6

The classical motion of the massless relativistic string

6.1 Introduction

In this chapter we start to consider the properties of the *massless relativistic string* (the MRS). We will begin with a simple situation in which the MRS plays the role of a constant force field, acting upon a 'charge' and an 'anticharge' placed at the endpoints of an open MRS. This means that the motion will be in one space dimension along the force direction. We will refer to it as the *yoyo-mode* for reasons that will become clear when it is exhibited.

In later chapters we will come back to more complex modes involving several dimensions. All these modes are used in the Lund model as semi-classical models for different high-energy interactions between hadrons. The yoyo-mode is used both to describe an e^+e^- annihilation event and as a simple model for stable hadrons. In the last section of this chapter we provide a possible dynamical analogy between the QCD vacuum and superconductivity as a justification for using string dynamics to describe hadronic states and interactions.

In the yoyo-mode the two charges at the endpoints of the string move like point particles, i.e. the momentum of the state is localised in these endpoint particles of the MRS force field. At any moment the total energy of the state can be decomposed into the energy in the force field, corresponding to a linearly rising potential, and the kinetic energies of the particles at the endpoints. We will use the situation to exhibit in detail the *causality* and the *relativistic covariance* properties of the MRS.

In the Lund model the endpoints of an open MRS are always identified with triplet, 3, or antitriplet, $\overline{3}$, color charges, i.e. with quark, q, or antiquark, $\overline{q}$, properties. In connection with the description of baryonic particles, cf. Chapter 13, we will consider more complex charge configurations.

https://doi.org/10.1017/9781009401296.006 Published online by Cambridge University Press

6.2 The MRS as a constant force field

1 The equations of motion

The equations of motion in relativistic particle dynamics are, in general, complex in a consistent theory. The finiteness of the maximal velocity, i.e. that of light, implies a *causality requirement*. A message about changes in the system, such as e.g. the change in the state of motion of a charge somewhere, takes a finite time to be transmitted to any other part of the system. Consequently, the reaction of the system to the change, i.e. the ensuing force action, is of a retarded character.

More precisely, some cause at the origin at time t_0 will affect what happens at a point $\mathbf{R}$ only after a message has been able to reach that far. If this moves with the velocity of light, $c = 1$, in a straight line, it will cause an effect at time t with $t = t_0 + |\mathbf{R}|$. The calculations including the retarded times then become rather complicated.

There is one particular situation, that of a *constant force*, that is easy to work with (because then the retardation effects are not noticeable). The historical start of what is now known as the Lund string model was based upon the consideration of such a force, [14]. We only later learned that the ensuing motion is a simple variety of the modes of the MRS [24].

If we consider the motion of a relativistic particle in space-time (t, x), with rest mass m, energy E and momentum p, under the influence of a constant force $-\kappa$, we have the force equation

$$\frac{dp}{dt} = -\kappa. \tag{6.1}$$

The solution is evidently

$$p = p(t) = p_0 - \kappa t \equiv \kappa(t_0 - t) \tag{6.2}$$

The velocity of the particle is

$$\frac{dx}{dt} = \frac{p}{E} = \frac{dE}{dp}, \quad E = \sqrt{p^2 + m^2} \tag{6.3}$$

(The first equation of (6.3) corresponds to one of Hamilton's equations, the hamiltonian being given by the relativistic particle energy.)

From Eqs. (6.1) and (6.3) it is possible to obtain an equation for the variation of the energy with respect to the space coordinate, if we use the chain rule for differentiation:

$$\frac{dE}{dx} = \left(\frac{dE}{dp}\right)\left(\frac{dp}{dt}\right)\frac{dt}{dx} = \frac{dp}{dt} = -\kappa \tag{6.4}$$

This equation has, similarly, a simple solution:

$$E = E(x(t)) = E_0 - \kappa x \equiv \kappa(x_0 - x) \tag{6.5}$$

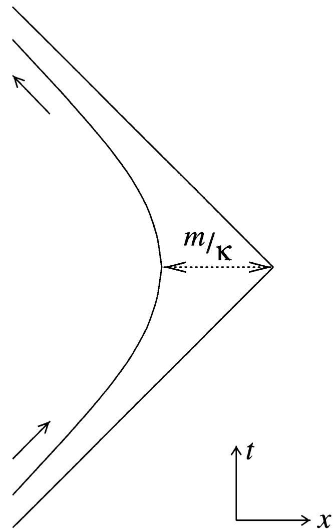

Fig. 6.1. The motion in space and time of a particle with mass m under the influence of a constant force $-\kappa$. The distance between the hyperbola and the intersection (t_0, x_0) between the asymptotes is m/κ.

From the relationship between energy, momentum and mass we conclude that the orbit of the motion is

$$m^2 = E^2 - p^2 = \kappa^2 \left[(x_0 - x)^2 - (t_0 - t)^2 \right] \tag{6.6}$$

i.e. a hyperbola in space-time, centred at (t_0, x_0) and with a size parameter m/κ (see Fig. 6.1).

At large negative times the particle comes in from the region of large negative space coordinates with its momentum pointing along the positive coordinate axis. The momentum decreases and the particle is, at time $t = t_0$, momentarily at rest at the *classical turning-point* $x - x_0 = -m/\kappa$. Afterwards it moves with increasingly negative momentum back to large negative space coordinates.

We note that if the mass vanishes then the particle will move along the lightcones $|t - t_0| = x_0 - x$ throughout and it will look as though it 'bounces' back (changing from velocity $+c$ to $-c$ with vanishing energy and momentum at the origin (t_0, x_0)).

We will use massless particles from now on because of the simplifications in the ensuing pictures of the motion. We would like to stress, however, that the dynamics we are going to consider is basically independent of this assumption (cf. the considerations in Chapter 12).

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2 The Schwinger model and confinement

A particularly interesting dynamical situation arises when there is a constant force and a linearly rising potential; this occurs in one-spacedimensional electrodynamics. There are also in three space dimensions situations that can be approximated by one-dimensional dynamics, e.g. the field between two condensor plates.

Then the usual four-vector potential $A_\mu = (A_0, \mathbf{A})$ can by a gauge choice be arranged so that only the scalar potential $A_0 \equiv V$ is nonvanishing. The single component of the electric field $\mathscr{E} = -dV/dx$ will in a charge-free region fulfil Gauss's law, i.e.

$$\frac{d^2V}{dx^2} = 0 \tag{6.7}$$

which means exactly a linear potential. This constant force is approximately realised in a capacitor.

A quantised version of one-dimensional electrodynamics was investigated by Schwinger, [101]. He was able to show that an electric field coupled to massless fermion particles is (essentially, i.e. leaving aside some peculiar modes) *equivalent to a free, non-interacting, but massive, quantum field theory.*

The quanta of this field are massive and electrically neutral. Their mass is a function of the electric charge, $m^2 = g^2/\pi$. Note that the charge g, as defined by a gaussian 'integral' (in a one-dimensional world there are no transverse dimensions to integrate over)

$$g = \mathscr{E} \tag{6.8}$$

does not have the same dimensions as in the usual three-dimensional case. The dimensions of the electric field $\mathscr{E}$ can be read out from the usual energy density requirement, that half the square of the field strength is equal to the energy density, $dE/dx = \mathscr{E}^2/2$. This means that the electric field has (energy) dimensions $\dim \mathscr{E}$ equal to 1. Therefore g^2 has the dimensions of a squared mass in this case.

The fact that the quanta are electrically neutral is very surprising because it seems as if the original electric charges have vanished. It turns out, however, that *the resulting free-quantum field, ϕ, corresponds to a dipole density*. The original massless fermions are arranged two by two with a positive and a negative charge bound together as a dipole.

This is a realisation of confinement, i.e. the original massless fermions are not observable by themselves but only in particular combinations. In the Schwinger model the original fermions and antifermions can only occur in pairs as bound states with one of each kind.

In this one-dimensional setting this means that one of the charges must be to the left of the other, thereby producing a dipole moment.

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We may compare this with with the case of colored quanta, where the hadronic states are built from color combinations corresponding to no-color singlets. In the Lund fragmentation model the hadrons are modelled by the massless relativistic string, corresponding to a color field spanned between two endpoints associated with quark (color-3) and antiquark (color-$\bar{3}$) charges (the 'ultimate dipoles' in Chapters 7–14).

We will also introduce this dipole character in the description of multi-gluon bremsstrahlung in the dipole cascade model (Chapters 16–18). In this case the emitting current has only a direction and a very small space extension. Similarly in the linked dipole chain model, which describes the properties of deep inelastic scattering (Chapter 20) we will again find the same dipole structures, describing the (squared) wave functions of the hadrons (the structure functions).

In the Schwinger-model case confinement is related to the infinitely rising field energy necessary in order that a charge should be moved away from all the other charges. In our calculations in subsection 1 we found a constant energy density along the whole negative axis beyond where the particle reaches its classical turning point.

We will carry the model on a little further to a simulated particle-production situation, like the one described in [39]. These authors investigated the situation where an external current is composed of a $\pm g$ charged pair. The charges set out at the time $t = 0$ in opposite directions along the single space dimension, the 1-axis. We assume that they move with velocity $v = c = 1$. This means that there is a current (j_0^{ext}, j_1^{ext}), where

$$j_0^{ext} = g\epsilon(x_1)\delta(\epsilon(x_1)x_1 - t), \quad j_1^{ext} = g\delta(\epsilon(x_1)x_1 - t) \tag{6.9}$$

(note the appearance of the sign function $\epsilon = \pm 1$, depending upon the sign of its argument, which describes the way the charges $\pm g$ move). This current corresponds to an external dipole density

$$\phi^{ext} = \frac{g}{m}\Theta(t + x_1)\Theta(t - x_1) \tag{6.10}$$

where the fields are normalised somewhat differently from that in [39]. Our choice is in accordance with the one-dimensional equivalent to the fields introduced in Chapter 3; thus the quantum field ϕ is, using $\omega \equiv \omega(k) = \sqrt{k^2 + m^2}$ and L for the length of the one-dimensional 'quantisation box',

$$\phi(x_1, t) = \sum_k \frac{1}{\sqrt{2\omega L}} \{a \exp i(kx_1 - \omega t) + a^* \exp[-i(kx_1 - \omega t)]\} \tag{6.11}$$

Then we may write out the equations of motion for the fields, the Klein-Gordon equation

$$(\Box + m^2)\phi = m^2\phi^{ext} \tag{6.12}$$

perform all the operations for quantisation and solve the equations to obtain as solution a coherent-state field like those of Chapter 3. Then the quanta in every state will be distributed in a Poissonian manner with an excitation probability described by the mean occupation number $\bar{n}(k)$ (cf. Eqs. (3.25), (3.41)):

$$h = \frac{1}{\sqrt{2\omega L}} \int_0^\infty dt \int dx m^2 \phi^{ext} \exp[i(kx - \omega t)] = \frac{1}{\sqrt{2\omega L}} \left(\frac{2g}{m}\right), \quad (6.13)$$
$$\bar{n} = |h|^2 = \frac{4g^2}{2\omega L m^2} = \frac{2\pi}{\omega L}$$

We have performed the integral in the first line by adding a small negative imaginary part to ω (remember the three-act scenario described in the first section of Chapter 3) and used the relationship between the mass and the coupling constant in the second line.

This means that when we go to the limit $L \to \infty$ we obtain for $\bar{n}$

$$\bar{n}\Delta n \to \bar{n} dk \frac{L}{2\pi} = \frac{dk}{\omega} \equiv dy \quad (6.14)$$

in terms of the rapidity variable y. This is nothing other than the wee parton spectrum of Feynman or, if you like, the distribution of photons in the method of virtual quanta in Chapter 2.

Consequently, an external excitation in the Schwinger model tends to spread as a Poissonian fluctuating production of dipole quanta of average size one quantum per unit rapidity!

3 *The yoyo-mode at rest*

As a classical model corresponding to Schwinger's dipole quanta we consider the motion of a system of two massless particles, a q- and a $\bar{q}$-particle, which are acting upon each other with an attractive constant force.

In Fig. 6.2 we consider the situation when the q and $\bar{q}$ go apart with the same energy E_0 from a common origin but in opposite directions. Such a system evidently has a total energy $E_{tot} = 2E_0$. This coincides with the system mass m as the total momentum vanishes.

According to the results in subsection 1 the particles will move along the two different lightcones and each will lose energy-momentum κ per length and time unit. The starting situation corresponds to the q and $\bar{q}$ each having lightcone energy-momentum $2E_0$.

The ensuing motion can most easily be described in terms of a series of fixed-time snapshots (the lines on the right-hand side of the figure, although the space-time picture given on the left of Fig. 6.2 provides a total view of the system):

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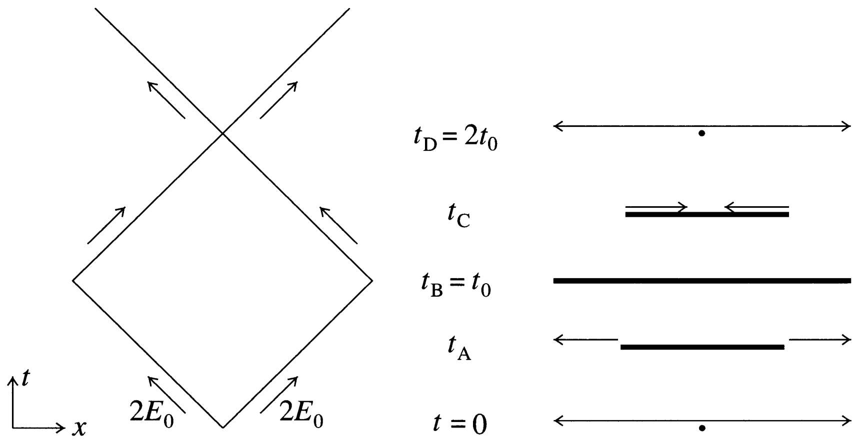

Fig. 6.2. On the left, a space-time diagram for the motion of a $q\bar{q}$-state, in which the particles always have the same energy, i.e. the yoyo-mode at rest. The different times mentioned in the text are shown, with the length of the arrowed lines corresponding to the energy of the particles and the length of the thicker lines corresponding to their separation, i.e. the field in between.

A After a time t_A ($< E_0/\kappa \equiv t_0$) they will be a distance $2t_A$ apart, each with energy $E_0 - \kappa t_A$. The 'lost' energy has gone into the force field, which now has energy κ times its length, i.e. $2\kappa t_A$.

B At time $t_B = t_0 = E_0/\kappa$ they have lost all their energy and they will then turn back and move towards each other.

C At the time ($2t_0 >$) $t_C > t_0$ they will be at a separation $4t_0 - 2t_C$, each with energy $\kappa t_C - E_0$. This energy has been obtained from the force field, which now is dragging them towards each other.

D At time $t_D = 2t_0$ they will meet again but this time they have exchanged their modes of motion compared to the starting point. The q moves along the original $\bar{q}$-direction and vice versa.

As can be seen by a straightforward extrapolation of the argument, after the time $4t_0 = 4E_0/\kappa \equiv 2E_{tot}/\kappa$ the q- and $\bar{q}$-particles will come back exactly to the starting position. Actually a little thought will tell us that the system is always in the same mode of motion at the times t and $t + 2E_{tot}/\kappa \equiv t + t_{per}$. This fact that *the period of motion is equal to* $t_{per} = 2E_{tot}/\kappa$ *is true for all modes of the* MRS, as we will see later.

Another general property of the MRS is that *the total area* $\tilde{A}$ *spanned by the force field in space-time during one period is related to the squared*

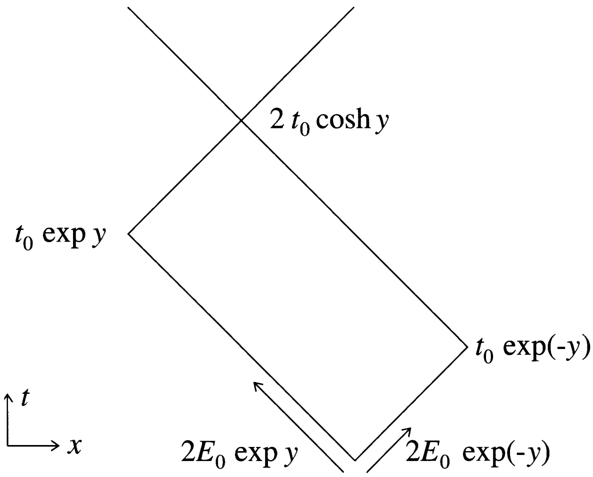

Fig. 6.3. The yoyo-mode after a Lorentz boost along the positive direction; the times and the lightcone energy-momenta from Fig. 6.2 are shown in the new system.

mass of the system. It is easy to see that the relationship is

$$\kappa^2 \tilde{A} = \kappa^2 8 \frac{E_0^2}{2\kappa^2} = m^2 \tag{6.15}$$

in our case; there are exactly eight identical triangles with side and height lengths $t_0 = E_0/\kappa$.

In this particular mode the q and the $\overline{q}$ will continue to move in and out along the lightcones and the name 'yoyo-mode' has a self-evident meaning. The energy and momentum are at different times divided in different ways between the endpoint particles and the force field. We note for future reference that, averaged over a period, half of the energy is in the endpoint particles and half of it is in the field. This is the same result for energy sharing between the quarks and the gluons in a hadron that we quoted in Chapter 5 from the experimental results.

4 Lorentz covariance and causality properties

The model is Lorentz-covariant; we will now demonstrate this by an explicit calculation.

We will consider the situation after we have boosted the system (see Fig. 6.3) longitudinally, i.e. along its axis, with the rapidity y. Then the q-particle, which moves along the positive direction, will by the corre-

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sponding Lorentz transformation change its original (positive) lightcone component $2E_0$ to $2E_0 \exp(-y)$ according to the results in Chapter 2. For the $\overline{q}$ we obtain correspondingly for the negative lightcone component that $2E_0 \to 2E_0 \exp y$.

Thus the total system energy, which at the origin is completely in the $q\overline{q}$-pair, changes from E_{tot} to $E_{tot} \cosh y \equiv E'_{tot}$. The system is now moving with a total momentum $-E_{tot} \sinh y \equiv P'_{tot}$.

It is not obvious that the force equation, Eq. (6.1), is Lorentz-invariant. But it is easy to show this property for our massless particles, which move along the lightcones $x = \pm t$ with energies and momenta $E = \pm p$; the plus and minus signs are valid for particles moving to the right and the left, respectively. In this case the time and the momentum component of such a particle in a different frame are

$$t' = t \exp(\pm y), \quad p' = p \exp(\pm y) \tag{6.16}$$

and we immediately obtain that

$$\frac{dp'}{dt'} = \frac{dp}{dt} \tag{6.17}$$

A more general but also more complex argument could be based upon the properties of the electromagnetic field and its interactions with particles; then all dynamical variables evidently have simple covariance properties. The constant force will occur in one-dimensional QED as mentioned in connection with the discussion of the Schwinger model.

Thus, in the new frame the particles will also be acted upon by a constant force of the same size. The main difference is that the q now has a diminished, and the $\overline{q}$ an increased, original energy. Therefore, in this case they will not stop at the same time. Again using the equal-time snapshot technique we have, from Fig. 6.3,

A The q will stop and turn around at time $t'_A = t_0 \exp(-y)$ (at the space point $t_0 \exp(-y)$) and after that move behind the $\overline{q}$ at a distance $2t_0 \exp(-y)$.

During the ensuing motion the $\overline{q}$ is losing its energy to the field and the q will be increasing its energy from the field, both of them at the same rate. In somewhat vivid language the q 'eats', and the $\overline{q}$ 'spits out', the field as they move along.

B At time $t'_B = t_0 \exp y$ (at the spot $x' = -t_0 \exp y$) the $\overline{q}$ has used up its energy and turns around towards the q.

From Fig. 6.3 we also deduce the following three properties:

C1 The two particles will meet again at time $t_0 \exp y + t_0 \exp(-y) = 2t_0 \cosh y$.

C2 The meeting point has x'-coordinate given by $-2t_0 \sinh y$.

C3 By the time they arrive at the meeting point the two particles have exchanged their energies and momenta, i.e. the q has gained exactly as much energy as the $\overline{q}$ has lost, and vice versa (although the gain and loss have not occurred at the same times but rather through the field).

After a second such yoyo 'round' the q- and $\overline{q}$-particles will be back at their original energy-momentum conditions.

The time it has taken is, however, longer than in the rest system, i.e. instead of $4t_0$ it is $4t_0 \cosh y$. But we note that the period is again given by twice the total energy divided by κ: $2E'_{tot}/\kappa = 2E_{tot}/\kappa \cosh y$. This is the MRS version of the time-dilation effect, described in Chapter 2.

The Lorentz-contraction phenomenon implies that the field sizes are correspondingly always shorter. We note, however, that *the Lorentz-contraction and the time-dilation effects combine in such a way that the space-time size spanned by the field during the period will again satisfy Eq. (6.15).* We leave the proof of this statement to the reader.

Finally, we note from the above exercise that during such a full period the system has moved a distance $\delta x'$ from the origin to the meeting point:

$$\delta x' = 2[t_0 \exp(-y) - t_0 \exp y] = -4t_0 \sinh y \equiv 2P'_{tot}/\kappa. \tag{6.18}$$

This is another general property of the MRS: *during a period $t_{per} = 2E_{tot}/\kappa$ the system will be translated by the vector $\mathbf{x}_{per} = 2\mathbf{P}_{tot}/\kappa$.*

There are two comments to add to this result:

- when the system is at rest as in the previous subsection then $\mathbf{P}_{tot} = 0$ by definition of 'at rest';

- the system will move during a period as if it had a mean velocity $\mathbf{x}_{per}/t_{per} = \mathbf{P}_{tot}/E_{tot}$, which is just the usual velocity for a particle with energy-momentum $(E_{tot}, \mathbf{P}_{tot})$.

This moving extended system contains three parts and behaves in a surprising manner. The two particles are moving with the velocity of light in the same or opposite directions and therefore contain both energy and momentum. There is, further, the field, which throughout seems to be longitudinally at rest, i.e. it contains only energy and no momentum. But the field nevertheless does change its position because it only exists in the region between the charges!

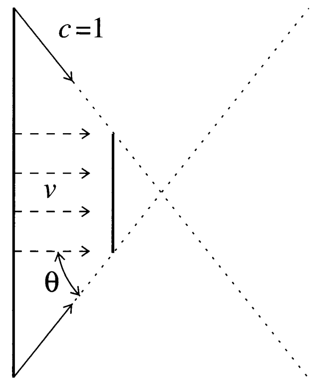

Fig. 6.4. The yoyo-mode after a Lorentz boost transverse to the field direction. The field is shown by the thick solid lines, the endpoints move with the velocity of light $c = 1$ along the thin solid lines and the field moves with velocity v along the direction of the broken lines. The dotted lines are the continuations of the motion of the endpoints.

With respect to causality we note that the two particles meet every half period, but meanwhile are often at spacelike distances with respect to each other. From Figs. 6.2 and 6.3 we note that each particle can while in motion in principle send away lightlike and even timelike messages 'via the field'; these can be received by its partner during the second part of the half-period. Thus the typical communication time can be short (when the particles move together in a strongly Lorentz-contracted string field) or long (when they move apart). It is necessary to introduce some kind of measurement procedure to define the notion of 'communication' and we will not speculate further on the subject at this point.

The result is, however, that there is always a finite delay time for any message travelling through the system. If one of the endpoint particles were acted upon by some outside agent then it would take some time before the other one would 'know'. This feature will be more noticeable when we consider the reaction of the yoyo system to an external momentum transfer, in Chapter 20.

5 *A transverse boost of the yoyo-mode*

It is instructive to consider the yoyo-state in a frame that is boosted transversely to the directions of motion of the two endpoints. We will then find that the field this time actually must contain also momentum. The situation is shown in Fig. 6.4 for two different times.

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We are now going to analyse the situation using the following two rules.

1 The two endpoints always move with the velocity of light.

2 The energy-momentum is conserved throughout and we will see that it is even *locally* conserved owing to causality.

The left-hand vertical line in Fig. 6.4 corresponds to a time when in its rest frame the system is stretched out as far as possible, i.e. at a turning time for the two endpoint charges. Then, in a frame where the string is moving with velocity v with respect to the rest frame the field contains both energy and momentum. If the field length is $2l$ then its rest frame energy is $2\kappa l$. In the moving frame that means (cf. Chapter 2):

$$E = 2\kappa l\gamma(v), \;\; P = 2\kappa l v\gamma(v) \tag{6.19}$$

where $\gamma(v) = 1/\sqrt{1-v^2}$. Note that the force field does not change its shape or size as it is boosted transversely. Equation (6.19) evidently gives the total energy-momentum of the system.

After a time δt (measured in the new frame) the endpoints have moved the distance δt and a point in the middle of the field has moved $v\delta t$. From Fig. 6.4 we conclude that the velocity v is related to the angle θ by

$$v = \cos\theta \tag{6.20}$$

The length of the force field is now $2(l - \delta t \sin\theta)$ and therefore the energy-momentum of the field is proportionally smaller.

In particular the field energy has decreased by an amount

$$\delta E = 2\kappa\delta t \sin\theta\gamma(v) \tag{6.21}$$

Using Eq. (6.20) we obtain

$$\gamma(v) = \frac{1}{\sin\theta} \tag{6.22}$$

which implies that

$$\delta E = 2\kappa\delta t \tag{6.23}$$

This field-energy loss is easy to understand from what we already know. It means that each endpoint particle will obtain (from the field) an increased energy $\delta e = \kappa\delta t$ while it moves the distance δt.

Further, we note that the momentum of the field along the boost direction has decreased by

$$\delta P = 2\kappa\delta t \sin\theta v\gamma(v) = 2\kappa\delta t \cos\theta \tag{6.24}$$

(again using Eqs. (6.20), (6.22)). This is the amount of momentum $\delta p \cos\theta$ which the q- and $\overline{q}$-particles have picked up along that direction.

They have also (in order to fulfil the masslessness condition $\delta e = |\delta p|$) acquired two compensating components $\pm\delta p \sin\theta$ along the field direction. In this way one can describe the force from the field on the particles as $\pm\kappa \sin\theta = \pm\kappa\sqrt{(1-v_\perp^2)}$ in the moving frame. We have seen before that the force is not changed for Lorentz boosts along the string but, owing to the time dilation effect, it is in this way affected by transverse boosts.

Consequently the energy-momentum is redistributed between the endpoint particles and the field in a *local* way. Once again we can talk of them as 'eating' or 'spitting out' the field in their neighborhoods.

From the two rules given above it is possible to trace any complicated motion of the force field, as we will see in later sections. A useful exercise at this point is to consider the necessary Lorentz transformations and the ensuing motion if one were to boost the 'flat' yoyo-mode in a direction between the longitudinal (exhibited in the previous subsection) and the transverse as discussed here.

You will then notice that it is only the transverse part of the field velocity (transverse, of course, with respect to the field direction) that plays a role for the field momentum. This means that the field only contains momentum with respect to its transverse motion, i.e. longitudinal momentum-carrying modes of the field do not exist for the MRS field (but they do occur for the endpoints). This is once again quite in accordance with good old classical string motion, where only transverse degrees of freedom play a role.

6.3 The QCD vacuum as a color superconductor

Both the Schwinger model and QCD are confining in the sense that the real charges (respectively electromagnetic and color) cannot be isolated from each other and only occur in particular singlet combinations. Confinement is, however, also expected to lead to restrictions on the spatial extension of the force fields between the charges. Calculations in the lattice approximation of QCD tend to confirm this behaviour.

The MRS, as a model of a confining force field in which the charges are identified as the endpoints, evidently has both these properties. In this section we will provide a motivation for the use of the MRS in hadron dynamics. We introduce a color superconductor as a simple model for the QCD vacuum state. We will also briefly mention another well-known model, the *bag model* for hadrons, and point out its relation to the MRS.

1 The London equations and types I and II superconductors

Electromagnetic superconductors have many wonderful properties and we mention only a few here:

- According to condensed-matter physics there is a tiny attractive interaction between two electrons close to the fermi surface, owing to the exchange of phonons associated with the crystal lattice of the material. Therefore there exists a (very) loosely bound state of two electrons, a *Cooper pair*, with spin 0. The spatial extent of the state, called ξ, is often in the μm range, i.e. it may be of macroscopic size. Due to this bosonic nature many such states may overlap in space and behave as a degenerate (although charged) Bose gas. The pairs move freely through the material and there is no resistance.
- According to Lenz's law an applied magnetic field will produce a (super) current of Cooper-pair states that will expel the applied field. Thus a magnetic field will only have an exponentially falling penetration depth (called λ) in a superconductor. If the temperature or the field is increased beyond a critical size, the states will be excited and break up and there is thus a phase transition from the superconducting to the normal state.

Due to the relative sizes of ξ and λ, such ordinary superconductors have one of two rather different behaviours at the critical point. We will now consider the two cases, called types I and II superconductors. *The shape of the normal-state field regions depends upon the superconductor type.*

If $\xi \gg \lambda$ the boundary regions between the superconducting state and the rest will be empty because neither the magnetic field nor the Cooper pairs can spread there. These regions are then inactive from a dynamical viewpoint. Nature will according to the gospel of thermodynamics then try to *minimise the boundaries* of a type I superconductor.

At the opposite extreme, $\lambda \gg \xi$, both the Cooper-pair density and the magnetic fields can populate the boundary region and Nature will consequently *maximise the boundaries* between the superconducting and the normal state in a type II superconductor.

It is known, [98], that there are in QCD possible color magnetic field configurations with energy below the no-particle state. In these states gluon combinations take the place of the Cooper pairs in an electromagnetic superconductor and the color electric field is in this case neutralised by the vacuum fields. The sizes of the corresponding lengths ξ and λ are not known from first principles. If the QCD vacuum corresponds to such a state then the appearance of color charges and fields in between them will correspond to regions with normal-state properties. Such regions will then be surrounded by such a vacuum color superconductor. In particular the boundary regions between the superconducting and normal states are interesting.

For the type I superconductor, the region where the (color) field expands (the normal-state region) will have boundaries that are as small as possible. For a localised excitation, the field will arrange itself as an (isolated) 'resonant cavity field', cf. Jackson, with standing waves inside this, in general, spherical region. The total field energy is proportional to the volume and we note that a sphere has the smallest boundary-to-volume ratio possible.

If the field has a longitudinal extension then the whole field will stay inside a cylindrical 'wave guide'. Once again the field energy will be proportional to the volume and if the longitudinal size is given then a connected cylinder shape will have minimal surface area.

There are, in QCD, analogy models for the two cases. The first corresponds to an isolated hadronic state, containing valence-quark color charges and color field energy organised into a spherical *bag*. The second corresponds to the production of an outward-moving $q\bar{q}$-state with its field energy organised into a *flux tube*. We will not go into details here but the basic idea involves introducing a 'bag-pressure' from the vacuum. This is neutralised at the boundary by the pressure from the fields inside so that there is a stable boundary.

To explain the different behaviour of a type II superconductor we consider a slab of matter (width L) in an (electromagnetic) superconducting state. Both for types I and II there is a minimal critical field, $\mathscr{B}_{c1}$, for which the superconducting state breaks down. We assume the field exists inside a region of total area A. Outside A there is still a superconducting state. For a type I superconductor the region will be homogeneous and the boundary region will have area $R_I = 2\sqrt{\pi A}L$. The whole field passes through A and so the total energy deposited in the slab is $E = \mathscr{B}_{c1}^2 AL$ and the total flux is $\Phi = \mathscr{B}_{c1}A$. For the type II case there is also a second critical field strength, $\mathscr{B}_{c2} > \mathscr{B}_{c1}$. For a field strength in between $\mathscr{B}_{c1}$ and $\mathscr{B}_{c2}$, the region will be penetrated by many thin *vortex-line* fields *each of a quantised size*. The core size is typically ξ and there is a weak repulsive interaction which keeps the vortex lines apart so that the field strength will vary inside A.

We may for simplicity consider the area A as divided into n circular non-connected regions. You will then find the same flux and the same energy deposit but the boundary region now has area $R_{II} \sim \sqrt{n}R_I$. Thus to maximise the boundary it is profitable to subdivide the region. When the field strength is greater than $\mathscr{B}_{c2}$ the whole region becomes filled with vortex lines and it will behave as for the type I case.

We shall exhibit a few steps in the London theory of superconductivity, [91], and in particular show the quantisation of the flux lines.

We consider a constant Cooper-pair density $n(\mathbf{x}, t)$ and a corresponding current $\mathbf{j}(\mathbf{x}, t) = -2en\mathbf{v}$, with $\mathbf{v}$ the velocity field. The continuity equation as well as the Lorentz force law will give (with a Cooper-pair mass m and charge $-2e$) for the stable state

$$\nabla \mathbf{j} = \nabla \mathbf{v} = 0, \quad \frac{d\mathbf{v}}{dt} = -\frac{2e}{m}(\mathscr{E} + \mathbf{v} \times \mathscr{B}) \tag{6.25}$$

The total change in time of the velocity field should be regarded as the change in time for a fixed coordinate plus the change in the coordinate for a fixed time; thus

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) \tag{6.26}$$

Then the Lorentz force law is equivalent to

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{2e}{m}\mathscr{E} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) = \mathbf{v} \times \left(\nabla \times \mathbf{v} - \frac{2e}{m}\mathscr{B}\right) \tag{6.27}$$

We may now apply the differential vector operator 'curl' ($\nabla\times$) on both sides of this equation and note that, according to Faraday's induction law (cf. Chapter 2), $\nabla \times \mathscr{E} = -\partial\mathscr{B}/\partial t$ and also that $\nabla \times \nabla a = 0$ for any function a.

Then one obtains the resulting equation for the vector $\mathscr{L}$:

$$\frac{\partial \mathscr{L}}{\partial t} = \nabla \times (\mathbf{v} \times \mathscr{L}) \quad \text{where} \quad \mathscr{L} = \nabla \times \mathbf{v} - \frac{2e\mathscr{B}}{m} \tag{6.28}$$

When both fields and current vanish $\mathscr{L} = \mathbf{0}$. The Londons, [91], made the fundamental assumption that *$\mathscr{L}$ should always vanish inside a superconductor*. This implies immediately an equation for the magnetic field because a vanishing $\mathscr{L}$ means that

$$\mathscr{B} = -\frac{m}{4ne^2}(\nabla \times \mathbf{j}) = -\frac{m}{4ne^2}(\nabla \times (\nabla \times \mathscr{B})) = \frac{m}{4ne^2}\triangle\mathscr{B} \tag{6.29}$$

Equations (6.28), (6.29) are known as the *London equations*. To exhibit the result in (6.29) we have used Ampère's law (assuming a static situation, $\partial\mathscr{E}/\partial t = 0$) and also the absence of magnetic charges (cf. Chapter 2). The solutions to Eq. (6.29) correspond to magnetic fields which are exponentially falling with a rate equal to the parameter λ mentioned above, which

https://doi.org/10.1017/9781009401296.006 Published online by Cambridge University Press

fulfils:

$$\lambda = \sqrt{\frac{m}{4ne^2}} \tag{6.30}$$

The inverse of this λ is identical to the plasma frequency we met in the discussion of the behaviour of the dielectricity in Chapter 2 (although here, for the Cooper pairs, the charge is $-2e$).

2 Solutions of the differential equation

We will need a particular solution of Eq. (6.29), i.e. the one corresponding to cylindrical symmetry around the 3-axis, with no variation along that axis, $\boldsymbol{\mathscr{B}} = \mathscr{B}\mathbf{e}_3$ with $\partial\mathscr{B}/\partial x_3 = 0$. We will solve that equation at the same time as we also exhibit the behaviour of the Feynman propagator in spacelike regions (as promised in Chapter 3).

Let us consider symmetrical solutions $f \equiv f(x^2)$ to the equation

$$(\triangle_{2d} - A_{2d}^2)f = 0 \tag{6.31}$$

(for $x^2 > 0$) where $2d$ is the dimension of the space and A_{2d} is a positive number. It is instructive to note that in both of the following cases,

$$\triangle_{2d} = \sum_{j=1}^{2d} \frac{\partial^2}{\partial x_j^2}, \quad x^2 = \sum_{j=1}^{2d} x_j^2 \tag{6.32}$$

$$\triangle_{2d} = \sum_{j=1}^{2d-1} \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial t^2}, \quad x^2 = \sum_{j=1}^{2d-1} x_j^2 - t^2$$

we obtain directly the following equation in $z = x^2$:

$$4\left(z\frac{d^2 f}{dz^2} + d\frac{df}{dz}\right) - A_{2d}^2 f = 0 \tag{6.33}$$

Assuming that the solution is of the kind

$$f(z) = (\zeta)^{2\alpha} g(\zeta) \text{ where } \zeta = \sqrt{z} > 0 \tag{6.34}$$

the equation can be brought into the form (dots mean derivatives with respect to ζ)

$$\zeta^2 \ddot{g} + (2d + 4\alpha - 1)\zeta\dot{g} + [4\alpha(d + \alpha - 1) - A_{2d}^2\zeta^2]g = 0 \tag{6.35}$$

Then if we choose $\alpha = (1-d)/2$ we obtain a modified Bessel differential equation,

$$\zeta^2\ddot{g} + \zeta\dot{g} - [(1-d)^2 + A_{2d}^2\zeta^2]g = 0 \tag{6.36}$$

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For the case we started with, i.e. $d = 1$ with $A_{2d} = \lambda$ in Eq. (6.30), we have

$$f \equiv \mathscr{B} = CK_0\left(\frac{\sqrt{x^2}}{\lambda}\right) \tag{6.37}$$

where K_0 is the modified Bessel function of rank 0, which is exponentially falling and behaves for large values of its argument as follows:

$$K_0(x) \simeq \sqrt{\frac{\pi}{2x}}\exp(-x) \tag{6.38}$$

In order that $\mathscr{B}$ should be a proper magnetic field the normalisation constant C must have (energy) dimension 2.

For the Feynman propagator *for spacelike values of* x^2 we obtain ($d = 2$ and $A_{2d} = m$) the same exponential falloff as in Eq. (6.38) but a power in front:

$$\Delta_F(x,m) \propto \frac{m}{\sqrt{x^2}}K_1(m\sqrt{x^2}) \tag{6.39}$$

3 *The quantisation of the magnetic flux*

The result in Eq. (6.37) has a logarithmic singularity for $x^2 = 0$:

$$\mathscr{B} \simeq C\log(\lambda/\sqrt{x^2}) \tag{6.40}$$

The corresponding magnetic flux, Φ, through the 12-plane is

$$\Phi = \int dx_1 dx_2 \mathscr{B} = 2\pi C\int_0^\infty x dx K_0\left(\frac{x}{\lambda}\right) = 2\pi C\lambda^2 = \left(\frac{2\pi}{2e}\right)\frac{Cm}{n} \tag{6.41}$$

We note that the quantity Cm/n is a dimensionless number (n, being a three-dimensional space density, then has energy dimension 3 using our ordinary convention with $c = \hbar = 1$).

We also note that the Cooper-pair (super)current $\mathbf{j}$ is given by

$$\mathbf{j} = \nabla \times \mathscr{B} = -\mathbf{e}_\phi \frac{d\mathscr{B}}{d\sqrt{x^2}} \tag{6.42}$$

where the derivative can be expressed in terms of the modified Bessel function K_1 and therefore again falls off exponentially in directions normal to the 3-axis. *It is, however, singular*, like $1/\sqrt{x^2}$, *along the* 3-*axis.*

We also note that *the current flows around the origin, i.e. the* 3-*axis.* (The unit vector $\mathbf{e}_\phi$ circulates around this axis in the direction of increasing azimuthal angle ϕ.) Thus the Cooper pairs circulate, thereby producing a magnetic field similar to that in a solenoid. This is the reason for the nonvanishing magnetic flux through the 12-plane and the singularity along the 3-axis.

In order to understand what is going on we go back to the London condition for a superconducting state, $\mathscr{L} = \mathbf{0}$, which we write as

$$m\mathscr{L} = \nabla \times m\mathbf{v} - 2e\mathscr{B} = \nabla \times (m\mathbf{v} - 2e\mathbf{A}) = 0 \tag{6.43}$$

where we have introduced the vector potential $\mathbf{A}$. In Chapter 11 we will study this expression further and show that the canonical momentum of a particle with kinetic momentum $m\mathbf{v}$ and charge $-2e$ is, in an electromagnetic field,

$$\mathbf{p} = m\mathbf{v} - 2e\mathbf{A} \tag{6.44}$$

We further note that the flux Φ in Eq. (6.41) is given by

$$-2e\Phi = \int dx_1 dx_2 m\mathscr{L} = \oint d\mathbf{s} \cdot \mathbf{p} \tag{6.45}$$

Here we have used Stoke's theorem. This result was noted by F. London and he interpreted it correctly, along the lines of a Bohr-Sommerfeld quantisation condition: the integral should be equal to an integer times Planck's constant h. In this way we obtain that the combination Cm/n in Eq. (6.41) is an integer, N, and that the flux $\Phi = -N/2e$.

The result may at first sight seem like witchcraft. The vector $\mathscr{L}$ was assumed to vanish, according to the London prescription, *inside the superconductor*. The fact that its surface integral is nonvanishing and in particular equal to an integer times a flux unit must then mean that the whole surface is *not inside the superconductor*. We have already pointed out that there is a singularity for the solution along the 3-axis. In other words there is a thin 'hole' along the axis and we may conclude that it should be of the order $\xi \ll \lambda$ and correspond to a lack of Cooper pairs. This is a *vortex line*.

F. London suggested on the basis of these results that it should be possible to produce a *magnetic flux trap*. Suppose that we have a ring of matter in a normal state inside a magnetic field and that we then bring the ring into a superconducting state. This will produce a supercurrent of Cooper pairs in the ring. Further suppose that after this we remove the magnetic field and investigate the magnetic flux *through the hole in the ring*, caused by the supercurrent (which must continue inside the superconductor because there is no 'stopping force'!). A set of clever experiments, [49], were later performed, which justified both the flux trapping and, in particular, the quantisation of the flux.

We conclude that the solution we have obtained, which corresponds to a *vortex line*, penetrates the superconductor to a small depth and contains a definite flux corresponding to an integer times the inverse charge of a Cooper pair. This corresponds to the typical type II superconductor breakdown. The superconductor is penetrated by as many isolated vortex lines as the field flux permits and we now understand the subdivision of the slab discussed above.

A dynamical vortex line, i.e. one connected to moving charges must have a dynamics very similar to the MRS and therefore if the QCD vacuum state has the properties of a superconductor type II our use of the MRS as a model for the color electric force field is natural. We will later consider the question of the width of the Lund string field, cf. Chapter 11, and will find that its radius is typically 0.3–0.4 fm.