. We apply Green's formula to ψ using the Green's function g for $\mathbf{C} - \bar{D}$. For $z \in W - S$ we have (cf. [3], (2.2.17))

$$\psi(z)$$
$$= \frac{1}{2\pi} \int_{W-S} [\Delta H(\zeta) \cdot g(z - \zeta) + 2 \nabla H(\zeta) \cdot \nabla g(z - \zeta)] u(\zeta) d\xi d\eta$$

$$- \frac{1}{2\pi} \int_{\partial W} [\psi(\zeta) \nabla g(z - \zeta) - \nabla\psi(\zeta) g(z - \zeta)$$

$$+ 2u(\zeta) \nabla H(\zeta) g(z - \zeta)] ds(\zeta) - \sigma(z),$$

$\zeta = \xi + i\eta$, where $\sigma(z)$ (cf. [3], p. 62) is essentially harmonic in $\mathbf{C} - \bar{D}$, Δ_ζ denotes the Laplacian and ∇_ζ the gradient with respect to ζ. Set

$$\psi(z) = I_1(z) + I_2(z) - \sigma(z),$$

where $I_1(z)$ is the first integral and $I_2(z)$ the second in the above representation of ψ. We note that the integral I_1 reduces to one over V since both ΔH and ∇H vanish on G_1, $H \equiv 0$ on $\mathbf{C} - G_2$, and $V \cap S = \emptyset$. Similarly the integral I_2 reduces to one over $\partial W \cap G_2$. Hence

$$I_1(z) = \frac{1}{2\pi} \int_V [\Delta H(\zeta) g(\zeta, z) + 2 \nabla_\zeta H(\zeta) \cdot \nabla_\zeta g(\zeta, z)] u(\zeta) d\xi d\eta$$

and

$$I_2(z) = -\frac{1}{2\pi} \int_{\partial W \cap G_2} [\psi(\zeta) \nabla_\zeta g(\zeta, z) - g(\zeta, z) \nabla_\zeta \psi(\zeta)$$

$$+ 2u(\zeta) g(\zeta, z) \nabla_\zeta H(\zeta)] ds(\zeta).$$

First note that

$$\| I_1(z) \|_{K_1 \cup K_2 \cup K} \leqq C \| u \|_K,$$

where the constant C is independent of u. This holds since

$$|I_1(z)| \leqq \|u\|_V \int_V E$$

$$\leqq 2\|u\|_K \int_{\mathbf{C} - \bar{D}} E = 2\|u\|_K \int_{\Omega_2 - \Omega_1} E,$$

where

$$E = [\, |\Delta H(\zeta)| \, |g(z - \zeta)| + 2|\nabla H(\zeta)| \, |\nabla g(z - \zeta)| \,] d\xi d\eta.$$

For $z \in K_1 \subsetneq \Omega_1 \subset \Omega_2$ both g and ∇g are bounded for $\zeta \in \Omega_2 - \Omega_1$ (cf. [3], p. 72). So

$$\| I_1(z) \|_{K_1} \leqq C \|u\|_K,$$

where C is independent of u. On K_2 the same reasoning applies since if U is a bounded open neighborhood of $\overline{\Omega_2 - \Omega_1}$ such that $\bar{U} \cap K_2 = \emptyset$ then for all $z \in K_2$ we have

$$\|g\|_{K_2 \times \overline{\Omega_2 - \Omega_1}} < \infty \quad \text{and} \quad \|\nabla g\|_{K_2 \times \overline{\Omega_2 - \Omega_1}} < \infty$$

(cf. [3], p. 72-3). For $z \in K$ we note that

$$\int_{\Omega_2 - \Omega_1} [\,|\Delta H(\zeta)|\,|g(z-\zeta)| + 2|\nabla H(\zeta)|\,|\nabla g(z-\zeta)|\,]d\xi d\eta \leqq C$$

for a constant C depending only on D, K_1, and K_2, since the integral is a continuous function of z. So

$$\|I_1\|_K \leqq C\|u\|_K, \quad C \text{ independent of } u.$$

Hence

$$(5) \qquad \|I_1\|_{K \cup K_1 \cup K_2} \leqq C\|u\|_K,$$

where the constant C is independent of u.

Rewrite $I_1(z)$ as

$$(6) \qquad I_1(z) = \frac{1}{2\pi}\int_V [\Delta H(\zeta) g(z, \zeta)\,]u(\zeta)d\xi d\eta$$
$$+ \frac{1}{\pi}\int_V [\nabla_\zeta H(\zeta) \cdot \nabla_\zeta g(z, \zeta)\,]u(\zeta)d\xi d\eta.$$

Now $I_1(z) \in C^1(V)$ by Lemma 4.1 [4]. Furthermore by this same lemma we have

$$D_z^i \int_V [\Delta H(\zeta) g(z, \zeta)\,]u(\zeta)d\xi d\eta = \int_V \Delta H(\zeta)[D_z^i g(z, \zeta)\,]u(\zeta)d\xi d\eta,$$

for $z \in K$, where D_z^i is the first partial with respect to the ith coordinate of $z = z(x, y)$. Now

$$g(z, \zeta) = \Gamma(z - \zeta) + h(\zeta)$$

where

$$\Gamma(z - \zeta) = -\log|z - \zeta|,$$

and $h(\zeta)$ is the harmonic part of g. Then

$$\left\|\int_V \Delta H(\zeta)u(\zeta)D_z^i g(z, \zeta)d\xi d\eta\right\|_K$$
$$\leqq \left\|\int_{S(H)\cap V} \Delta H(\zeta)u(\zeta)D_z^i g(z, \zeta)d\xi d\eta\right\|_K,$$

where $S(H)$ = support of H

$$\leqq \left\|\int_{S(H)\cap V} |\Delta H(\zeta)u(\zeta)D_z^i g(z, \zeta)d\xi d\eta|\right\|_K$$
$$\leqq \tilde{C}\|u\|_V$$
$$\leqq C\|u\|_K, \quad C = \text{constant},$$

using the estimate

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$$|D_z^i\Gamma(z - \zeta)| \leqq \frac{1}{2\pi}|z - \zeta|^{-1}$$

(cf. [**4**]), where C is independent of u. For the second integral in (6) we need only worry about the logarithmic terms in

$$u(\zeta)\,\nabla_\zeta H(\zeta) \cdot \nabla_\zeta g(z, \zeta) = \sum_{i=1}^{2} [D_\zeta^i H \cdot D_\zeta^i g]u(\zeta).$$

Since

$$D_\zeta^i\Gamma(z - \zeta) = -D_z^i\Gamma(z - \zeta) \quad \text{and} \quad \partial V \in C^1$$

(i.e., the divergence theorem applies to V) we have by Lemma 4.2 [**4**] that for $z \in K$,

$$(7) \quad D_z^j \int_V [D_\zeta^i H(\zeta) \cdot D_\zeta^i\Gamma(z - \zeta)]u(\zeta)d\xi d\eta$$
$$= D_z^j \int_V [-D_z^i\Gamma(z - \zeta)][(D_\zeta^i H(\zeta)) \cdot u(\zeta)]d\xi d\eta$$
$$= -\int_V [D_z^{ij}\Gamma(z - \zeta)][(D_\zeta^i H(\zeta)) \cdot u(\zeta) - (D_z^i H(z)) \cdot u(z)]d\xi d\eta$$
$$+ D_z^i H(z) \cdot u(z) \int_{\partial V} D_z^i\Gamma(z - \zeta)\nu_j(\zeta)ds_\zeta$$

where

$$D_z^{ij} = D_z^j(D_z^i),$$

ν_j denotes the jth component of the outer unit normal, and ds_ζ is the differential of arc length.

Applying the Taylor's formula and Cauchy inequality to u and noting that $D_\xi^i H$ satisfies a Lipschitz condition we have

$$|(D_\xi^i H(\xi)) \cdot u(\xi) - (D_z^i H(z)) \cdot u(z)|$$
$$\leqq |D_\xi^i H(\xi)|\,|u(\xi) - u(z)| + |u(z)|\,|D_\xi^i H(\xi) - D_z^i H(z)|$$
$$\leqq ||D_\xi^i H||_V |z - \xi|(\,||D^1 u||_V + ||D^2 u||_V) + ||u||_V C_1 |z - \xi|$$
$$\leqq C_2|z - \xi|(\,||D^1 u||_K + ||D^2 u||_K + ||u||_K)$$

for $z \in K$, $\xi \in V$, and a constant C_2 is independent of u.
This last estimate together with the inequality

$$|D^{ij}\Gamma(z - \zeta)| \leqq \frac{1}{\pi}|z - \zeta|^{-2}$$

cf. [**4**] implies the first term in (7) satisfies

$$\left|\int_V [D_z^{ij}\Gamma(z - \zeta)][D_\zeta^i H(\zeta)u(\zeta) - D_\zeta^i H(z)u(z)]d\xi d\eta\right|$$

$$\leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K),$$

where C is independent of u.

As for the second integral in (7), we again see that

$$|D_z^i \Gamma(z - \zeta)| \leqq \frac{1}{2\pi}|z - \zeta|^{-1}$$

implies that for $z \in K$ and $\zeta \in \partial V$,

$$\left| D_z^i H(z) \cdot u(z) \int_{\partial V} D_z^i \Gamma(z - \zeta)\nu_j(\zeta) ds_\zeta \right|$$
$$\leqq \|D_\zeta^i H\|_{S(H)} \|u\|_V \hat{C} \leqq C\|u\|_K,$$

where C is again independent of u. Thus

$$\|D_z^i I_1\|_K \leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K),$$

where the constant C is independent of u.

We next estimate $D_z^i I_1$ for $z \in K_1 \cup K_2$, using Leibnitz's rule. Now

$$(8) \quad D_z^i I_1(z)$$
$$= \frac{1}{2\pi} \int_V \Delta H(\zeta) D_z^i g(z, \zeta) u(\zeta) + 2D_z^i [(\nabla_\zeta H(\zeta) \cdot \nabla_\zeta g(z, \zeta)) u(\zeta)$$
$$= \frac{1}{2\pi} \int_V [\Delta H(\zeta) D_z^i g(z, \zeta) u(\zeta) + 2u(\zeta) \nabla_\zeta H(\zeta) \cdot \nabla_\zeta D_z^i g(z, \zeta)] d\xi d\eta.$$

The integral in (8) reduces to one over $V \cap (\Omega_2 - \Omega_1)$, since $H \equiv 0$ outside Ω_2 and $H \equiv 1$ on Ω_1. Now $|\nabla_z g(z, \zeta)|$ is bounded by M say, for

$$\zeta \in V \cap \overline{\Omega_2 - \Omega_1} \quad \text{and} \quad z \in K_1 \cup K_2$$

(cf. [3], p. 73). Applying the Poisson formula one gets

$$\|D_z^j \nabla_\zeta g\|_{(K_1 \cup K_2) \times (\overline{\Omega_2 - \Omega_1})} \leqq \tilde{M} < \infty,$$

where $\tilde{M}$ is independent of u. Hence for $z \in K_1 \cup K_2$ we have

$$|D_z^i I_1(z)| \leqq \left| \int_{V \cap (\overline{\Omega_2 - \Omega_1})} \Delta H(\zeta) D_z^i g(z, \zeta) u(\zeta) \right|$$
$$+ \left| \int_{V \cap (\overline{\Omega_2 - \Omega_1})} 2u(\zeta) \nabla_\zeta H(\zeta) \cdot D_z^j \nabla_\zeta g \right|$$
$$\leqq M\|u\|_V \|\Delta H\|_{\overline{\Omega_2 - \Omega_1}} + 2\|u\|_V \cdot \|\nabla H\|_{\overline{\Omega_2 - \Omega_1}} \cdot \tilde{M}$$
$$\leqq C\|u\|_K$$

where C is independent of u.

This together with our estimate of $D_z^i I_1$ on K gives

(9) $$\|D_z^i I_1\|_{K_1 \cup K_2 \cup K} \leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K)$$

as desired, where the constant C is independent of u.

If we set $h_0(z) = I_2(z) - \sigma(z)$ then from (5) and (9) we have

$$\|h_0(z) - \psi\|_{K \cup K_1 \cup K_2} = \|I_1\|_{K \cup K_1 \cup K_2} \leqq C\|u\|_K$$

and

$$\|D^i(h_0 - \psi)\|_{K \cup K_1 \cup K_2} = \|D_z^i I_1\|_{K \cup K_1 \cup K_2}$$
$$\leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K).$$

Furthermore because of inequalities (1), (2), (3), and (4) we deduce

$$\|h_0 - u\|_{K \cup K_1} \leqq C\|u\|_K,$$
$$\|h_0\|_{K \cup K_2} \leqq C\|u\|_K,$$
$$\|D^i(h_0 - u)\|_{K \cup K_1} \leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K),$$

and

$$\|D^i h_0\|_{K \cup K_2} \leqq C(\|u\|_K + \|D^1 u\|_K + \|D^2 u\|_K).$$

Finally, if we approximate h_0 by a Riemann sum, we get our desired essentially harmonic approximation h (cf. **[3]**, p. 64 and p. 76).

3. We shall now turn to the result alluded to in Section 1.

THEOREM 1. *Let G be a domain in the complex plane $\mathbf{C}$ such that $\mathbf{C} - \bar{G} \neq \emptyset$ contains the closure of an open disk D_0. Let F be a relatively closed subset of G such that $F = \overline{F^0}$ and $G^* - F$ is connected and locally connected. Let E be a subset of $\partial F \cap \partial G$. Let $f \in C^1(F)$ be harmonic in the interior of F. Suppose that there are functions h_n harmonic in G such that*

$$\|f - h_n\|_F \to 0 \quad \text{and} \quad \|D^i(f - h_n)\|_F \to 0, \quad i = 1, 2.$$

If f extends continuously to $F \cup E$, then there is a sequence m_n of functions harmonic on Ω and continuous on $F \cup E$ such that

$$\|f - m_n\|_F \to 0.$$

Proof. We may without loss of generality assume that G is bounded, for if z_0 is the center of an open disk in $\mathbf{C} - \bar{G}$, then the general case can be reduced to this one by inverting with respect to this disk. Let $\{G_n\}$ denote a canonical exhaustion (cf. **[8]**) of G. For each $n = 1, 2, 3, \ldots$, we choose a positive number a_n associated with $K_{1n} = \bar{G}_n$, $K_{2n} = (\mathbf{C} - D_0) - G_{n+1}$, and $K_n = F_n \equiv F \cap \bar{G}_{n+1}$ so that $1 < a_1 < a_2 < \ldots$. Since $F^0 \cap G_{n+1} = \emptyset$ for at most a finite number of n, we may assume $K_n^0 \neq \emptyset$ for each n. If $\epsilon > 0$ is given, we select positive $\epsilon_1, \epsilon_2, \ldots$ so that

$$\epsilon_1 > \epsilon_2 > \dots \quad \text{and} \quad \sum_{n=1}^{\infty} \epsilon_n < \epsilon/2.$$

From our hypotheses we have

$$(10) \quad \|f - h_k\|_F < \frac{\epsilon_n}{4a_n} \quad \text{and}$$

$$(11) \quad \|D^i(f - h_k)\|_F < \frac{\epsilon_n}{4a_n}, \quad i = 1, 2,$$

for all k sufficiently large. By relabeling the subscripts of $\langle h_k \rangle$ if necessary we may assume (10) and (11) hold for h_n itself. By Lemma 6 [**2**] there exist functions q_n essentially harmonic on $\mathbf{C}$ such that

$$(12) \quad \|h_n - q_n\|_{\bar{G}_{n+1}} < \frac{\epsilon_n}{4a_n} \quad \text{and}$$

$$(13) \quad \|D^i(h_n - q_n)\|_{\bar{G}_{n+1}} < \frac{\epsilon_n}{4a_n}, \quad \begin{array}{l} i = 1, 2, \\ n = 1, 2, \dots \end{array}.$$

Hence from (10), (12) we have

$$(14) \quad \|f - q_n\|_{F_n} \leqq \|f - h_n\|_{F_n} + \|h_n - q_n\|_{F_n} < \frac{\epsilon_n}{2a_n},$$

while (11) and (13) imply

$$(15) \quad \|D^i(f - q_n)\|_{F_n} \leqq \|D^i(f - h_n)\|_{F_n} + \|D^i(h_n - q_n)\|_{F_n} < \frac{\epsilon_n}{2a_n}.$$

Now (14) implies

$$\|q_{n+1} - q_n\|_{F_n} \leqq \|q_{n+1} - f\|_{F_n} + \|f - q_n\|_{F_n}$$
$$< \frac{\epsilon_{n+1}}{2a_{n+1}} + \frac{\epsilon_n}{2a_n} < \frac{\epsilon_n}{a_n},$$

while (15) similarly yields

$$\|D^i(q_{n+1} - q_n)\|_{F_n} < \frac{\epsilon_n}{a_n}.$$

Let D be a disk such that $\bar{D} \subset \mathbf{C} - \bar{G}$. Applying Lemma 1 to the functions q_n, q_{n+1} relative to $\mathbf{C} - \bar{D}$ and to the sets K_{1n}, K_{2n}, and K_n, we obtain essentially harmonic functions r_n on $\mathbf{C} - \bar{D}$ such that

$$(16) \quad |r_n(z) - q_n(z)| < \epsilon_n, \quad z \in K_{1n} \cup K_n,$$

$$|r_n(z) - q_{n+1}(z)| < \epsilon_n, \quad z \in K_{2n} \cup K_n,$$

$$\left|\frac{\partial(r_n - q_n)}{\partial x}\right| < \epsilon_n, \quad z \in K_{1n} \cup K_n,$$

$$\left|\frac{\partial(r_n - q_n)}{\partial y}\right| < \epsilon_n, \quad z \in K_{1n} \cup K_n,$$

$$\left|\frac{\partial(r_n - q_{n+1})}{\partial x}\right| < \epsilon_n, \quad z \in K_{2n} \cup K_n,$$

$$\left|\frac{\partial(r_n - q_{n+1})}{\partial y}\right| < \epsilon_n, \quad z \in K_{2n} \cup K_n.$$

Let

$$t_n(z) = \sum_{\nu=1}^{n-1} (r_\nu(z) - q_{\nu+1}(z)).$$

Since

$$(\mathbf{C} - \bar{D}_0) - G \subseteq (\mathbf{C} - \bar{D}_0) - G_n, \quad n = 1, 2, \dots$$

the sequence $\{t_n\}$ converges uniformly in $(\mathbf{C} - \bar{D}_0) - G$ by the second inequality in (16). In particular, if we set

$$t(\zeta) = \lim_n t_n(\zeta), \quad \zeta \in \partial G,$$

then t is continuous on ∂G.

Consider $\zeta \in \partial G \subseteq K_{2n}$ and $z \in K_{2n}$. Let $\zeta = \xi + i\eta$, $z = x + iy$. Then

$$\begin{aligned}(r_j(z) &- q_{j+1}(z)) - (r_j(\zeta) - q_{j+1}(\zeta)) \\ &= \frac{\partial(r_j - q_{j+1})}{\partial x}(\zeta) \cdot (\xi - x) \\ &+ \frac{\partial(r_j - q_{j+1})}{\partial y}(\zeta) \cdot (\eta - y) + \delta_1^j(\xi - x) + \delta_2^j(\eta - y)\end{aligned}$$

where

$$\delta_1^j, \delta_2^j \to 0 \quad \text{as } |\xi - x| \to 0 \quad \text{and} \quad |\eta - y| \to 0.$$

Hence from (16) we have

$$\begin{aligned}|t_n(z) - t_n(\zeta)| \leqq \sum_{j=1}^{n-1} &\epsilon_j(|\xi - x| + |\eta - y|) \\ &+ \sum_{j=1}^{n-1} \delta_1^j|\xi - x| + \delta_2^j|\eta - y|\end{aligned}$$

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$$\leqq 2|\zeta - z|\epsilon + |\zeta - z| \sum_{j=1}^{n-1} (\delta_1^j + \delta_2^j).$$

For fixed n, we may choose z close to ζ so that

$$\delta_1^j + \delta_2^j < \epsilon_j, \quad 1 \leqq j \leqq n - 1,$$

whenever $|z - \zeta| < \delta(n)$. Then for $k \geqq n$, consider

$$z \in F_{k+1} - F_k \subseteq K_{2n},$$

where k is chosen sufficiently large, say $k \geqq N(n)$ so that

$$|z - \zeta| < \delta(n).$$

Then for

$$z \in (F_{k+1} - F_k) \cap \{z: |z - \epsilon| < \delta(n)\}$$

we have

$$|t_n(z) - t_n(\zeta)| \leqq 2|z - \zeta|\epsilon + |\zeta - z|\frac{\epsilon}{2}$$

$$= \frac{5}{2}\epsilon|z - \zeta|,$$

while the second inequality in (16) implies

$$|t_n(\zeta) - t(\zeta)| < \sum_n^\infty \epsilon_\nu.$$

Hence, we have

$$(17) \quad |t_n(z) - t(\zeta)| < \frac{5}{2}\epsilon|z - \zeta| + \sum_n^\infty \epsilon_\nu, \; z \in F_{k+1} - F_k, \; k \geqq N(n).$$

Let

$$m(z) = \sum_{\nu=1}^{k+1} (r_\nu(z) - q_{\nu+1}(z)) + q_{k+2}(z)$$

$$+ \sum_{\nu=k+2}^{\infty} (r_\nu(z) - q_\nu(z)).$$

Note that $m(z)$ is independent of k. Then for $z \in F_{k+1} - F_k$, $k \geqq N(n)$, we have by (14), (16), (17) and the fact $z \in K_{2\nu}$, $n \leqq \nu \leqq k + 1$, that

$$
\begin{aligned}
&|m(z) - f(z) - t(\zeta)| \\
&\leqq |t_n(z) - t(\zeta)| + \sum_{\nu=n}^{k+1} |r_\nu(z) - q_{\nu+1}(z)| \\
&+ |q_{k+2}(z) - f(z)| + \sum_{\nu=k+2}^{\infty} |r_\nu(z) - q_{\nu(z)}| \\
&\leqq \frac{5}{2}\epsilon|z - \zeta| + \sum_{\nu=n}^{\infty} \epsilon_\nu + \sum_{\nu=n}^{k+1} \epsilon_\nu + \epsilon_{k+2} + \sum_{\nu=k+2}^{\infty} \epsilon_\nu.
\end{aligned}
$$

Hence the difference between $m(z) - f(z)$ and $t(\zeta)$ converges to 0 if $n \leqq k$ increases and $z \in K_{2n} \cap (F_{k+1} - F_k)$ converges on F to a point ζ of $\bar{F} - F$. Note m is essentially harmonic on G. Also

$$|m(z) - f(z)| < \epsilon \quad \text{for all } z \in F$$

(cf. [**2**], p. 181). So if we define $g(z) = m(z) - f(z)$ for $z \in F$ and $g(z) = t(z)$ for $z \in \bar{F} - F$, then

$$|g(z)| \leqq \epsilon \quad \text{for } z \in \bar{F}.$$

Thus the approximating functions m can be extended to $F \cup E$. Finally, since $G^* - F$ is connected and locally connected, we can apply a pole pushing argument (cf. [**2**], p. 181) to the poles of m to complete the proof of Theorem 1.

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