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CONTINUITY OF THE SCATTERING TRANSFORMATION FOR THE KORTEWEG DE VRIES EQUATION

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Introduction. It is at present well known that, if $q(x, t)$ is a solution of the Korteweg de Vries (K d V) equation

(1) $\quad q_{xxx} - 6qq_x + q_t = 0$

such that $q(x, 0) = q_0(x)$, where $q_0(x)$ behaves reasonably at infinity, and if

$$(R(k, t), \lambda_j(t), c_j(t)), \quad j = 1, \ldots, n,$$

are the scattering data (see [4]) corresponding to $\mathbf{q}(x, t)$, then

$$R(k, t) = \exp(8ik^3 t)R(k, 0)$$

(2) $$c_j(t) = \exp(8\lambda_j^3 it)c_j(0)$$

$$\lambda_j(t) = \lambda_j(0) = \lambda_j.$$

The bijective map F which associates the scattering data

$$(R(k, t), \lambda_j(t), c_j(t))$$

to $q(x, t)$, is called the scattering transformation. The knowledge of F and its inverse F^{-1} allows us to solve the K d V equation. In fact, we have

$$q(x, t) = F^{-1}((R(k, t), \lambda_j(t), c_j(t)), \quad j = 1, \ldots, n$$

and

$$(R(k, 0), \lambda_j(0), c_j) = F(q_0(x)).$$

This article is a part of a study of the properties of continuity of F and F^{-1}. Here, we are going to define, in the space of solutions $q(x, t)$ and in the space of scattering data, relatively natural topologies in relation to which F will be continuous. Because of the particularly simple behaviour in t of the scattering data, we will study the continuity of the map

$$F:q \rightarrow (R(k), \lambda_j, c_j).$$

This work can be motivated by several points, in addition to purely intrinsic reasons.

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First off, for the characterization of the infinite dimensional manifold formed by the solutions of the K d V equation, the study of the continuity of the map F is necessary, before looking at the more complex question of its differentiability.

Secondly, the study of the continuity of F and of F^{-1} is a useful tool for looking at the approximation of solutions of the K d V equation by known solutions.

Finally, in studying the closure of the set of the admissible scattering data, and using the continuity of F^{-1}, one could envisage to construct, by passing to the limit, mathematical objects more general than functions, solutions of K d V in some generalized sense.

In the first part, we shall recall the map F. On this subject one can consult classical references such as: [**3**] and [**4**] and the articles cited there.

In the second part, we discuss certain natural topologies on the space of the functions $q(x, t)$ and on the space of scattering data topologies which make the map F_0 associated to F continuous and in the third part we study the continuity of F.

1. The scattering theory. Let us consider the Schroedinger equation

(3) $$-f_{xx} + qf = k^2 f$$

where $q(x)$ is a locally integrable function such that: $q \in L_1^1$ i.e.,

$$\|q\|_1^1 = \int_{-\infty}^{\infty} (1 + |x|)\,|q(x)|\,dx < \infty.$$

Let $f_1(x, k)$ and $f_2(x, k)$ be the Jost functions, i.e., solutions of (3) with the following asymptotic behaviour

$$f_1(x, k) \simeq e^{ikx} \quad x \to +\infty$$
$$f_2(x, k) \simeq e^{-ikx} \quad x \to -\infty.$$

Let us note that if $k \neq 0$, we obtain for the Wronskian:

$$W(f_1(x, k), f_1(x, -k)) = -2ik$$

and

$$W(f_2(x, k), f_2(x, -k)) = +2ik.$$

In particular, the solutions $f_1(x, k)$, $f_1(x, -k)$ are linearly independent if $k \neq 0$. Hence we can write $f_2(x, k)$ as a linear combination of $f_1(x, k)$ and $f_1(x, -k)$:

$$f_2(x, k) = b_1(k)f_2(x, k) + a_1(k)f_2(x, -k).$$

Similarly we obtain

$$f_1(x, k) = b_2(k)f_2(x, k) + a_2(k)f_2(x, -k).$$

From this we conclude easily:

$$a_1(k) = \frac{1}{2ik} W(f_1(x, k), f_2(x, k)) = a_2(k),$$

$$b_1(k) = -b_2(k).$$

Let us set $a(k) = a_2(k)$ and $b(k) = b_2(k)$.

Let us consider, now, the map F_0 which to $q(x)$ associates the couple of functions $(a(k), b(k))$; one shows (see **[1]** and the references therein) that, if $q(x, t)$ is a solution of the equation (1), such that $q(x, 0) = q(x)$, then

$$F_0(q(x, t)) = (a(k, t), b(k, t)), \quad \text{and}$$

$$a(k, t) = a(k, 0)$$

$$b(k, t) = e^{-k^3 t} b(k, 0).$$

Let us introduce, to simplify the presentation, the two functions

$$m_1(x, k) = e^{-ikx} f_1(x, k) \quad \text{and} \quad m_2(x, k) = e^{ikx} f_2(x, k).$$

These functions are solutions, respectively, of the differential equations

$$m_1'' + 2ikm_1' = qm_1 \quad \text{and} \quad m_2'' - 2ikm_2' = qm_2$$

and satisfy the boundary conditions

$$\lim_{x \to \infty} m_1(x, k) = 1 \quad \text{and} \quad \lim_{x \to -\infty} m_2(x, k) = 1.$$

One obtains

$$m_1(x, k) = b_2(k) e^{-2ikx} m_2(x, k) + a_2(k) m_2(x, -k)$$

and

$$m_2(x, k) = b_1(k) e^{2ikx} m_1(x, k) + a_1(k) m_1(x, -k).$$

Let us introduce the kernel

$$D_k(y) = \frac{1}{2ik}(e^{2iky} - 1);$$

we obtain the $m_1(x, k)$ as a solution of the integral equation

$$\text{(4)} \qquad m_1(x, k) = 1 + \int_x^\infty D_k(u - x) q(u) m_1(u + k) du$$

for each k such that $\operatorname{Im} k \geqq 0$. Similarly $m_2(x, k)$ is a solution of the integral equation

$$m_2(x, k) = 1 + \int_{-\infty}^x D_k(x - u) q(u) m_2(u, k) du.$$

Taking the limit as $x \to +\infty$, and making the identifications we obtain:

$$b(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2iku} q(u) m_1(u, k) du$$

and

$$(5) \quad a(k) = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} q(u) m_1(u, k) du$$

$\operatorname{Im} k \geqq 0, k \neq 0.$

In the following, we are going to state certain well known properties of the solutions $m_1(x, k)$, in order to establish the behaviour of $b(k)$ and $a(k)$.

In the sequel, when there is no ambiguity, we shall denote $m_1(x, k)$ by $m(x, k)$.

Let us return to the equation (4), which is an equation of Volterra; we solve it by iteration, and obtain after some majorations:

$$(6) \quad \begin{aligned} |m(x, k)| &\leqq 1 + \frac{1}{|k|} \eta(x) e^{\eta(x)/|k|} \\ |m(x, k)| &\leqq 1 + \gamma(x) e^{\gamma(x)}. \end{aligned}$$

where:

$$\eta(x) = \int_x^{\infty} |q(u)| du$$

$$\gamma(x) = \int_x^{\infty} (u - x) |q(u)| du.$$

We have also from (4)

$$|m(x, k)| \leqq 1 + \int_0^{\infty} u |q(u)| \, |m(u, k)| du - x \int_x^{\infty} |q(u)| \, |m(x, k)| du.$$

But as for $u \geqq 0$, $1 + \gamma(u) e^{\gamma(u)} \leqq 1 + \gamma(0) e^{\gamma(0)}$, this gives

$$|m(x, k)| \leqq 1 + \gamma(0) + \gamma^2(0) e^{\gamma(0)} - x \int_x^{\infty} |q(u)| \, |m(u, k)| du.$$

Let

$$C = 1 + \gamma(0) + \gamma^2(0) e^{\gamma(0)}.$$

Then $C \geqq 1$. Let us set

$$M(x, k) = \frac{m(x, k)}{C(1 + |x|)}$$

and

$$p(x) = (1 + |x|)q(x)|,$$

where $p(x)$ is integrable. We obtain

$$\begin{aligned}|M(x, k)| &\leqq \frac{C}{C(1 + |x|)} - \frac{x}{C(1 + |x|)} \int_x^\infty p(u)\,|M(u, k)|du \\ &\leqq 1 + \frac{|x|}{C(1 + |x|)} \int_x^\infty p(u)\,|M(u, k)|du \\ &\leqq 1 + \int_x^\infty p(u)\,|M(u, k)|du.\end{aligned}$$

Resolving by iteration one obtains

$$\begin{aligned}|M(x, k)| &\leqq \exp\left(\int_x^\infty p(u)du\right) \\ &\leqq \exp \int_{-\infty}^\infty (1 + |u|)\,|q(u)|du = \exp \|q\|_1^1;\end{aligned}$$

i.e.,

(7) $$|m(x, k)| \leqq \exp \|q\|_1^1 C(1 + |x|).$$

2. Continuity of the map F_0. Let now $\mathscr{M}$ be the space of all the complex valued functions $f(z)$ which are analytic in the half-plane $\operatorname{Im} z > 0$ and continuous in $\operatorname{Im} z \geqq 0$ except at the point $z = 0$. Let us define on $\mathscr{M}$ the topology of uniform convergence on compact sets. Let $\mathscr{N}$ be the space of functions $g(x)$, continuous on the real line except perhaps at $x = 0$, and equipped with the topology of uniform convergence on compact sets.

THEOREM 1. *The map F_0 that associates to q the couple $(a(k), b(k))$ is a continuous function from L_1^1 into the product space $\mathscr{M} \times \mathscr{N}$.*

Proof. Let us note first that $a(k)$ is indeed an element of $\mathscr{M}$. In fact,

$$a(k) = 1 - \frac{1}{2ik} \int_{-\infty}^\infty q(u)m(u, k)du.$$

As $m(u, k)$ is analytic in the half plane $\operatorname{Im} k \geqq 0$ and continuous in $\{k|\operatorname{Im} k \geqq 0\} - \{0\}$, this is true about $a(k)$ as well.

Similarly, starting from the expression

$$b(k) = \frac{1}{2ki} \int_{-\infty}^\infty e^{2ikx}q(x)m(k, x)dx,$$

we conclude that $b(k)$ is continuous on $\mathbf{R} - \{0\}$.

Let now $q_n(x)$ be a sequence in L_1^1, converging to $q(x)$. Let $m_n(x, k)$ be the solution of the equation

$$m'' + 2ikm' = q_n m$$

such that

$$\lim_{x \to \infty} m_n(x, k) = 1.$$

Set also

$$(a_n(k), b_n(k)) = F_0(q_n).$$

From (5) we obtain

$$a(k) - a_n(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} (q_n(u)m_n(u, k) - q(u)m(u, k))du$$

that is:

$$|a(k) - a_n(k)| \leqq \frac{1}{2|k|} \left| \int_{-\infty}^{\infty} |q_n(u)| m_n(u, k) - m(u, k)|du \right| + \frac{1}{2|k|} \int_{-\infty}^{\infty} |m(u, k)| |q_n(u) - q(u)|du.$$

From (7) we have:

$$(8) \qquad |a(k) - a_n(k)| \leqq \frac{1}{2|k|} \int_{-\infty}^{\infty} |q_n(u)| |m_n(u, k) - m(u, k)|du + \frac{1}{2|k|} C\|q_n - q\|_1^1$$

but from (4) we obtain

$$(m_n(x, k) - m(x, k)) = \int_x^{\infty} D_k(u - x)(q_n(u)m_n(u, k) - q(u)m(u, k))du$$

$$|m_n(x, k) - m(x, k)| \leqq \int_x^{\infty} |D_k(u - x)| |q_n(u)| |m_n(u, k) - m(u, k)|du + \int_{-\infty}^{+\infty} |D_k(u - x)| |q_n(u) - q(u)| |m(u, k)|du$$

$$|m_n(x, k) - m(x, k)| \leqq \frac{1}{|k|} \int_x^{\infty} |q_n(u)| |m_n(u, k) - m(u, k)|du + \frac{C}{|k|}\|q_n - q\|_1^1.$$

Resolving this inequality of Volterra by iteration we have

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$$|m_n(x, k) - m(x, k)| \leqq \frac{C}{|k|} \|q_n - q_1\|_1^1 \exp \frac{1}{|k|} \int_x^\infty |q_n(u)| du$$

or

$$(9) \quad |m_n(x, k) - m(x, k)| \leqq \frac{C}{|k|} \|q_n - q\|_1^1 e^{1/|k| \|q_n\|_1}.$$

Substituting (9) into (8) we have

$$|a(k) - a_n(k)| \leqq \frac{C}{2|k|} \|q_n - q\|_1^1 \left[1 + \frac{1}{|k|} \|q_n\|_1 e^{\|q_n\|_1/k}\right]$$

but

$$\|q_n\|_1 \leqq \|q_n - q\|_1 + \|q\|_1 \leqq \|q_n - q\|_1^1 + \|q\|_1$$

from which

$$|a(k) - a_n(k)| \leqq \frac{C}{2|k|} \|q_n - q\|_1^1 \left[1 + \frac{\|q_n - q\|_1^1}{|k|} e^{\|q_n - q\|_1^1} e^{\|q\|_1} + \frac{\|q\|_1}{|k|} e^{\|q\|_1} e^{\|q_n - q\|_1^1}\right].$$

This implies that if K is any compact set such that

$$K \subseteq \{k \mid \operatorname{Im} k \geqq 0\} - \{0\},$$

if $q_n \to q$ in the topology of L_1^1, $a_n(k)$ will converge uniformly in K toward $a(k)$. It is clear that we have similar inequalities for $b(k)$, when $\operatorname{Im} k = 0$, $k \neq 0$, which proves the theorem.

COROLLARY. *Let us consider now the solutions $q(x, t)$ of the* K d V *equation, such that $q(x, 0) = q(x)$, and the map F_0 that associates $(a(k, t), b(k, t))$ to $q(x, t)$. We obtain immediately the following results*:

(a) *If $q_n(x)$ converges in L_1^1 toward $q(x)$, then $F_0(q_n(x, t))$ will tend toward $F_0(q(x, t))$ in the topology of $\mathcal{M} \times \mathcal{N}$, uniformly in t.*

(b) *If for a fixed t, $q_n(x, t)$ tends to $q(x, t)$ in L_1^1, then $F_0(q_n(x, t))$ tends to $F_0(q(x, t))$ in $\mathcal{M} \times \mathcal{N}$.*

3. Continuity of the scattering map.

THEOREM 2. *Let $q_n(x)$ be a sequence of functions that converges toward $q(x)$ in L_1^1, let*

$$r(k) = \frac{a(k)}{b(k)} \text{ and } r_n(k) = \frac{b_n(k)}{a_n(k)}$$

and let $\alpha_1, \ldots, \alpha_l$ be the zeros of $a(k)$. Then $\{r_n(k)\}$ converges to $r(k)$ uniformly on compact sets in $\mathbf{R} - \{0\}$, *and for all i, there exists a complex* *sequence $\alpha_{i,n}$ such that*

$$\lim_{n\to\infty} \alpha_{i,n} = \alpha_i$$

and such that $\alpha_{i,n}$ is a zero of $a_n(k)$.

Proof. As the functions $a_n(k)$ and $a(k)$ are $\neq 0$ outside of the axis $\operatorname{Re} k = 0$ (see [**3**] and [**4**]), it is clear that, taking into account the Theorem 1, the quotient $b_n(k)/a_n(k)$ converges uniformly on all compact sets of $\mathbf{R} - \{0\}$ to $b(k)/a(k)$.

We know that the number of zeros of $a(k)$ is finite on the real axis. If α_i is a zero of $a(k)$, there exists a closed disk centered at α_i with radius r, such that $a(k) \neq 0$ elsewhere in the disk. On the other hand, as $a_n(k)$ is a sequence of holomorphic functions that converges uniformly on compact sets in the upper half plan to the function $a(k)$, according to the Hurwitz's theorem [**6**], starting from a certain N all the functions $a_n(k)$ have a unique zero inside the disk. Let $\alpha_{i,n}$ be these zeros, i.e., $a_n(\alpha_{i,n}) = 0$. Let us show that $\alpha_{i,n} \to \alpha_i$.

Given $\epsilon > 0$, consider the disk centered at α_i and of radius ϵ. For ϵ sufficiently small, according to Hurwitz's theorem, $\exists N_0$ such that for $n > N_0$, $a_n(k)$ has only one zero inside the disk. Hence

$$n > N_0 \Rightarrow |\alpha_i - \alpha_{i,n}| < \epsilon.$$

The α_i can be considered as eigenvalues of the self-adjoint operator associated with the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + q(x)\psi = \alpha\psi$$

in the Hilbert space of the square integrable functions on the real line. Let f_j be a Jost function associated with the eigenvalue α_j. The normalization coefficient c_j is defined by the relation

$$\frac{1}{c_j} = \int_{-\infty}^{\infty} (f_j(x, i\alpha_j))^2 dx$$

or, using the functions $m_j = e^{\alpha_j x} f_j(x, i\alpha_j)$

$$\frac{1}{c_j} = \int_{-\infty}^{\infty} e^{-2\alpha_j x} m_j^2(x, i\alpha_j) dx.$$

THEOREM 3. *Let $q_n(x)$ be a sequence of functions converging to $q(x)$ in L_1^1, $\alpha_1, \dots, \alpha_l$ the zeros of $a(k)$, and let $\alpha_{n,j}$ be a complex sequence, such that*

$$a_n(\alpha_{n,j}) = 0 \textit{ for all } j \textit{ and}$$

$$\lim_{n\to\infty} \alpha_{n,j} = \alpha_j \quad \textit{for all } j.$$

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Let c_j be the normalization coefficient that corresponds to α_j, $c_{n,j}$ the one corresponding to $\alpha_{n,j}$. Then

$$\lim_{n\to\infty} c_{n,j} = c_j.$$

Proof. Let $f_{n,i}(x, k)$, $i = 1, 2$, be the Jost functions for $q_n(x)$. If $k = \alpha_{n,j}$ then

$$(2i\alpha_{n,j})^{-1} W(f_{n,1}(x, \alpha_{n,j}), f_{n,2}(x, \alpha_{n,j})) = \alpha_n(\alpha_{n,j}) = 0$$

and

$$f_{n,1} \text{ is proportional to } f_{n,2},$$

i.e.,

$$f_{n,2}(x, \alpha_{n,j}) = \gamma_{n,j} f_{n,1}(x, \alpha_{n,j}).$$

Furthermore we have (see [**3**] page 148)

$$-\dot{f}''_{n,2} + q\dot{f}_{n,2} = k^2\dot{f}_{n,2} + 2kf_{n,2}k \neq 0$$

where

$$\dot{f}_{n,2} \text{ means } \frac{d}{dk} f_{n,2}(x, k).$$

Then

$$\frac{d}{dx} W(f_{n,1}, \dot{f}_{n,2}) = 2kf_{n,1}f_{n,2}.$$

For $k = \alpha_{n,j}$ we obtain, $f_{n,1}$ being proportional to $f_{n,2}$,

$$2\alpha_{n,j} \int_{-\infty}^{x} f_{n,1}(t, \alpha_{n,j}) f_{n,2}(t, \alpha_{n,j}) dt = W(f_{n,1}, \dot{f}_{n,2}).$$

Similarly we have

$$2\alpha_{n,j} \int_{x}^{\infty} f_{n,1}(t, \alpha_{n,j}) f_{n,2}(t, \alpha_{n,j}) dt = W(\dot{f}_{n,1}, f_{n,2}).$$

So

$$\begin{aligned}
2\alpha_{n,j} &\int_{-\infty}^{+\infty} f_{n,1}(t, \alpha_{n,j}) f_{n,2}(t, \alpha_{n,j}) dt \\
&= W(\dot{f}_{n,1}, f_{n,2}) + W(f_{n,1}, \dot{f}_{n,2}) \\
&= \frac{d}{dk} W(f_{n,1}(x, k), f_{n,2}(x, k)) |_{k=\alpha_{n,j}} \\
&= \frac{d}{dk}(2ika_n(k)) |_{k=\alpha_{n,j}} \\
&= [2ia_n(\alpha_{n,j}) + 2i\alpha_{n,j}\dot{a}_n(\alpha_{n,j})]
\end{aligned}$$

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whence

$$\dot{a}_n(\alpha_{n,j}) = \frac{1}{i}\int_{-\infty}^{+\infty} f_{n,1}(t, \alpha_{n,j}) f_{n,2}(t, \alpha_{n,j}) dt.$$

Or

$$\dot{a}_n(\alpha_{n,j}) = \frac{\gamma_{n,j}}{i}\int_{-\infty}^{+\infty} f_{n,1}^2(x, \alpha_{n,j}) dx.$$

Then

$$\frac{1}{c_{n,j}} = \frac{i}{n,j}\,\dot{a}_n(\alpha_{n,j}) \quad \text{for all } j.$$

Similarly we have

$$\frac{1}{c_j} = \frac{i}{\gamma_j}\dot{a}(\alpha_j).$$

Let us show now that for all k, $\operatorname{Im} k > 0$, $\dot{a}_n(k)$ converges to $\dot{a}(k)$ uniformly on compact sets, when $q_n(x)$ converges to $q(x)$ in L_1^1: we have

$$[\dot{a}(k) - \dot{a}_n(k)] = -\frac{1}{k}[a(k) - a_n(k)] + \int_{-\infty}^{+\infty}\left[\frac{q_n\dot{m}_n - q\dot{m}}{2ik}\right]du$$

$$\frac{1}{2ik}\int_{-\infty}^{+\infty}(q_n\dot{m}_n - q\dot{m})du = \frac{1}{2ik}\int_{-\infty}^{+\infty}[q_n - q]\dot{m}\,du + \frac{1}{2ik}\int_{-\infty}^{+\infty} q_n[\dot{m}_n - \dot{m}]du.$$

Let us estimate $|\dot{m}(x, k)|$. From

$$\dot{m}(x, k) = \int_x^{\infty} D_k(t - x)q(t)\dot{m}(t, k)dt + \int_x^{\infty}\dot{D}_k(t - x)q(t)m(t, k)dt$$

using the estimate

$$|D_k(y)| < \frac{1}{|k|} \quad \text{for } \operatorname{Im} y \geqq 0 \text{ and}$$

$$|\dot{D}_k(y)| \leqq 1 + \frac{2}{|k|^2} \quad \text{for } \operatorname{Im} y \geqq 0$$

we obtain

$$|\dot{m}(x, k)| \leqq \frac{1}{|k|} \int_x^\infty |q(t)| \ |\dot{m}(t, k)| dt + \left(1 + \frac{2}{|k|^2}\right) c \|q\|_1^1 e^{\|q\|_1^1},$$

where $c = 1 + \gamma(0) + \gamma^2(0)e^{\gamma(0)}$.

Solving by iteration we have $|\dot{m}(x, k)| \leqq E(k)$ where

$$E(k) = \left(1 + \frac{2}{|k|^2}\right) c \|q\|_1^1 \exp\left(\|q\|_1^1 + \frac{\|q\|_1}{|k|}\right).$$

Therefore

$$\left|\frac{1}{2ik} \int_{-\infty}^{+\infty} [q_n - q] m du\right| \leqq \frac{1}{2|k|} E(k) \|q_n - q\|_1,$$

and

$$|\dot{m}_n(x, k) - \dot{m}(x, k)| \leqq \frac{1}{|k|} \int_x^\infty |q_n(t)| \ |\dot{m}_n - \dot{m}| dt$$
$$+ \frac{E(k)}{|k|} \|q_n - q\|_1 + \left(1 + \frac{2}{|k|^2}\right) \|q_n\|_1 \frac{c}{|k|} \|q_n - q\|_1^1 e^{\|q_n\|_1/|k|}$$
$$+ \left(1 + \frac{1}{|k|^2}\right) c e^{\|q\|_1^1} \|q_n - q\|_1^1.$$

Solving by iteration we obtain

$$|\dot{m}_n(x, k) - \dot{m}(x, k)| \leqq F(k) \|q_n - q\|_1^1 e^{\|q_n\|_1/|k|},$$

where

$$F(k) = \frac{E(k)}{k} + \left(1 + \frac{2}{|k|^2}\right) c \left(\frac{\|q_n\|_1}{|k|} \exp\left(\frac{\|q_n\|_1}{|k|}\right) \exp \|q\|_1^1\right).$$

Therefore

$$\left|\frac{1}{2ik} \int_{-\infty}^{\infty} q_n [\dot{m}_n - \dot{m}] du\right| \leqq \|q_n\|_1 F(k) e^{\|q_n\|_1/k} \|q_n - q\|_1^1$$

and

$$|\dot{a}(k) - \dot{a}_n(k)| \leqq \frac{1}{|k|} |a(k) - a_n(k)| + \frac{1}{2|k|} E(k) \|q_n - q\|_1$$
$$+ F(k) e^{\|q_n\|_{1/k}} \|q_n - q\|_1^1.$$

As $\dot{a}(k)$ and $\dot{a}_n(k)$ are continuous for Im $k > 0$ (see **[3]**) we conclude that $\dot{a}_n(\alpha_{n,j})$ will converge to $\dot{a}(\alpha_j)$ when q_n converges to q in L_1^1.

Let us show now that $\gamma_{n,j}$ converges to γ_j. Indeed there exists an x_0 such that

$$f_1(x_0, \alpha_j) \neq 0.$$

Furthermore from (9) we conclude that $f_{n,1}(x_0, k)$ converges uniformly to $f_1(x_0, k)$ on compact sets of $\{k \mid \text{Im } k > 0\}$ when q_n converges to q in L_1^1. Then $f_{n,1}(x_0, \alpha_{n,j})$ will converge to $f_1(x_0, \alpha_j)$. By the same argument we prove that $f_{n,2}(x_0, \alpha_{n,j})$ converges to $f_2(x_0, \alpha_j)$. Then

$$\gamma_{n,j} = \frac{f_{n,2}(x_0, \alpha_{n,j})}{f_{n,1}(x_0, \alpha_{n,j})}$$

will converge to γ_j and

$$\frac{i\dot{a}_n(\alpha_{n,j})}{\gamma_{n,j}} \text{ will converge to } \frac{i\dot{a}(\alpha_j)}{\gamma_j}$$

i.e.,

$$\lim_{n\to\infty} c_{n,j} = c_j.$$

Definition (Scattering map). Let F be the map that associates to $q \in L_1^1$ the triple

$$F(q) = (r(k), (\alpha_1, \ldots, \alpha_l), (c_1, \ldots, c_l))$$

as defined above. The map F is called the *scattering* transform. The zeros $\{\alpha_i\}$ of $a(k)$ are ordered on the imaginary axis so that their absolute values form a decreasing sequence.

Definition (Topology τ on the image $F(L_1^1)$. Let us introduce on $F(L_1^1)$ a topology τ saying that a triple sequence

$$(r_n(k), (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{l(n),n}), (c_{1,n}, \ldots, c_{l(n),n}))$$

converges to $(r(k), (\alpha_1, \ldots, \alpha_m), (c_1, \ldots, c_m))$ if

1) $r_n(k)$ converges to $r(k)$ uniformly on the compact sets of $\mathbf{R} - \{0\}$;

2) $$\lim_{n\to\infty} \alpha_{i,n} = \alpha_i$$

with the convention that if $l(n) < m$ then we complete the $l(n)$-triplet $(\alpha_{1n}, \ldots, \alpha_{l(n),n})$ with $m - l(n)$ zeros, and

$$\lim_{n\to\infty} \alpha_{l(n),n} = 0 \quad \text{if } l(n) > m.$$

3) $$\lim_{n\to\infty} c_{i,n} = c_i$$

there we also complete the $l(n)$-triplet with zeros if $m - l(n) > 0$. We impose no conditions on

$$\lim_{n\to\infty} c_{l(n),n}.$$

THEOREM 4. *Let D be a domain in L_1^1, bounded in the topology of L^1; i.e., $\exists d > 0$ such that*

$$D \subseteq \left\{ q \in L_1^1 \middle| \int_{-\infty}^{+\infty} |q(t)|dt < d \right\}.$$

Then F is continuous in D with respect to the topology τ.

Proof. Let q_n be a sequence of functions in D converging to q in L_1^1. Let $a_n(k)$ and $a(k)$ be the corresponding functions. From (5) we conclude

$$|a_n(k) - 1| \leqq \frac{1}{2|k|} \int_{-\infty}^{\infty} |q_n(u)|\ |m_n(u, k)|du$$

and

$$|a(k) - 1| \leqq \frac{1}{2|k|} \int_{-\infty}^{\infty} |q(u)|\ |m(u, k)|du;$$

from (6) we obtain

$$|m_n(u, k)| \leqq 1 + \frac{1}{|k|} \left(\int_{-\infty}^{\infty} |q_n(u)|du \right) e^{1/|k| \int_{-\infty}^{\infty} |q_n(u)|du}$$

i.e.,

$$|m_n(u, k)| \leqq 1 + \frac{d}{|k|} e^{d/|k|};$$

similarly

$$|m(u, k)| \leqq 1 + \frac{d}{|k|} e^{d/|k|}.$$

We then obtain the following inequalities:

$$|a_n(k) - 1| \leqq \frac{d}{2|k|} + \frac{d^2}{|k|^2} e^{d/|k|}$$

and

$$|a(k) - 1)| \leqq \frac{d}{2|k|} + \frac{d^2}{|k|^2} e^{d/|k|}.$$

Hence $\exists \kappa$ such that for $|k| > \kappa$

$$(10) \quad |a_n(k) - 1| < 1 \quad \forall n$$

$$|a(k) - 1| < 1.$$

Let us recall that the zeros of $a(k)$ as well those of $a_n(k)$ are finite in number, simple and are situated on the positive imaginary axis (i.e.,

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Im $k > 0$ and Re $k = 0$).

We conclude that there exists $\kappa > 0$ such that all zeros of $a(k)$ and of $a_n(k)$ are situated on the interval between 0 and $i\kappa$ on the imaginary axis.

Let α_l be the zero of $a(k)$ such that $|\alpha_l|$ is minimum. The zeros of $a_n(k)$ being separated $\forall\epsilon' > 0$, $\epsilon' < |\alpha_l|$, $\exists\epsilon$ such that $\epsilon \leqq \epsilon'$ and is not a zero of $a_n(k)$, $\forall n$. Consider the domain in the complex plane, bounded by the line Re $z = -\delta$, Re $z = \delta$, Im $z = \kappa$, the real axis and the upper half circle $|z| = \epsilon$, Im $z \geqq 0$, $(\delta > \epsilon)$. On the contour of this domain $a(k)$ and $a_n(k)$ are continuous and $\neq 0$; in addition, $a(k)$ and $a_n(k)$ are holomorphic in the domain. Then, by Hurwitz's theorem $\exists N$ such that $n > N \Rightarrow a(k)$ and $a_n(k)$ have the same number of zeros within the domain. Eventually, $a_n(k)$ may have zeros α_n outside the domain, but then $|\alpha_n| < \epsilon$; i.e., the sequence of these zeros converges to 0.

Applying Theorem 2 and Theorem 3 we conclude that F is continuous in the topology τ.

COROLLARY. *Consider now the solution $q(x, t)$ of* K d V *equation, satisfying the initial condition $q(x, 0) = q(x)$. Let*

$$F(q(t)) = (r(k, t), \alpha_j, c_j(t))$$

such as defined in (2). *Then we have immediately the following results*:

(a) *If for a fixed t, the sequence $\{q_n(x, t)\}$ in L_1^1 is uniformly bounded in L^1, and converges in L_1^1 to $q(x, t) \in L_1^1$, then $F(q_n(x, t))$ converges to $F(q(x, t))$ in the topology τ.*

(b) *If $q_n(x)$ converges in L_1^1 to $q(x)$ and the sequence $q_n(x)$ is uniformly bounded in L^1, then $F(q_n(x, t))$ converges to $F(q(x, t))$ in the topology τ uniformly in t.*

THEOREM 5. *Let $G_i (i = 1, 2)$ be domains in $L_2^1 \cap L_1^1$, bounded in the topology of L^1; i.e., $\exists d$ such that*

$$G_1 = \{q \in L_2^1 | \int_{-\infty}^{\infty} |q(t)|dt < d \quad \text{and} \quad \int_{-\infty}^{\infty} q(x)m(x, 0)dx = 0\}$$

$$G_2 = \{q \in L_2^1 | \int_{-\infty}^{\infty} |q(t)|dt < d \quad \text{and} \quad \int_{-\infty}^{\infty} q(x)m(x, 0)dx \neq 0\}.$$

Then F is a continuous map from G_i equipped with topology of L_1^1 to

$$C \times \bigoplus_{i=1}^{\infty} \mathbf{R}^i \times \bigoplus_{i=1}^{\infty} \mathbf{R}^i$$

where C represents the space of complex valued functions continuous on

$\mathbf{R} - \{0\}$, *considered with the topology of uniform convergence on the compact sets of* $\mathbf{R} - \{0\}$. *On* $\bigoplus_{i=1}^{\infty} \mathbf{R}^i$ *we consider the following norm*

$$\|\alpha_{1,1}, (\alpha_{2,1}, \alpha_{2,2}), \ldots, (\alpha_{l,1}, \ldots, \alpha_{l,l})\| = \sum_{i=1}^{l} \sup_j |\alpha_{i,j}|.$$

Proof. From (10) we conclude that there exists $\kappa > 0$ such that all the zeros of $a(k)$ and $a_n(k)$ are situated on the interval $(0, i\kappa)$ of the imaginary axis. From [3, Theorem 1] it is known that if $q \in L_2^1$ then $\frac{\partial}{\partial k} m(x, k)$ exists and is continuous for $\operatorname{Im} k \geqq 0$. Let us suppose that q and q_n are in G_1; therefore $a(k)$ is continuous for $k = 0$ and

$$\lim_{k \to 0} a(k) \neq 0.$$

Hence on the contour of the rectangle with vertices $(-\delta, \delta, \delta + i\kappa, -\delta + i\kappa)$ $a(k)$ and $a_n(k)$ are continuous and $\neq 0$; in addition $a(k)$ and $a_n(k)$ are holomorphic in the interior of this rectangle. By Hurwitz's theorem $\exists N$ such that $n > N \Rightarrow a(k)$ and $a_n(k)$ have the same number of zeros within the rectangle. As all the zeros are within the rectangle, we conclude that, starting from a certain N, $a_n(k)$ and $a(k)$ have the same number of zeros. The continuity of F now follows directly from Theorem 2 and Theorem 3.

Let us suppose now that q and q_n are such that

$$\nu = \int_{-\infty}^{+\infty} q(x) m(x, 0) dx \quad \text{and} \quad \nu_n = \int_{-\infty}^{+\infty} q_n(x) m_n(x, 0) dx$$

then ν and ν_n are finite and $\neq 0$. We have:

$$ka(k) = k - \frac{\nu}{2i} - \frac{1}{2i} \int_{-\infty}^{+\infty} q(t)(m(t, k) - m(t, 0))dt;$$

$m(x, k)$ is continuous for $\operatorname{Im} k \geqq 0$ then $ka(k)$ and $ka_n(k)$ are also continuous on the contour of the rectangle with vertices $(-\delta, \delta, \delta + i\kappa, -\delta + i\kappa)$. $ka(k)$ and $ka_n(k)$ are $\neq 0$ on this contour, and the zeros of $\kappa a(k)$ are the same as the zeros of $a(k)$. Then by Hurwitz's theorem, starting from a certain N, $a_n(k)$ and $a(k)$ have the same number of zeros. This implies that F is continuous (Theorem 2 and Theorem 3).

COROLLARY. *We have immediately the following proposition for* $q(x, t)$, *the solution of* K d V *equation such that* $q(x, 0) = q(x)$

(a) *if for a fixed t, the sequence* $q_n(x, t)$ *in* G_i, $(i = 1, 2)$ *converges in* L_1^1 *to* $q(x, t)$ *in* G_i *then* $F(q_n(x, t))$ *converges to* $F(q(x, t))$ *in*

$$C \times \bigoplus_{i=1}^{\infty} \mathbf{R}^i \times \bigoplus_{i=1}^{\infty} \mathbf{R}^i.$$

(b) *if* $q_n(x)$ *converges in* L_1^1 *to* $q(x)$ *and if* $q_n(x)$ *and* $q(x) \in G_i$ $(i = 1, 2)$ *then* $F(q_n(x, t))$ *converges to* $F(q(x, t))$ *in the topology of*

$$C \times \bigoplus_{i=1}^{\infty} \mathbf{R}^i \bigoplus_{i=1}^{\infty} \mathbf{R}^i$$

uniformly in t.

Conclusion. In this article we have given conditions for the continuity of the scattering transformation. In order for us to obtain more general results for the continuity of the inverse transformation, we would have to establish conditions for the differentiality of the scattering transformation and make use of an inverse function theorem.

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