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ON THE CANCELLATION PROBLEM OF ZARISKI

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Let K_1 and K_2 be extension fields over a field K with $charK = p > 0$. Assume $L = K_1(x_1) = K_2(x_2) \supset K$ where x_i is transcendental over K_i, for $i = 1,2$. In this paper we prove that if K_1 is a perfect field, then $K_1 = K_2$.

Let K_1 and K_2 be finitely generated extensions of a field K and let x_i be transcendental over K_i, $i = 1,2$. The cancellation problem of Zariski [**5**] asks if $K_1(x_1) = K_2(x_2)$, must K_1 and K_2 be K-isomorphic? In general the answer is no [**1**]. However there are some special cases in which the answer is yes [**2, 3, 4, 5**]. For example, it is known that the answer is yes if $charK = 0$ and $x_1 = x_2$ [**2, 5**]. But for the case of a finite base field, very little is known. In this paper we shall prove the answer is yes for an important case of a finite base field, that is, if $charK = p > 0$ and K_1 is a perfect field, then $K_1 \cong K_2$. In this case we have a stronger result, namely $K_1 = K_2$.

THEOREM. *Let K_1 and K_2 be extension fields over a field K with $charK = p > 0$. Assume $L = K_1(x_1) = K_2(x_2) \supset K$ where x_i is transcendental over K_i, for $i = 1,2$. If K_1 is a perfect field, then $K_1 = K_2$.*

REMARK. In [**2, 3, 4, 5**], it is assumed that K_1 and K_2 are finitely generated extensions of K. However, in our Theorem this assumption in not required.

We start with a lemma.

LEMMA. *Let K_1 and K_2 be fields as in the Theorem. If K_1 is a perfect field, then so is K_2.*

PROOF: Let φ be the Frobenius automorphism of L so that $\varphi(a) = a^p$ for all $a \in L$, where $p = charK > 0$. Then $\varphi(L) = L^p = K_1^p(x_1^p) = K_2^p(x_2^p)$. Since $K_1^p = K_1$, $K_1(x_1^p) = K_2^p(x_2^p)$. Thus $[K_2(x_1) : K_2^p(x_2^p)] = [K_1(x_1) : K_1(x_1^p)] = p$. However $p = [K_2(x_2) : K_2^p(x_2^p)] = [K_2(x_2) : K_2(x_2^p)] \times [K_2(x_2^p) : K_2^p(x_2^p)] = p \times [K_2(x_2^p) : K_2^p(x_2^p)]$. So $[K_2(x_2^p) : K_2^p(x_2^p)] = 1$, that is, $K_2^p(x_2^p) = K_2(x_2^p)$. This implies that $K_2^p = K_2$. $\square$

PROOF OF THEOREM: Let K_1K_2 be the compositum of K_1 and K_2 in L. Then $L = K_1K_2(x_1, x_2)$ since $K_1K_2(x_1, x_2) \subset L$ and $L \subset K_1K_2(x_1, x_2)$ by the definition of

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compositum. First we show that L is a transcendental extension over K_1K_2. By the Lemma K_2 is also a perfect field. So $L^{p^n} = K_1^{p^n}\left(x_1^{p^n}\right) = K_2^{p^n}\left(x_2^{p^n}\right) = K_1\left(x_1^{p^n}\right) = K_2\left(x_2^{p^n}\right)$ for every positive integer n. Thus $K_1K_2 \subset L^{p^n}$ for every positive integer n.

$$
\begin{array}{ll}
L & = K_1(x_1) = K_2(x_2) = K_1K_2(x_1, x_2) \\
| & \\
L^p & = K_1(x_1^p) = K_2(X_2^p) \\
\vdots & \\
L^{p^n} & = K_1(x_1^{p^n}) = K_2(x_2^{p^n}) \\
\vdots & \\
K_1K_2 & \\
\end{array}
$$

$$
\begin{array}{ccc}
 & K_1K_2 & \\
/ & & \backslash \\
K_1 & & K_2 \\
\backslash & & / \\
 & K &
\end{array}
$$

But $[L : L^{p^n}] = p^n$ for every positive integer n. So L is an infinite dimensional extension over K_1K_2. Since $L = K_1K_2(x_1, x_2)$, L must be a transcendental extension over K_1K_2. Now we claim that K_1K_2 must be algebraic over K_i, for $i = 1, 2$. Otherwise $1 = tr.d_{K_i}K_i(x_i) = tr.d_{K_i}K_1K_2 + tr.d_{K_1K_2}L \geqslant 2$, for $i = 1, 2$. Since K_i is algebraically closed in L, for $i = 1, 2$, we conclude that $K_1K_2 \subset K_i$, for $i = 1, 2$. Hence $K_1K_2 = K_1 = K_2$. □

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