ESTIMATION OF RANDOM CHANGES IN THE EARTH'S ROTATION

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Abstract. The usual procedures for estimating the motion of the Earth's pole use a least squares data reduction procedure to estimate the coefficients in a time series solution for the coordinates of the pole. Whether optical data or radar tracking data from near Earth satellites is used, the presence of random accelerations in the equations which describe the motion of the Earth's pole will lead to errors if the data is reduced in the classical least squares manner. This investigation presents a technique for estimating the polar motion in the presence of unmodeled accelerations. The unmodeled acceleration is represented by a first order stationary Gauss-Markoff process which can be separated into a timewise correlated component and a purely random component. Using this model, a sequential estimation procedure is developed for estimating the orientation of the pole, the components of Earth's angular velocity and the magnitude of the components of the unmodeled acceleration. The application of the method using both optical and radar tracking data from near Earth satellites is discussed.

The rotational motion of the Earth about its center of mass is influenced by the effects of the atmosphere, the oceans, the solid lithosphere, the fluid components of the Earth's interior and the lunar, solar and planetary gravitational torques. These components interact in various complex modes to exchange energy and momentum. As a consequence, the Earth rotates about an axis whose orientation changes continuously and the rate of rotation of the Earth about its instantaneous axis is not constant when measured against the time indicated by an atomic clock.

The dynamical system which describes the Earth's rotational motion can be separated into the modeled components of the forcing moment, unknown or unmodeled components and purely random components. Discrepancies between the predicted motion of the Earth's pole and the observed motion can be attributed to (1) errors in the model used to describe the Earth's motion, (2) errors in the observations used to determine the Earth's motion and (3) deficiencies in the methods used to reduce the observations.

The primary data source for determining the Earth's polar motion is optical observations of the 'fixed' stars. However, several new sources are providing data of significantly higher accuracy than the present optical data. These sources include the observations of near Earth satellites, the lunar laser ranging experiments and the use of very long base-line interferometry. With these data sources it is natural to consider procedures for improving the accuracy achieved in reducing the observation. In this summary, present procedures for determining the Earth's polar motion are reviewed briefly and a proposed extention of these methods to account for errors induced by unmodeled accelerations is described.

At the present time, there are two organizations which determine the position of the pole using optical data, the IPMS in Mizusawa-shi, Japan (Yumi, 1970), and the BIH

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in Paris, France (Guinot,1970). A third organization, the United States Naval Weapons Laboratory, Dahlgren, Virginia uses satellite doppler data (Anderle and Beuglass, 1970). The polar motion is determined by each of these organizations using observations of the variation in latitude at observation points around the Earth.

For example, the coordinates of the poles are determined by the IPMS using observations of the variation in latitude obtained at the five International Latitude Service (ILS) stations. These stations are positioned around the Earth at approximately the same latitude of 38°8′. The coordinates of the pole, (x,y), and the non-polar variation of latitude, z, are determined from the relation

$$\Delta\phi_i = x\cos\lambda_i + y\sin\lambda_i + z, \qquad i = 1, \ldots, k \tag{1}$$

where $\Delta\phi_i = \phi_i - \Phi_i$, ϕ_i is the observed latitude of the ith station, Φ_i is the mean latitude of the ith station and λ_i is the longitude of the ith station and k represents the number of stations. If the notation,

$$\mathbf{y}_k = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_k \end{bmatrix}, \quad \mathbf{H}_k = \begin{bmatrix} h_{11} & h_{12} & 1 \\ h_{21} & h_{22} & 1 \\ \vdots & \vdots & \vdots \\ h_{k1} & h_{k2} & 1 \end{bmatrix}, \quad \mathbf{v}_k = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2}$$

is used, where $h_{i1} = \cos\lambda_i$ and $h_{i2} = \sin\lambda_i$, the relation between the observations and the polar positions can be expressed in matrix notation as

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{v}_k. \tag{3}$$

The least squares solution for the components of the Earth's pole of rotation is obtained then as

$$\hat{\mathbf{x}}_k = (\mathbf{H}_k^T\mathbf{H}_k)^{-1}\,\mathbf{H}_k^T\mathbf{y}_k = \mathbf{P}_k\mathbf{H}_k^T\mathbf{y}_k \tag{4}$$

provided $k \geq 3$, i.e., provided the number of observations is, at least, equal to the number of coordinates to be determined. If one additional observation

$$\mathbf{y}_{k+1} = \mathbf{h}_{k+1}\mathbf{x} + \mathbf{v}_{k+1} \tag{5}$$

becomes available, the least square estimate of the polar position is given by

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{H}_k^T\mathbf{H}_k + \mathbf{h}_{k+1}^T\mathbf{h}_{k+1})^{-1}\,(\mathbf{H}_k^T\mathbf{y}_k + \mathbf{h}_{k+1}^T\mathbf{y}_{k+1}) \tag{6}$$

In each of Equations (4) and Equations (6), the estimate is obtained by inverting a (3×3) matrix. While the inversion of a (3×3) matrix is not a difficult task, if $\hat{\mathbf{x}}_k$ has been computed, $\hat{\mathbf{x}}_{k+1}$ can be computed with only a scalar division. If the observations are being used to determine other model parameters, then the state vector may be of higher dimension and the matrix inversion may require considerable effort.

Using the definition $\mathbf{P}_{k+1} = (\mathbf{P}_k + \mathbf{h}_{k+1}^T\mathbf{h}_{k+1})^{-1}$, a well known matrix identity (Liebelt, 1967) can be used to express $\mathbf{P}_{k+1}$ as follows

$$\begin{aligned} \mathbf{P}_{k+1} &= [\mathbf{I} - \mathbf{K}_{k+1}\mathbf{h}_{k+1}]\,\mathbf{P}_k \\ \mathbf{K}_{k+1} &= \mathbf{P}_k\mathbf{h}_{k+1}^T\,[\mathbf{h}_{k+1}\mathbf{P}_k\mathbf{h}_{k+1}^T + \mathbf{I}]^{-1}. \end{aligned} \tag{7}$$

Substituting $\mathbf{P}_{k+1}$ into Equation (6) and rearranging the terms leads to the following expression for $\hat{\mathbf{x}}_{k+1}$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mathbf{K}_{k+1} \left[\mathbf{y}_{k+1} - \mathbf{h}_{k+1} \hat{\mathbf{x}}_k\right]. \tag{8}$$

The dimension of the matrix to be inverted in Equations (7) and (8) is equal to the number of observations included in the vector $\mathbf{y}_{k+1}$. If only a single observation is added, then the new estimate is obtained by a simple scalar division. This procedure for computing the estimate is referred to as a *sequential* estimation procedure as contrasted with the *batch* procedure represented by Equation (6).

The primary advantage of using Equations (7) and (8) rather than Equation (6) for estimating the state, with the observation-state relationship represented by Equation (1), lies in the reduction in the matrix computation. However, Equations (1) represent a determination of the polar motion based on geometric considerations only. To obtain maximum benefit of the more accurate observations now being made available, the dynamical effects of the Earth's motion must be introduced. In addition, the observations should attempt to obtain better estimates of such parameters as station locations, geopotential terms, etc. Squires *et al.* (1969) discuss the errors which occur when a batch processor is used to reduce observations obtained from a satellite influenced by unmodeled accelerations. The study concludes that the batch algorithm is insensitive to the unmodeled accelerations and, as a consequence, arrives at erroneous results. The approach discussed in the previous paragraph can be extended to obtain an algorithm for estimating the motion of the Earth's poles, using a model that includes the effects of both random and unmodeled accelerations on the Earth's motion. The application of the algorithm to the problem of determining the trajectory and the unmodeled accelerations acting on a lunar orbiting satellite are discussed in Tapley and Ingram (1970), Ingram and Tapley (1971), and Tapley and Ingram (1971). The algorithm is applied to the problem of estimating the Earth's polar motion by assuming that the observational data are range and range-rate observations of a near-Earth satellite.

The equations of the dynamical system, i.e., the rotational motion of the Earth and the motion of the satellite, can be described by a set of first order non-linear differential equations of the following functional form:

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v}, & \dot{\mathbf{v}} &= \mathbf{a}_m (\mathbf{r}, \mathbf{J}, \mathbf{r}_m, \mathbf{r}_s) \\ \dot{\boldsymbol{\alpha}} &= \mathbf{f}(\boldsymbol{\omega}, \boldsymbol{\alpha}), & \dot{\boldsymbol{\omega}} &= \mathbf{g}(\boldsymbol{\omega}, \mathbf{J}, \mathbf{r}_m, \mathbf{r}_s, \boldsymbol{\varepsilon}) \end{aligned} \tag{9}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity components of the satellite, $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are the Euler angles and angular velocity components of the Earth's motion, $\mathbf{r}_m$ and $\mathbf{r}_s$ are position vectors of the Moon and Sun, $\mathbf{J}$ is a vector of model parameters such as J_2 or μ and $\boldsymbol{\varepsilon}$ is an unmodeled random acceleration which influences the Earth's rotational motion. Equation (9) can be expressed as

$$\dot{\boldsymbol{\eta}} = \mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\varepsilon}, t) \tag{10}$$

where $\boldsymbol{\eta}$ is a m-vector which represents the components of the position and velocity of the satellite, the Euler angles and the components of the angular velocity of the Earth, unknown parameters in the model which are to be estimated, and $\boldsymbol{\varepsilon}(t)$ is a 3-vector of unmodeled or random forcing terms. In the subsequent analysis $\boldsymbol{\varepsilon}(t)$ is modeled as the superposition of a purely random component and an exponentially time-wise correlated component. A process which satisfies these assumptions can be described by the following differential equation

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{B}\boldsymbol{\varepsilon} + \mathbf{u} \tag{11}$$

where $\mathbf{B}$ is a (3×3)-matrix of constant correlation coefficients and $\mathbf{u}$ is a random forcing term assumed to be distributed with zero mean and known covariance, i.e.,

$$E[\mathbf{u}] = 0, \quad E[\mathbf{u}(t)\,\mathbf{u}^T(\tau)] = \mathbf{Q}(t)\,\delta(t-\tau) \tag{12}$$

where $\delta(t-\tau)$ is the Dirac delta function, defined by

$$\delta(t) = \lim_{\sigma \to 0} \delta_\sigma(t)$$

$$\delta_\sigma(t) = \begin{cases} 0, \, |t| < \sigma, \\ \dfrac{1}{2\sigma}, \, |t| < \sigma. \end{cases}$$

This definition implies that

$$\int_{t_1}^{t_2} \delta(t-\tau)\,\mathrm{d}\tau = \begin{cases} 1, \, t_1 < t < t_2 \\ \frac{1}{2}, \, t_1 = t \quad \text{or} \quad t = t_2 \\ 0, \, \text{otherwise}. \end{cases}$$

The covariance matrix in Equation (12) implies the further assumption that $\mathbf{u}(t)$ is not correlated in time. If the n-vector $\mathbf{X}^T = [\boldsymbol{\eta}^T \vdots \boldsymbol{\varepsilon}^T]$ is defined, then Equations (10) and (11) can be combined to obtain

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{u}, t), \qquad \mathbf{X}(t_0) = \mathbf{X}_0. \tag{13}$$

The observation-state relations can be expressed as

$$\mathbf{Y}_i = \mathbf{G}(\mathbf{X}_i, t_i) + \mathbf{v}_i, \qquad i = 1, \ldots, k. \tag{14}$$

where $\mathbf{v}_i$ is the noise in the observations. Since the observations are non-linear functions of the state, and since in general an analytic solution of Equation (13) is not available, the observations at the various times must be related by numerically integrating Equations (13). For simplicity, Equations (13) and (14) are linearized, under the assumption that the true motion of the dynamical system differs by only a small amount from the modeled motion. The linearized relations are given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(t)\,\mathbf{x} + \mathbf{C}(t)\,\mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}_i &= \mathbf{H}_i \mathbf{x}_i + \mathbf{v}_i, \qquad i = 1, \ldots, k \end{aligned} \tag{15}$$

where $\mathbf{u}(t)$ satisfies Equation (12) and $\mathbf{x}_0$ is, in general, unknown. The symbol * in

$\mathbf{H}_i = (\partial \mathbf{G}/\partial \mathbf{X}_i)^*$ and $\mathbf{A}(t) = (\partial \mathbf{F}/\partial \mathbf{X}_i)^*$ indicates that the quantities are evaluated using the modeled motion.

Assuming that the observation noise is distributed with zero mean and known variance, i.e., $E[\mathbf{v}_i] = 0$ and $E[\mathbf{v}_i \mathbf{v}_i^T] = \mathbf{R}_i$, the best linear unbiased minimum variance estimator for the state, $\hat{\mathbf{x}}_k$, at time t_k given the observation $\mathbf{y}_i$, $i = 1, \ldots, k$ can be determined by a repeated application of a sequential algorithm similar to that given by Equations (7) and (8).

Assume that the first $k-1$ observations have been processed to obtain $\hat{\mathbf{x}}_{k-1}$ and $\mathbf{P}_{k-1}$ and then given the observation $\mathbf{y}_k$, consider the problem of determining the best estimate of $\hat{\mathbf{x}}_k$. The algorithm for computing $\hat{\mathbf{x}}_k$ is obtained as follows:

(1) Integrate the modeled or reference trajectory from t_{k-1} to t_k, using the equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t), \qquad \mathbf{X}_{k-1} = \mathbf{X}^*_{k-1}. \tag{16}$$

(2) Propagate the estimate and associated covariance obtained at t_{k-1} forward to t_k using the relation

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{A}(t)\,\bar{\mathbf{x}}, \qquad \bar{\mathbf{x}}(t_{k-1}) = \hat{\mathbf{x}}_{k-1} \\ \dot{\bar{\mathbf{P}}} &= \mathbf{A}(t)\,\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{A}^T(t) + \mathbf{Q}, \qquad \bar{\mathbf{P}}(t_{k-1}) = \mathbf{P}_{k-1}. \end{aligned} \tag{17}$$

(3) Evaluate the weighting matrix $\mathbf{K}_k$ using the relation

$$\mathbf{K}_k = \bar{\mathbf{P}}_k \mathbf{H}_k^T [\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k]^{-1} \tag{18}$$

where $\mathbf{R}_k = E[\mathbf{v}_k\, \mathbf{v}_k^T]$ and $\mathbf{H}_k = (\partial \mathbf{G}/\partial \mathbf{X}_k)^*$.

(4) Given $\mathbf{y}_k = \mathbf{Y}_k - \mathbf{G}(\mathbf{X}_k^*, t_k)$, evaluate $\hat{\mathbf{x}}_k$ and $\mathbf{P}_k$ using the relation

$$\begin{aligned} \hat{\mathbf{x}}_k &= \bar{\mathbf{x}}_k + \mathbf{K}_k [\mathbf{y}_k - \mathbf{H}_k \bar{\mathbf{x}}_k] \\ \mathbf{P}_k &= [\mathbf{I} - \mathbf{K}_k \mathbf{H}_k]\, \bar{\mathbf{P}}_k. \end{aligned} \tag{19}$$

(5) Then given an observation $\mathbf{y}_{k+1}$, return to step (1) and repeat the process. Initial values of $\bar{\mathbf{x}}_0 = 0$ and $\mathbf{P}_0$ set equal to a large symmetric positive definite matrix can be used as starting values for the algorithm. The advantages of using the algorithm described by Equations (16) through (19) over the usual minimum variance estimate are the following:

(1) The effects of the unmodeled errors in the equations of motion for the system can be included in a convenient manner.

(2) The effects of the linearization can be minimized by rectifying the modeled trajectory at each observation point to include the knowledge gained by processing the data at that time, i.e. by selecting the initial condition for the integration indicated in step (1) as $\bar{\mathbf{X}}_{k-1} = \mathbf{X}^*_{k-1} + \hat{\mathbf{x}}_{k-1}$.

(3) The history of unmodeled acceleration, $\boldsymbol{\varepsilon}(t)$, is determined as a direct output of the estimation process. Additional analysis should yield the source of the unmodeled acceleration and this information can be used to improve the basic dynamical model.

An additional advantage associated with utilizing the approach described in the previous discussion lies in the more realistic description of the observation-state relationship. The expressions for the range and range-rate observations for the problem described by Equation (9) are

$$\varrho = [(\mathbf{r} - \mathbf{r}_s)\cdot(\mathbf{r} - \mathbf{r}_s)]^{\frac{1}{2}} + v_\varrho$$
$$\dot{\varrho} = \frac{1}{\varrho}[(\mathbf{r} - \mathbf{r}_s)\cdot(\mathbf{r} - \mathbf{r}_s)] + v_{\dot{\varrho}} \tag{20}$$

where v_ϱ and $v_{\dot{\varrho}}$ are noise components in the observation. If the position and velocity of the satellite, $\mathbf{r}$ and $\dot{\mathbf{r}}$, and the tracking station, $\mathbf{r}_s$ and $\dot{\mathbf{r}}_s$, respectively are expressed in rectangular cartesian coordinates, then the coordinates of the tracking station in the inertial coordinate systems can be expressed in terms of the tracking station in the Earth-fixed coordinate system by the following relations:

$$\begin{bmatrix} U_{s_i} \\ V_{s_i} \\ W_{s_i} \end{bmatrix} = S^T(\theta, \psi, \phi) \begin{bmatrix} \omega_y z_{s_i} - \omega_z y_{s_i} \\ \omega_z x_{s_i} - \omega_x z_{s_i} \\ \omega_x y_{s_i} - \omega_y x_{s_i} \end{bmatrix}; \quad \begin{bmatrix} X_{s_i} \\ Y_{s_i} \\ Z_{s_i} \end{bmatrix} = S^T(\theta, \psi, \phi) \begin{bmatrix} x_{s_i} \\ y_{s_i} \\ z_{s_i} \end{bmatrix} \tag{21}$$

where $S^T(\theta, \psi, \phi)$ is the usual transformation matrix for transforming relations in the body fixed coordinate system to the inertial coordinate system. Through Equations (20), the effects of the Earth's angular velocity are related directly to the observations. As a consequence, the range-rate observation should be sensitive, in a direct manner, to changes in the angular velocity. Furthermore, since the coordinates of the tracking station are functions of the Euler angles, (θ, ψ, ϕ), and since the differential equations describing these quantities depend on the angular velocity, it is anticipated that the observations of range also will be sensitive to the angular velocity changes.

In summary, the utilization of a dynamical model for the Earth's rotation gives an improved means of reducing the observations to obtain the polar motion. Such a procedure allows observations made at different points in time to be related in a dynamically consistent manner. The Earth's motion is best modeled as a non-linear dynamical system influenced by random and/or unmodeled accelerations. The usual least squares or minimum variance algorithms which process the observations in batch form are subject to errors caused by unmodeled forces However, the method presented in this summary can be used to estimate the motion of the Earth's pole as well as the unmodeled forces influencing the Earth's motion. The algorithm described in this summary has been programmed for use on the Control Data Corporation 6400/6600 at The University of Texas at Austin. Numerical results obtained during further study will be presented at a later date.

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