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HAMILTONIAN CYCLES IN PRODUCTS OF GRAPHS

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Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G; let K_n denote the complete graph with n vertices and let $K_{n,m}$ denote the complete bipartite graph on n and m vertices. A *Hamiltonian cycle* (*Hamiltonian path*, respectively) in a graph G is a cycle (path, respectively) in G that contains all the vertices of G. A graph G is called *Hamiltonian* if it contains a Hamiltonian cycle. The *path number* $m(G)$ of a graph G is defined as the minimum number of disjoint paths needed to cover all the vertices of G (see [3], [1] and [4]). Clearly, a graph G has a Hamiltonian path if and only if $m(G)=1$; every Hamiltonian graph has a Hamiltonian path, hence a graph G is non-Hamiltonian if $m(G)\geq 2$.

The *Cartesian product* $G\text{x}H$ and the *strong product* $G\bar{\text{x}}H$ of two graphs G and H are defined by:

$$V(G\text{x}H) = V(G\bar{\text{x}}H) = V(G)\text{x}V(H)$$

and

$$E(G\text{x}H) = \{(u_1, v_1)(u_2, v_2) \mid u_1, u_2 \in V(G),\ v_1, v_2 \in V(H) \text{ and either}$$

$$u_1 = u_2 \text{ and } v_1v_2 \in E(H) \text{ or else } v_1 = v_2 \text{ and } u_1u_2 \in E(G)\},$$

while

$$E(G\bar{\text{x}}H) = E(G\text{x}H) \cup \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in V(G),\ v_1, v_2 \in V(H),$$

$$u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\}.$$

Let $\text{x}G^n$ ($\bar{\text{x}}G^n$, respectively) denote the graph $G\text{x}G\text{x}\cdots\text{x}G$ (n times) ($G\bar{\text{x}}G\bar{\text{x}}\cdots\bar{\text{x}}G$, n times, respectively).

G. Sabidussi ([6], Lemma 2.3) proved that if G is an n-connected graph and H is a k-connected graph, then $G\text{x}H$ is $(n+k)$-connected; he also proved that if G is a connected graph and H is a cycle, then $G\text{x}H$ is Hamiltonian provided

$$\overline{\overline{V(H)}} \geq 2\overline{\overline{V(G)}}-2$$

(for a proof, apply Lemma 2.7 of [6], with $Y=Z=$the cycle H). Recently, M. Rosenfeld and D. Barnette [5] proved that if G is a connected graph and H is a cycle, then $G\text{x}H$ is Hamiltonian provided the maximum degree of the vertices of G is not more than the number of the vertices of H; for related results, see [2].

The purpose of this note is to present few properties of products of graphs, concerning the existence of a Hamiltonian cycle in them.

THEOREM 1. *If for a graph G and a natural number n, the graph $\mathrm{x}G^n$ is Hamiltonian, then the graph $\mathrm{x}G^k$ is Hamiltonian for all $k \geq n$.*

Proof. It is obviously enough to prove that $\mathrm{x}G^{n+1}$ is Hamiltonian whenever $\mathrm{x}G^n$ is. Suppose $\mathrm{x}G^n$ is Hamiltonian, and G has m vertices. Let H be a Hamiltonian cycle in $\mathrm{x}G^n$. The graph G is clearly connected (see [6], Lemma 2.2), and the maximum degree of G is less than m; the graph H has m^n vertices, hence by [5] the graph $G\mathrm{x}H$ is Hamiltonian; since the graph $G\mathrm{x}H$ spans $\mathrm{x}G^{n+1}$, it follows that $\mathrm{x}G^{n+1}$ is Hamiltonian.

THEOREM 2. *For every n and k, $n, k \geq 1$, there exists an n-connected graph $G = G(n, k)$, such that the graph $\bar{\mathrm{x}}G^k$ is non-Hamiltonian.*

Proof. Let n and k be given, and let t be an arbitrary natural number. The path number $m(\bar{\mathrm{x}}(K_{1,t})^n)$ of the graph $\bar{\mathrm{x}}(K_{1,t})^n$ satisfies the inequality

$$m(\bar{x}(K_{1,t})^n) \geq 2t^n - (t+1)^n;$$

to prove it, let u be the t-valent vertex of $K_{1,t}$ and let $v_1, \ldots, v_t$ be the 1-valent vertices of $K_{1,t}$. A vertex of $\bar{\mathrm{x}}(K_{1,t})^n$ is an n-tuple $(y_1, \ldots, y_n)$, where $y_i \in V(K_{1,t})$ for all $1 \leq i \leq n$; two vertices $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ of $\bar{\mathrm{x}}(K_{1,t})^n$ form an edge of $\bar{\mathrm{x}}(K_{1,t})^n$ if and only if $y_i \neq z_i$ for at least one index i, and for all j, $1 \leq j \leq n$, $y_j = z_j$ or $y_j z_j \in E(K_{1,t})$. Let a vertex $(y_1, \ldots, y_n)$ of the graph $\bar{\mathrm{x}}(K_{1,t})^n$ be of type 1 if $u \in \{y_1, \ldots, y_n\}$, where the last set has less than or equal to n elements; the rest of the vertices of the graph $\bar{\mathrm{x}}(K_{1,t})^n$ belong to type 2. The graph $\bar{\mathrm{x}}(K_{1,t})^n$ has t^n vertices of type 2 and $(t+1)^n - t^n$ vertices of type 1. Every edge of the graph $K_{1,t}$ has the vertex u as an end point, hence every edge of the graph $\bar{\mathrm{x}}(K_{1,t})^n$ has at least one vertex of type 1 as an end point; every pair of vertices of type 2 in a simple path in $\bar{\mathrm{x}}(K_{1,t})^n$ is therefore separated by a vertex of type 1; every simple path in the graph $\bar{\mathrm{x}}(K_{1,t})^n$ contains at most one more vertex of type 2 than those of type 1, hence $t^n - (t+1)^n - t^n = 2t^n - (t+1)^n \leq m(\bar{\mathrm{x}}(K_{1,t})^n)$.

As a consequence, $m(\bar{\mathrm{x}}(K_{1,t})^{nk}) \geq 2t^{nk} - (t+1)^{nk} = f(t)$, where $f(t)$ is a polynomial in t of degree nk; since $n, k \geq 1$, there exists a natural number s for which $f(s) > 2$.

The graph $G = G(n, k) = \bar{\mathrm{x}}(K_{1,s})^n$ is an n-connected graph, as follows from the connectedness of the graph $K_{1,s}$ by induction on n, using Lemma 2.3 of [6]; moreover $m(\bar{\mathrm{x}}G^k) > 2$, hence $\bar{\mathrm{x}}G^k$ is non-Hamiltonian.

REMARK. The idea of dividing the vertex set of a graph in order to show that the graph is non-Hamiltonian is due to T. A. Brown [3].

THEOREM 3. *The graph $\mathrm{x}(K_{n,m})^k$ is non-Hamiltonian for all $k \geq 1$, provided $n \neq m$.*

Proof. Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be the partition of the vertex set $V(K_{n,m})$ of $K_{n,m}$. The vertices of the graph $\mathrm{x}(K_{n,m})^k$ are all the k-tuples $(y_1, \ldots, y_k)$

with $y_i \in V(K_{n,m})$; two vertices of $\times(K_{n,m})^k$, $(y_1, \ldots, y_k)$ and $(z_1, \ldots, z_k)$, form an edge of that graph if $y_i \neq z_i$ for exactly one index i, $1 \leq i \leq k$, and where $y_i z_i \in E(K_{n,m})$. Let a vertex $(y_1, \ldots, y_k)$ of the graph $\times(K_{n,m})^k$ belong to type 1 if the number of indices j, for which $y_j \in \{u_1, \ldots, u_n\}$, is even; let the rest of the vertices of the graph $\times(K_{n,m})^k$ belong to type 2. All the edges of the graph $K_{n,m}$ are of the form $u_i v_j$, hence every edge of the graph $\times(K_{n,m})^k$ has exactly one end point of type 1 and the other one of type 2; a straightforward calculation shows that the number of vertices of type 1 exceeds the number of vertices of type 2 by $(n-m)^k$, hence $m(x(K_{n,m})^k) \geq (n-m)^k$; it follows that if $n \neq m$ the graph $\times(K_{n,m})^k$ is non-Hamiltonian.

Every spanning subgraph of a non-Hamiltonian graph is itself non-Hamiltonian; as a result, we have the following corollaries.

COROLLARY 1. *If T is a tree spanning the graph $K_{n,m}$, with $n > m \geq 1$, then $\times T^k$ is non-Hamiltonian for all $k \geq 1$.*

COROLLARY 2. *If T is a tree with an odd number n of vertices, and $n \geq 3$, then $\times T^k$ is non-Hamiltonian for all $k \geq 1$.*

Is it true that for every connected graph G with at least two vertices, there exists an integer $k = k(G)$ for which the graph $\bar{\times} G^k$ is Hamiltonian?

REFERENCES

1. D. Barnette, *Trees in Polyhedral graphs*, Canad. J. Math. **18** (1966), 731–736.

2. M. Behzad and S. E. Mahmoudin, *On Topological Invariants of the Products of Graphs*, Canad. Math. Bull. **12** (1969), 157–166.

3. T. A. Brown, *Simple paths on Convex Polyhedra*, Pacific J. Math. **11** (1961), 1211–1214.

4. B. Grünbaum, *Convex Polytopes*, J. Wiley, New York, 1967.

5. M. Rosenfeld and D. Barnette, *Hamiltonian Circuits in Certain Prisms*, Discrete Math. **5** (1973), 389–394.

6. G. Sabidussi, *Graphs with given group and given graph-theoretical properties*, Canad. J. Math. **9** (1957), 515–525.

7. ——, *Graph Multiplication*, Math. Z. **72** (1960), 446–457.

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