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C-NODAL SURFACES OF ORDER THREE

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The problem of describing a surface of order three can be said to originate in the mid-nineteenth century when A. Cayley discovered that a non-ruled cubic (algebraic surface of order three) may contain up to twenty-seven lines. Besides a classification of cubics, not much progress was made on the problem until A. Marchaud introduced his theory of synthetic surfaces of order three in [**9**]. While his theory resulted in a partial classification of a now larger class of surfaces, it was too general to permit a global description. In [**1**], we added a differentiability condition to Marchaud's definition. This resulted in a partial classification and description of surfaces of order three with exactly one singular point in [**2**]–[**5**]. In the present paper, we examine C-nodal surfaces and thus complete this survey.

A surface F of order three is C-nodal if it is non-ruled, contains exactly one non-differentiable point v and the set of tangents of F at v is a non-degenerate cone of order two; that is, v is the vertex of the cone and any plane, not passing through v, intersects the cone in an oval.

The classification (2.4) is based upon the configuration of lines in a surface. In each of the subsequent sections, we describe a class of surfaces with a fixed number $l(v)$ of lines of F through v. In particular, we determine the distribution of the three types of differentiable points not lying on any line of the surface. In 3.3, 4.6, 5.10 and 6.13, we present a summary of the results in that section and an algebraic example.

1. Surfaces of order three.

Let P^3 be the real projective three-space. We denote the planes, lines and points of P^3 by the letters $\alpha, \beta, \ldots,$ $L, M, \ldots$ and $p, q, \ldots$ respectively. For a collection of flats $\alpha, L, p, \ldots,$ $\langle \alpha, L, p, \ldots \rangle$ denote the flat of P^3 spanned by them. For a set $\mathscr{M}$ in P^3, $\langle \mathscr{M} \rangle$ denotes the flat of P^3 spanned by the points of $\mathscr{M}$.

1.1 A (*plane*) *curve* Γ is the union of a finite collection of sets $C_\lambda(M)$ where the C_λ's are continuous maps from a line $M = \{m, m', \ldots\}$ into a plane α.

Let $\bar{C} = C_\lambda$. The line $T_m = \lim \langle \bar{C}(m), \bar{C}(m') \rangle$, as $m' \neq m$ tends to m, is the *tangent* of $\bar{C}$ at m. Let $\bar{C}$ be *differentiable*; that is, T_m exists and $|T_m \cap \bar{C}(M)| < \infty$ for every $m \in M$. We introduce (cf. [**1**], 1.3.3) the

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characteristic of $\bar{C}$ at m and the multiplicity with which a line $L \subset \alpha$ meets $\bar{C}$ at m. Then $\bar{C}$ is of *order* n if n is the supremum of the number of points of M, counting multiplicities, mapped into collinear points by $\bar{C}$.

If $\bar{C}$ is of order two [three], we denote $\bar{C}(M)$ by $S^1[F_*^1]$. We note that S^1 is a Jordan curve. For an exposition on F_*^1, we refer to [**1**], 1.4 and [**8**], pp. 1–7. If $\bar{C}(M)$ is a line [point], we consider $\bar{C}$ to have order one [two].

Γ is of *order* k if k is the supremum of the number of points of Γ, counting order on each C_λ, lying on any line not in Γ. If $k = 1$, then Γ is a (straight) line. If $k = 2$, then Γ is an S^1 or an isolated point or a pair of distinct lines. If $k = 3$, then Γ is (i) an F_1^* or (ii) the disjoint union of an F_1^* and an S' or a point or (iii) the union of a line and a Γ' of order two. We denote a Γ of order three satisfying (i) or (ii) by F^1.

1.2 A *surface of order three*, F, in P^3 is a compact, connected set such that every intersection of F with a plane is a curve of order $\leqq 3$ and some plane section is an F^1.

Let F be a surface of order three, $p \in F$. Let α be a plane through p. Then p is *regular* in $F[\alpha \cap F]$ if there is a line N in $P^3[\alpha]$ such that $p \in N$ and $|N \cap F| = 3$. Otherwise, p is *irregular* in $F[\alpha \cap F]$. An F has at most one irregular point and such a point is a cusp, double point or isolated point of some $\alpha \cap F$ ([**1**], 1.4).

A line T is a *tangent* of F at p if T is the tangent of some C_λ at m; $p = C_\lambda(m) \subset C_\lambda(M) \subset F$. Let $\tau(p)$ be the set of tangents of F at p. Then p is *differentiable* if p is regular (in F) and $\tau(p)$ is a plane $\pi(p)$; otherwise, p is *singular*.

We assume that every regular p is differentiable and $\pi(p)$ depends continuously on p.

We denote by $l(p)[l(p, \alpha)]$, the number of lines of $F[\alpha \cap F]$ passing through p and by $l(\alpha)$, the number of lines of $\alpha \cap F$. Clearly $l(\alpha) \leqq 3$. If $\mathscr{M} \subseteq F$ is not a point, we put

$$l(\mathscr{M}) = |\{L \subset P^3 | L \subset \mathscr{M}\}|.$$

Let p be differentiable. Then $p \in T \subset \pi(p)$ implies that either $T \subset F$ or $|T \cap F| \leqq 2$. Thus $l(p) = l(p, \pi(p))$ and p is irregular in $\pi(p) \cap F$. If $l(p) = 0$, then p is an isolated point, cusp or double point of $\pi(p) \cap F$ and we call p *elliptic*, *parabolic* or *hyperbolic* respectively. Let E, I and H denote the set of elliptic, parabolic and hyperbolic points of F respectively.

Let v be irregular (singular) in F. If F is *non-ruled*; that is, $l(F) < \infty$, then $v \in T \subset \tau(v)$ if and only if either $v \in T \subset F$ or $T \cap F = \{v\}$. Moreover, $\tau(v)$ is a plane or a union of two distinct planes or a cone of order two; cf. [**10**].

1.3 Let $\mathscr{A}$ be a closed, connected subset of an S^1 or an F_*^1. If the end points of $\mathscr{A}$ are distinct [equal], then $\mathscr{A}$ is a *subarc* [*subcurve*].

Let p be differentiable. Let $\mathscr{A}(p)$ be the set of all subarcs $\mathscr{A}$ of order two such that $p \in \mathscr{A} \not\subset \pi(p)$; $\{\mathscr{A}_1, \mathscr{A}_2\} \subset \mathscr{A}(p)$. Then $\mathscr{A}_1$ and $\mathscr{A}_2$ are *p-compatible* if there is a $\beta \subset P^3\backslash\{p\}$ and an open neighbourhood $U(p)$ of p in P^3 such that $U(p) \cap (\mathscr{A}_1 \cup \mathscr{A}_2)$ is contained in a closed half-space of P^3 bounded by $\pi(p)$ and β; otherwise, $\mathscr{A}_1$ and $\mathscr{A}_2$ are *p-incompatible*.

A pair of subarcs $\mathscr{A}$ and $\mathscr{A}'$ are *compatible* [*incompatible*] if there is a $p \in \mathscr{A} \cap \mathscr{A}'$ such that $\{\mathscr{A}, \mathscr{A}'\} \subset \mathscr{A}(p)$ and $\mathscr{A}, \mathscr{A}'$ are p-compatible [p-incompatible] ([**1**], 2.5.3).

Let $\mathscr{A}$ be a subarc or a subcurve, either of order two; $\alpha = \langle \mathscr{A} \rangle$. We define

$$e(\mathscr{A}) = \{p \in \alpha\backslash\mathscr{A} | p \text{ lies on a tangent of } \mathscr{A} \text{ at } r \text{ for some } r \in \mathscr{A}\}$$

and $i(\mathscr{A}) = \alpha\backslash\overline{e(\mathscr{A})}$. We note that $\alpha = i(\mathscr{A}) \cup \mathscr{A} \cup e(\mathscr{A})$ and $\mathscr{A} = S^1$ implies that $i(S^1)$ is the open disk in $\langle S^1 \rangle$ bounded by S^1.

1.4 Let $L \subset F$ and $r \in F\backslash L$ such that $\langle L, r \rangle \cap F$ consists of L and an S^1. We denote this S^1 by $S^1(L, r)$.

Let p, q, r and s be collinear points; $|\{p, q, r, s\}| = 4$. We say that p, q *separates* r, s if neither segment of $\langle p, q \rangle$ bounded by p and q contains both r and s; otherwise, p, q *does not separate* r, s. In an obvious manner, we extend these definitions to points on a subcurve; concurrent, coplanar lines and planes through a given line.

Let $\{p, q, r\} \subset \mathscr{A}$, $|\{p, q, r\}| = 3$ and $\mathscr{A}$ a subcurve. We denote by $\mathscr{A}(p, q, r)$ the subarc of $\mathscr{A}$ bounded by p and q and containing r.

Let $\mathscr{S}_n = \{1, 2, \dots, n\}$, n a positive integer.

Finally we note that when the meaning of a topological statement is clear, we do not indicate the topology (usually relative) involved.

1.5 By way of preparation for the classification and the descriptions, we list the following results.

1. Let F be non-ruled. Then $l(p) \leqq 6$ for any point $p \in F$ ([**11**]).

2. If p_1 and p_2 are irregular in F, then $\langle p_1, p_2 \rangle \subset F$ ([**1**], 2.2.6).

3. Let $\alpha \cap F$ be of order two. Then $\alpha \cap F = L \cup L'$, $L \neq L'$ and either $L' \subset \pi(p)$ for every regular $p \in L$ or $L \subset \pi(q)$ for every regular $q \in L'$ ([**1**], 2.2.3).

4. Let $\mathscr{A} \subset \alpha$ be a limit of subcurves or subarcs $\mathscr{A}_\lambda$ of order two. Then $|L \cap \mathscr{A}| \neq 3$ for each $L \subset \alpha$ ([**1**], 2.4.4).

5. Let $p_\lambda[\alpha_\lambda]$ be a sequence of points [planes] converging to $p[\alpha]$; $p_\lambda \in \alpha_\lambda$ for each λ.

a) If $\alpha \cap F$ is not of order two or $\alpha \cap F$ does not contain an isolated point, then $\lim (\alpha_\lambda \cap F) = \alpha \cap F$ ([**1**], 2.4.3).

b) If p_λ is a cusp [isolated point] of $\alpha_\lambda \cap F$ for each λ, then $l(p) = 0$ implies that p is cusp [isolated point or cusp] of $\alpha \cap F$ and $\alpha \cap F = L \cup S^1$ implies that $L \cap S^1 = \{p\}$ ([**1**], 2.4.6 and 2.4.9).

6. If p is regular in F and isolated in $\alpha \cap F$, then p is elliptic and $\alpha = \pi(p)$ ([**1**], 2.3.7).

7. Let $\mathscr{A}' \subset F$ such that $\mathscr{A}' \in \mathscr{A}(r)$ for each $r \in \mathscr{A}'$. Let L be a line such that $L \not\subset \langle \mathscr{A}' \rangle$ and for each $r \in \mathscr{A}'$, there is an $\mathscr{A}_r \in \mathscr{A}(r)$ with $L \subset \langle \mathscr{A}_r \rangle$. If $\mathscr{A}_r$ depends continuously on r, then $\mathscr{A}'$ and $\mathscr{A}_r$ are either compatible for all $r \in \mathscr{A}'$ or incompatible for all $r \in \mathscr{A}'$ ([**1**], 2.5.8).

8. Let r be regular and $\{\mathscr{A}, \mathscr{A}'\} \subset \mathscr{A}(r)$ such that

$$r \in \text{int}\ (\mathscr{A}) \cap \text{int}\ (\mathscr{A}') \quad \text{and} \quad r \notin \overline{e(\mathscr{A}) \cap e(\mathscr{A}')}.$$

Then $\mathscr{A}$ and $\mathscr{A}'$ are incompatible and if $l(r) = 0$, r is hyperbolic ([**5**], 2.5).

9. Let p be regular in F, $l(p) = 0$. Then (i) $p \in E$ if and only if $\mathscr{A}$ and $\mathscr{A}'$ are compatible for $\{\mathscr{A}, \mathscr{A}'\} \subset \mathscr{A}(p)$ and (ii) $p \in H$ if and only if there exist incompatible $\mathscr{A}$ and $\mathscr{A}'$ in $\mathscr{A}(p)$ such that $p \in \text{int}\ (\mathscr{A}) \cap \text{int}\ (\mathscr{A}')$ ([**1**], 2.5.5 and 2.5.7).

10. Every surface of order three contains a line ([**7**]).

11. Let G be an open region in F such that $\alpha \cap \bar{G} = \emptyset$ for some α, bd $(F \backslash G) =$ bd (G), $\langle \text{bd}\ (G) \rangle$ is a plane and each $r \in G$ is regular. Then $G \cap E \neq \emptyset$ ([**6**], 3.7).

12. Let F be non-ruled. Then $H \neq \emptyset$, H and E are open and

$$I = \{p \in \bar{H} \cap \bar{E} | l(p) = 0 \text{ and } p \text{ is regular}\}$$

is nowhere dense in F ([**6**], 3.8 and 3.9).

In view of 12: if we wish to describe an open region $X \subset F$ such that $l(r) = 0$ for each $r \in X$, we need only determine if $X \cap E \neq \emptyset$ or $X \subset H$. If $X \cap E \neq \emptyset$ and $X \not\subset E$, then $X \cap H \neq \emptyset$ usually follows by 1.5.5 b) applied to bd (X).

1.6 Let F be a non-ruled surface of order three containing exactly one irregular point v. In 1.2, we noted the possibilities for $\tau(v)$. We have already examined the case where $\tau(v)$ is a plane ([**3**]) and the case where $\tau(v)$ is a pair of planes ([**4**] and [**5**]). In this paper, we assume that $\tau(v)$ is a nondegenerate cone of order two.

We note that a cone of order two may degenerate into a line. If $\tau(v) = N$ and $N \cap F = \{v\}$, we call v a *peak* and F a surface with a peak; cf. [**2**]. We claim that these are the only non-ruled surfaces of order three containing exactly one irregular point v; that is, if $\tau(v)$ is a line N, then $N \cap F = \{v\}$.

Suppose $N \subset F$ and let $N \subset \beta$. From 1.2, $v \in L \neq N$ implies that $|L \cap F| = 2$. Hence either $\beta \cap F = N \cup N'$ where $v \notin N'$ or $\beta \cap F$ consists of N and an S^1 such that $N \cap S^1 = \{v\}$. Let $p \in N \backslash \{v\}$. Since $\pi(p)$ exists, $N \subset \pi(p)$ and p is irregular in $\pi(p) \cap F$, the preceding implies $\pi(p) \cap F = N \cup N_p$ where $N \cap N_p = \{p\}$. Then $N_p \neq N_q$ for $p \neq q$ in $N \backslash \{v\}$ yields that F contains infinitely many lines. Since F is non-ruled, this is a contradiction.

2. C-nodal surfaces.

2.0 Let F be a surface of order three. A point $v \in F$ is a *C-node* if v is irregular in F and $\tau(v)$ is a nondegenerate cone of order two with vertex v. F is *C-nodal* if F is non-ruled and has a C-node as its only irregular point.

Henceforth F is C-nodal with the C-node v. We denote $\tau(v)$ by K. From 1.5.1, $0 \leqq l(v) \leqq 6$.

From the definition of K, K is the common boundary of two disjoint open regions of F. It is clear that exactly one of these regions contains a line not meeting K. We denote this region by ext (K) and put int (K) $= P^3 \backslash \overline{\text{ext }(K)}$. Hence

$$P^3 = \text{int }(K) \cup K \cup \text{ext }(K).$$

Since K is of order two, any plane through v meets K in at most two lines. We note that K is not necessarily differentiable (n.n.d.) and hence $v \notin \alpha$ implies that $\alpha \cap K$ is a (n.n.d.) curve of order two; that is, any line $L \subset \alpha$ meets $\alpha \cap K$ in at most two points but $|L \cap K| = 1$ does not imply that L is a tangent of K.

2.1. Lemma. *Let β be a plane through v.*

1. *If $\beta \cap K$ consists of a pair of lines N_1 and N_2, then*

i) $(N_1 \cup N_2) \cap F = \{v\}$ *implies that v is the double point of $\beta \cap F$,*

ii) $N_1 \cup N_2 \subset F$ *implies that $\beta \cap F$ consists of three non-concurrent lines and*

iii) $N_i \subset F$ *and* $N_j \cap F = \{v\}$ *implies that $\beta \cap F$ consists of N_i and S^1 such that $|N_i \cap S^1| = 2$ and $N_j \cap S^1 = \{v\}$; $\{i, j\} \in \mathscr{S}_2$.*

2. *If $\beta \cap K$ consists of a line N, then*

i) $N \cap F = \{v\}$ *implies that v is the cusp of $\beta \cap F$ and*

ii) $N \subset F$ *implies that $\beta \cap F = N \cup S^1$ where $N \cap S^1 = \{v\}$.*

3. *If $\beta \cap K = \{v\}$, then v is the isolated point of $\beta \cap F$.*

Proof. Since $v \in L \not\subset K$ implies that $|L \cap F| = 2$, the assertions 1, 2 i) and 3 are immediate.

If $\beta \cap K = N \subset F$, then either $\beta \cap F = N \cup S^1$, $N \cap S^1 = \{v\}$, or $\beta \cap F = N \cup N'$, $N \notin N'$. In the latter case, $\beta = \pi(p)$ for $p \in N \backslash \{v\}$ by 1.5.2 and 1.5.3. Since $l(v) \leqq 6$, there is an $N' \subset K$ such that $N' \cap F = \{v\}$. By 1 iii),

$$\langle N, N' \rangle \cap F = N \cup S^1$$

and $N \cap S^1$ consists of v and say $p' \neq v$. Then $\pi(p') = \langle N, N' \rangle \neq \beta$, a contradiction.

2.2 If v is the double point of $\beta \cap F$, then $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ where $\mathscr{L} \cap (\mathscr{A}_1 \cup \mathscr{A}_2) = \{v\}$, $\mathscr{A}_1 \cap \mathscr{A}_2 = \{v, p\}$, p is the inflection point of $\beta \cap F$, $\{\mathscr{A}_1, \mathscr{A}_2\} \subset \mathscr{A}(p)$ and $\mathscr{L}$ is a subcurve of order two. We call $\mathscr{L}$

the *loop* of $\beta \cap F$. We note that any tangent of $\mathscr{L}\backslash\{v\}$ meets $\mathscr{A}_1 \cup \mathscr{A}_2$ and no tangent of $(\mathscr{A}_1 \cup \mathscr{A}_2)\backslash\{v\}$ meets $\mathscr{L}$.

If v is the cusp of $\beta \cap F$, then $\beta \cap F = \mathscr{A} \cup \mathscr{A}', \mathscr{A} \cap \mathscr{A}' = \{v, p\}$, p is the inflection point of $\beta \cap F$ and $\{\mathscr{A}, \mathscr{A}'\} \subset \mathscr{A}(p)$.

2.3 LEMMA. 1. *If* $(L \cup L') \cap K = \emptyset$, *then* $L \cap L' \neq \emptyset$.
2. $l(v) = 0, 2, 4$ *or* 6.

Proof. 1. Since $\langle L, v\rangle \cap F = L \cup \{v\}$ and $v \notin L'$, we have $L \cap L' \neq \emptyset$.
2. i) Let $K \cap F = M$ and $M \subset \beta$. From 2.1, $\beta \cap F = M \cup S^1$ such that $v \in S^1$. If $\beta \cap K = M$, then $M \cap S^1 = \{v\}$ and we put $\beta = \beta_v$ and $S^1 = S_v^1$. If $\beta \cap K \neq M$, then $M \cap S^1 = \{v, p\}$, $v \neq p$ and we put $\beta = \beta_p$ and $S^1 = S_p^1$.
By 1.5.4 and 1.5.5, $\lim \beta_p = \beta_q$ implies that

$$\lim S_p^1 = S_q^1, \lim i(S_p^1) = i(S_q^1) \quad \text{and} \quad \lim e(S_p^1) = e(S_q^1).$$

Then $\lim \beta_p = \beta_v$ and $M \cap S_v^1 = \{v\}$ yield that

$$\lim (M \cap i(S_p^1)) = \emptyset.$$

Let $q \in M\backslash\{v\}$. Then $M \cap S_q^1 = \{v, q\}$ and v and q are the end points of the disjoint open segments $M \cap i(S_q^1)$ and $M \cap e(S_q^1)$. By the preceding,

$$M \cap i(S_q^1) \subset M \cap i(S_p^1) \quad \text{for each } p \in M \cap e(S_q^1).$$

Hence as p tends to v in $M \cap e(S_q^1)$,

$$M \cap i(S_q^1) \subset \lim (M \cap i(S_p^1)).$$

This is a contradiction and thus $l(v) \neq 1$.
ii) Let $K \cap F = M_1 \cup M_2 \cup M_3$, $l(v) = 3$. By 2.1, there is a line $L_{ij} \subset \langle M_i, M_j\rangle \cap F$ such that $v \notin L_{ij}$, $i \neq j$ in $\mathscr{S}_3$. Clearly L_{12}, L_{13} and L_{23} are mutually disjoint.
Let $p \in M_3\backslash\{v\}$. Then

$$\alpha_p = \langle L_{12,p}\rangle \neq \langle M_1, M_2\rangle,$$

$\alpha_p \cap K$ is a (n.n.d.) curve of order two and

$$(\alpha_p \cap K) \cap F = \{L_{12} \cap M_1, L_{12} \cap M_2, p\}.$$

As $L_{12} \cap (M_3 \cup L_{13} \cup L_{23}) = \emptyset$, $l(p) = 1$ for $p \in M_3\backslash\{v\}$ implies that $\alpha_p \cap F = L \cup S^1$ such that $K \cap S^1 = \{p\}$. Put $S^1 = S_p^1$. Since S_p^1 and $\alpha_p \cap K$ are both curves of order two, it is clear that either $S_p^1 \subset \overline{\text{int }(K)}$ or $S_p^1 \subset \overline{\text{ext }(K)}$. As S_p^1 depends continuously on $p \in M_3\backslash\{v\}$, either $S_p^1 \subset \overline{\text{int }(K)}$ for all $p \in M_3\backslash\{v\}$ or $S_p^1 \subset \overline{\text{ext }(K)}$ for all $p \in M_3\backslash\{v\}$. It is easy to check that this is impossible and thus $l(v) \neq 3$.
iii) Let $K \cap F = \cup M_i$, $i \in \mathscr{S}_5$ and $l(v) = 5$. Then

$$\langle M_i, M_j\rangle \cap F = M_i \cup M_j \cup L_{ij}$$

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where $v \notin L_{ij}$, $i \neq j$ in $\mathscr{S}_5$. We note that $L_{ij} \cap L_{kl} \neq \emptyset$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$ and for $k \notin \{i, j\}$, $M_k \cap L_{ij} = \emptyset$.

Since $L_{13} \cap \langle L_{12}, L_{34} \rangle \neq \emptyset$, $L_{13} \cap (L_{12} \cup L_{34}) = \emptyset$ implies that there is a third line $L^* \subset \langle L_{12}, L_{34} \rangle \cap F$. Clearly L^* is not an L_{ij} and $L^* \cap L_{i5} = \emptyset$, $i \in \mathscr{S}_4$. Finally $M_5 \cap (L_{12} \cup L_{34}) = \emptyset$ yields that $L^* \cap M_5 \neq \emptyset$, $L^* \neq M_5$ and thus $\langle M_5, L^* \rangle \cap K = M_5$. This contradicts 2.1.2 ii) and hence $l(v) \neq 5$.

2.4 THEOREM. *Let F be C-nodal with the C-node v. Then F is one of the following types:* (1) $l(v) = 0$ *and* $1 \leqq l(F) \leqq 3$; (2) $l(v) = 2$ *and* $4 \leqq l(F) \leqq 5$; (3) $l(v) = 4$ *and* $l(F) = 11$ *and* (4) $l(v) = 6$ *and* $l(F) = 21$.

Proof. (1) If $l(v) = 0$, then 2.3.1 implies that the lines of F are coplanar and hence $l(F) \leqq 3$. By 1.5.10, $l(F) \geqq 1$.

(2) If $l(v) = 2$, then $l(F) \geqq 3$ from 2.1. Let

$$K \cap F = M_1 \cup M_2 \quad \text{and} \quad v \notin L_{12} \subset \langle M_1, M_2 \rangle \cap F.$$

From 2.1, any other line of F is disjoint from K and thus meets L_{12}. By 2.3.1, such lines are coplanar and thus $l(F) \leqq 5$. For the proof that $l(F) = 4$ or 5, we refer to 4.2.

(3) Let $K \cap F = \cup M_i$, $i \in \mathscr{S}_4$ and $l(v) = 4$. Then there is a line $L_{ij} \subset \langle M_i, M_j \rangle \cap F$ such that $v \notin L_{ij}$ for $i \neq j$ in $\mathscr{S}_4$ and $L_{ij} \cap L_{kl} \neq \emptyset$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. We note that these ten lines are the only lines of F meeting K.

As in the proof of 2.3 iii), there is a line $L_0 \subset \langle L_{12}, L_{34} \rangle \cap F$ such that $L_0 \cap K = \emptyset$. Suppose $L_1 \subset F$ such that $L_1 \neq L_0$ and $L_1 \cap K = \emptyset$. By 2.3.1, $\langle L_0, L_1 \rangle$ is a plane and hence $K \cap (L_0 \cup L_1) = \emptyset$ implies there is a line $L_2 \subset \langle L_0, L_1 \rangle \cap F$ such that $L_2 \cap M_i \neq \emptyset$, $i \in \mathscr{S}_4$. Then either $l(\langle L_2, v \rangle) \geqq 4$ or $v \in L_2$ and $l(v) \geqq 5$, a contradiction.

(4) Let $K \cap F = \cup M_i$, $i \in \mathscr{S}_6$ and $l(v) = 6$. Again

$$\langle M_i, M_j \rangle \cap F = M_i \cup M_j \cup L_{ij}, \quad v \notin L_{ij} \ (i \neq j)$$

and $L_{ij} \cap L_{kl} \neq \emptyset$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. These twenty-one lines are the only lines of F meeting K.

Let $L \subset F$ such that $L \cap K = \emptyset$. Then $L \cap \langle M_i, M_j \rangle \neq \emptyset$ and $L \cap (M_i \cup M_j) = \emptyset$ imply that $L \cap L_{ij} \neq \emptyset$, $i \neq j$ in $\mathscr{S}_6$. But $L \cap L_{12} \neq \emptyset \neq L \cap L_{34}$, $L_{56} \subset \langle L_{12}, L_{34} \rangle \cap F$ and $l(\langle L_{12}, L_{34} \rangle) = 3$ imply that L is one of L_{12}, L_{34} or L_{56}, a contradiction.

3. F with $l(v) = 0$.

3.0 Let F be C-nodal with the C-node v, $l(v) = 0$. We recall that

$$P^3 = \overline{\text{int } (K)} \cup K \cup \overline{\text{ext } (K)}$$

and every line meets $\overline{\text{ext } (K)}$. Let $F_1 = \text{int } (K) \cap F$ and $F_2 = \text{ext } (K) \cap F$. Then $K \cap F = \{v\}$ implies that $F = F_1 \cup F_2 \cup \{v\}$.

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Let $v \in N \subset \beta$ such that $N \backslash \{v\} \subset \text{int}(K)$. Then $\beta \cap K$ is a pair of distinct lines N_1 and N_2 and from 2.1, v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$. Since N_1 and N_2 are the tangents of $\beta \cap F$ at v, we obtain that either $\mathscr{L} \subset \overline{\text{int}(K)}$ and $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{ext}(K)}$ or $\mathscr{L} \subset \overline{\text{ext}(K)}$ and $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}(K)}$. It is well known that every line of β meets $\mathscr{A}_1 \cup \mathscr{A}_2$ and hence

$$\mathscr{L} = \beta \cap \bar{F}_1 \quad \text{and} \quad \mathscr{A}_1 \cup \mathscr{A}_2 = \beta \cap \bar{F}_2.$$

3.1 THEOREM. 1. *Every point of* F_1 *is elliptic.*
2. *Every line of* F *is contained in* F_2.

Proof. Clearly 1 implies 2.

Let $r \in F_1$. Then $r \neq v$, $l(v) = 0$ and 1.5.2 imply that $v \notin \pi(r)$ and $\pi(r) \cap K$ is a curve of order two. As $r \in F_1 \subset \text{int}(K)$ and $K \cap F = \{v\}$, we obtain that

$$r \in i(\pi(r) \cap K), \ (\pi(r) \cap K) \cap F = \emptyset \quad \text{and} \quad l(r) = 0.$$

It is immediate that $\pi(r) \cap F$ is disconnected and thus $r \in E$.

3.2 Since $F_1 \subseteq E$, we need only examine $\bar{F}_2$ to describe F completely. With slight modifications and $\bar{F}_1$ and v identified, the examination of $\bar{F}_2$ is a reiteration of the study of 'surfaces of order three with a peak' in [**2**].

3.3 SUMMARY. *Let* F *be C-nodal with the C-node* v, $l(v) = 0$. *Then*

$$F = F_1 \cup \{v\} \cup F_2$$

where $\bar{F}_i = F_i \cup \{v\}$, *every point of* F_1 *is elliptic and one of the following holds:*

1. $l(F_2) = 1$ *and every point* $p \in F_2$ *with* $l(p) = 0$ *is hyperbolic.*
2. $l(F_2) = 2$ *or* 3 *and* $F_2 = F_2' \cup F_2^*$ *where*
 i) F_2' *and* F_2^* *are disjoint regions with* $l(q) > 0$ *for* $q \in \overline{F_2'} \cap F_2^*$,
 ii) F_2' *is open*, $v \in \overline{F_2'}$ *and every point of* F_2' *is hyperbolic, and*
 iii) F_2^* *is closed, contains elliptic, parabolic and hyperbolic points and* $l(F_2^*) = l(F_2) = l(F)$.

Let P^3 be suitably coordinatized. The surface in P^3 defined by

$$x_0^3 - (x_1^2 + x_2^2 - x_0^2)x_3 = 0$$

satisfies 3.3.1 with $v \equiv (0, 0, 0, 1)$, line $L \equiv x_0 = x_3 = 0$ and $K \equiv x_1^2 + x_2^2 - x_0^2 = 0$. The surface defined by

$$x_0^3 + x_0x_1^2 - (x_1^2 + x_2^2 - x_0^2)x_3 = 0$$

satisfies 3.3.2 with $v \equiv (0, 0, 0, 1)$, lines $L_1 \equiv x_0 = x_3 = 0$, $L_2 \equiv x_0 - x_3 = x_2 + 2^{1/2}x_0 = 0$, $L_3 \equiv x_0 - x_3 = x_2 - 2^{1/2}x_0 = 0$ and $K \equiv x_1^2 + x_2^2 - x_0^2 = 0$. We refer to Figure 1 for a representation of F satisfying 3.3.2.

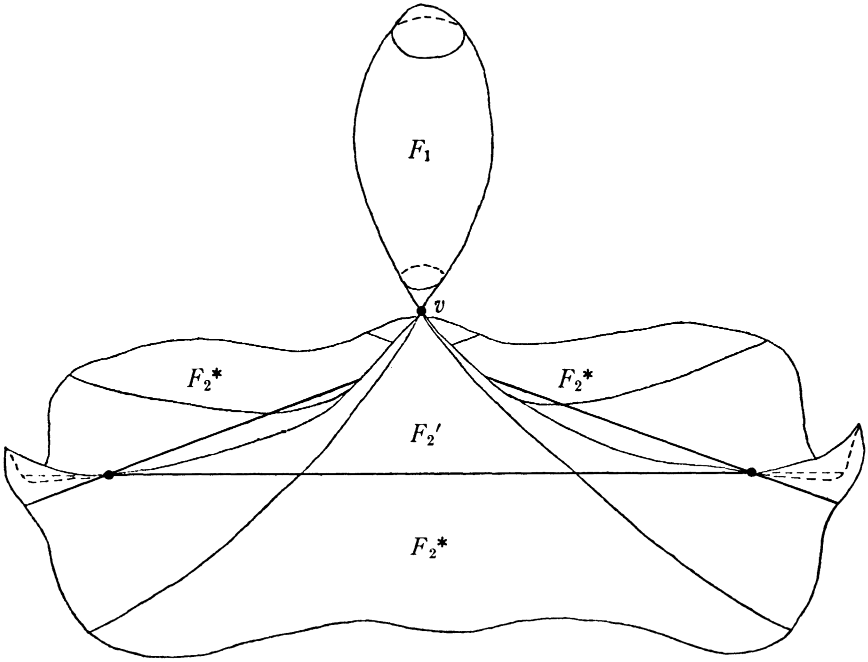

FIGURE 1

4. F with $l(v) = 2$.

4.0 Let F be C-nodal with C-node v, $l(v) = 2$ and $K \cap F = M_1 \cup M_2$. Then $\langle M_1, M_2 \rangle \cap F$ contains a line L_{12}, $v \notin L_{12}$ and (cf. the proof of 2.4) there are at most two other lines in F, neither of which meets K. We note that for $p \in L_{12} \backslash K$, $\pi(p) \cap K$ is a (n.n.d.) curve of order two meeting F at exactly $M_1 \cap L_{12}$ and $M_2 \cap L_{12}$. Therefore $l(p) = 1$ for all $p \in L_{12} \cap \text{int}(K)$.

4.1 LEMMA. 1. *Let* $p \in L_{12} \cap \text{int}(K)$. *Then* $l(\pi(p)) = 1$.

2. *Let* $p \in L_{12} \backslash K$ *such that* $\pi(p) \cap F = L_{12} \cup S^1$. *Then* $p \in \text{int}(K)$ *if and only if* $S^1 \subset \text{int}(K)$.

3. *There exists a* $p_0 \in L_{12} \cap \text{int}(K)$ *and a* $p_1 \in L_{12} \cap \text{ext}(K)$ *such that* $\pi(p_0) \cap F = \pi(p_1) \cap F = L_{12}$.

4. *Let* v *be the double point of* $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$. *Then* $\mathscr{L} \subset \text{int}(K)$ *if and only if* $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \text{ext}(K)$ *if and only if* $\beta \cap L_{12} \subset \text{ext}(K)$.

Proof. 1. If $l(\pi(p)) > 1$, then $l(p) = 1$ and p irregular in F imply that $l(\pi(p)) = 2$. Let $\pi(p) \cap F = L_{12} \cup L$, $p \notin L$. By 1.5.3, $L \subset \pi(p)$ yields that

$$L \subset \pi(L_{12} \cap M_1) \cap F = M_1 \cup M_2 \cup L_{12},$$

a contradiction.

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2. This is immediate since $p \in L_{12} \cap S^1$ and $S^1 \cap K = \emptyset$.

3. Clearly 1.5.5 and $l(\langle M_1, M_2 \rangle) = 3$ imply that there exist p' close to say $M_1 \cap L_{12}$ in both $L_{12} \cap \overline{\text{int}\ (K)}$ and $L_{12} \cap \overline{\text{ext}\ (K)}$ such that $\pi(p') \cap F$ consists of L_{12} and S^1. Now 1.5.5 and 2 yield 3.

4. Apply 2 and 3.

4.2 THEOREM. *There exist lines L_1 and L_2 in F such that*

$$(L_1 \cup L_2) \cap K = \emptyset.$$

Proof. Let v be the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$, $\beta \cap L_{12} = \{\bar{p}\} \subset \text{int}\ (K)$. Then $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}\ (K)}$ and $\mathscr{L} \subset \overline{\text{ext}\ (K)}$ by 4.1.4. From 2.2, there exists an $\bar{r} \in \mathscr{L}\backslash\{v\}$ such that $\bar{p} \in \pi(\bar{r})$ and thus $\bar{p} \in e(\mathscr{L})$. Let $p^* \in L_{12}$ such that $\langle p^*, \bar{r} \rangle \cap K = \emptyset$. Then $p^* \in \text{ext}\ (K)$, $\alpha = \langle v, p^*, r \rangle$ is a plane, $\alpha \cap K = \{v\}$ and v is the isolated point of $\alpha \cap F$.

Let H_1 and H_2 be the closed half-planes of β determined by $\langle \bar{r}, v \rangle$ and $\langle \bar{r}, \bar{p} \rangle$. Then $\langle \bar{r}, v \rangle \cap \mathscr{L} = \{\bar{r}, v\}$ and $\langle \bar{r}, \bar{p} \rangle \cap \mathscr{L} = \{\bar{r}\}$ yield that $\mathscr{L}_1 = H_1 \cap \mathscr{L}$ and $\mathscr{L}_2 = H_2 \cap \mathscr{L}$ are subarcs such that

$$\mathscr{L}_1 \cup \mathscr{L}_2 = \mathscr{L} \quad \text{and} \quad \mathscr{L}_1 \cap \mathscr{L}_2 = \{v, \bar{r}\}.$$

Let $\bar{p} \in N \subset \beta$ such that $|N \cap \mathscr{L}| = 2$. Then $v \notin N$ and $|N \cap \mathscr{L}_i| = 1$, $i \in \mathscr{S}_2$. Since the lines of F not meeting K are coplanar by 2.3.1, we obtain that, except for at most one plane, $\langle L_{12}, N \rangle \cap F$ consists of L_{12} and a curve S_N^1 of order two such that $S_N^1 \cap \mathscr{L}_i = N \cap \mathscr{L}_i$, $i \in \mathscr{S}_2$. Let $N \cap \langle v, \bar{r} \rangle = n$. Clearly $n \in i(\mathscr{L})$ and thus $\bar{p} \in e(\mathscr{L})$ implies that $\bar{p}$, n separates $N \cap \mathscr{L}_1$, $N \cap \mathscr{L}_2$. Finally we note that $\lim N = \langle \bar{p}, v \rangle$ implies that $\lim n = v$ and in particular $\lim \langle p^*, n \rangle = \langle p^*, v \rangle$.

Let $N = \langle \bar{p}, n \rangle$ be arbitrarily close to $\langle \bar{p}, v \rangle$. Since v is the isolated point of $\alpha \cap F$, $\langle p^*, n \rangle$ arbitrarily close to $\langle p^*, v \rangle$ implies that

$$\langle p^*, n \rangle \cap F = \{p^*\}.$$

Then $|\langle p^*, n \rangle \cap S_N^1| \leqq 1$ and the preceding yield that $n \in e(S_N^1)$ and $\bar{p} \in i(S_N^1)$. As $\bar{p} \in e(\mathscr{L})$ and $|\mathscr{L} \cap S_N^1| = 2$, we obtain that

$$e(\mathscr{L}) \cap e(S_N) = \emptyset$$

and thus $\mathscr{L}$ and S_N are incompatible by 1.5.8.

If $l(\langle L_{12}, N \rangle) > 1$ for some $N \subset \beta$ through $\bar{p}$, then 4.2. Hence we may assume that $\bar{p} \in N \subset \beta$ and $|N \cap \mathscr{L}| = 2$ imply that

$$\langle L_{12}, N \rangle \cap F = L_{12} \cup S_N^1.$$

Then the preceding and 1.5.7 yield that $\bar{p} \in i(S_N^1)$ for all such N. Since $\bar{p} \in \pi(\bar{r})$, we obtain that

$$\langle L_{12}, \bar{r} \rangle \cap F = L_{12} \cup \{\bar{r}\} \quad \text{or}$$

$$\langle L_{12}, \bar{r} \rangle \cap F = L_{12} \cup S^1$$

where $\bar{p} \in e(S^1)$ or $l(\bar{r}) > 0$. Clearly, each of the first two cases contradicts the continuity of the plane sections of F through L_{12} and thus $l(\bar{r}) > 0$. (If $l(\bar{r}) = 1$, then either $L_1 = L_2$ or $L_1 \cap L_2 \neq \{\bar{r}\}$.)

4.3 Let int $(K) \cap F = F_1$ and ext $(K) \cap F = F_2$. Then $K \cap F = M_1 \cup M_2$ implies that

$$\bar{F}_1 = F_1 \cup M_1 \cup M_2, \quad \bar{F}_2 = F_2 \cup M_1 \cup M_2,$$
$$L_1 \cup L_2 \subset F_2, \quad \bar{F}_1 \cap \bar{F}_2 = M_1 \cup M_2 \quad \text{and}$$
$$F = F_1 \cup F_2 \cup M_1 \cup M_2.$$

By 4.1.3, there is a $p_0 \in F_1$ such that $\pi(p_0) \cap F = L_{12}$ and thus $F_1 \backslash L_{12}$ consists of two open disjoint regions, say F_{11} and F_{12}. Clearly

$$F_1 = F_{11} \cup F_{12} \cup (F_1 \cap L_{12}) \quad \text{and} \quad \bar{F}_{11} \cap \bar{F}_{12} = \overline{(F_1 \cap L_{12})} \cup \{v\}.$$

If $L_1 \neq L_2$, let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the open half-spaces of P^3 determined by $\langle L_1, v\rangle$ and $\langle L_2, v\rangle$. We assume that $K\backslash\{v\} \subset \mathscr{P}_1$ and let $\mathscr{P}_1 \cap F_2 = F_{21}$, $\mathscr{P}_2 \cap F_2 = F_{22}$. Then

$$\text{bd } (F_{21}) = M_1 \cup M_2 \cup L_1 \cup L_2 \quad \text{and} \quad \text{bd } (F_{22}) = L_1 \cup L_2.$$

If $L_1 = L_2 = L$, let $F_{21} = F_2 \backslash L$ and $F_{22} = \emptyset$. In either case,

$$F_2 = F_{21} \cup F_{22} \cup L_1 \cup L_2.$$

4.4 THEOREM. *$F_{1i} \cap E \neq \emptyset$ and $v \in \overline{F_{1i} \cap E}$, $i \in \mathscr{S}_2$.*

Proof. Let $N \subset K$ such that $N \cap F = \{v\}$. Since K is a cone of order two, there is a plane γ through N such that $\gamma \cap K = N$ and γ is the limit of a sequence of planes β_λ such that v is the double point of $\beta_\lambda \cap F = \mathscr{L}_\lambda \cup \mathscr{A}_{1,\lambda} \cup \mathscr{A}_{2,\lambda}$ for each λ. Since $\gamma \cap L_{12} \subset$ ext (K), we may assume that

$$\beta_\lambda \cap L_{12} \subset \text{ext } (K) \text{ for each } \lambda.$$

Then $\mathscr{L}_\lambda \subset F_{11} \cup F_{12} \cup \{v\}$ for each λ by 4.1.4. Finally as $F_{11} \cap F_{12} = \emptyset$, we may assume that

$$\mathscr{L}_\lambda \subset F_{11} \cup \{v\} \quad \text{for each } \lambda.$$

From 2.1, v is the cusp of $\gamma \cap F = \mathscr{A}_1 \cup \mathscr{A}_2 \subset$ ext $(K) \cup \{v\}$. Hence $\lim \beta_\lambda = \gamma$, 1.5.5 a) and 1.5.4 imply that

$$\lim (\mathscr{A}_{1,\lambda} \cup \mathscr{A}_{2,\lambda}) = \mathscr{A}_1 \cup \mathscr{A}_2 \quad \text{and} \quad \lim \mathscr{L}_\lambda = \{v\}.$$

We note that F_{11} is a bounded open region satisfying 1.5.11. Thus for each λ, $\mathscr{L}_\lambda$ is the boundary of an open region $F_{11}(\mathscr{L}_\lambda) \subset F_{11}$ satisfying 1.5.11. Since $\lim \mathscr{L}_\lambda = \{v\}$ implies that $\lim \overline{F_{11}(\mathscr{L}_\lambda)} = \{v\}$, we obtain that $v \in \overline{F_{11} \cap E}$.

The preceding argument is symmetric in F_{11} and F_{12}.

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4.5 THEOREM. 1. *If* $F_{22} \neq \emptyset$, *then* $F_{22} \cap E \neq \emptyset$.

2. *Every* $r \in F_{21}$ *such that* $l(r) = 0$ *is hyperbolic.*

Proof. 1. We recall that $F_{22} \neq \emptyset$ implies that $L_1 \neq L_2$. If $L_1 \cap L_2 \cap L_{12} = \emptyset$, then $F_{22} \backslash L_{12}$ clearly consists of two open, disjoint triangular regions satisfying 1.5.11.

If $|L_1 \cap L_2 \cap L_{12}| = 1$, then bd $(F_{22}) = L_1 \cup L_2$ yields that, for $r \in F_{22}$, $l(r) = 0$ and $\langle L_{12}, r \rangle \cap F_{22}$ consists of either the isolated point r (hence $r \in E$) or an S^1 disjoint from L_{12}. In the latter case, 4.1.3 and the continuity of the plane section of F_{22} through L_{12} imply that there is an $r' \in F_{22}$ such that

$$\langle L_{12}, r' \rangle \cap F = L_{12} \cup \{r'\}.$$

2. Let $r \in F_{21}$, $l(r) = 0$. Since $\langle v, p, r \rangle$ is a plane for $p \in L_{12} \cap \text{int } (K)$, we may assume that $r \in \beta = \langle v, \bar{p}, \bar{r} \rangle$; cf. the proof of 4.2. Then $\{r, \bar{r}\} \subset \mathscr{L}$. Since $l(r) = 0$ and $\bar{p} \in \pi(\bar{r})$, either $r = \bar{r}$ or

$$\langle L_{12}, r \rangle \cap F = L_{12} \cup S_N^1$$

where $N = \langle \bar{p}, r \rangle$, $|N \cap \mathscr{L}| = 2$ and $r \in \mathscr{L} \cap S_N^1$.

If $F_{22} = \emptyset$, then $L_1 = L_2 = L$ and $\langle L_{12}, L \rangle \cap F = L_{12} \cup L$. As $\bar{p} \in \pi(\bar{r})$, 1.5.3 implies that $L \cap \mathscr{L} = \{\bar{r}\}$ and $r \neq \bar{r}$. If $F_{22} \neq \emptyset$, then $v \in \bar{F}_{21} \backslash \bar{F}_{22}$ implies that $\bar{F}_{21} \cap \mathscr{L}$ is the subarc containing v and bounded by $L_1 \cap \mathscr{L}$ and $L_2 \cap \mathscr{L}$. Clearly

$$|\langle \bar{p}, r' \rangle \cap \mathscr{L}| = 2 \quad \text{for } \bar{r} \in (\bar{F}_{21} \cap \mathscr{L}) \backslash \{v\}$$

and thus $\bar{p} \in \pi(\bar{r})$ implies that $\bar{r} \in F_{22} \cap \mathscr{L}$ and $r \neq \bar{r}$. It now readily follows from the proof of 4.2 that $\mathscr{L}$ and S_N^1 are incompatible and thus $r \in H$ by 1.5.9.

4.6 SUMMARY. *Let* F *be* C*-nodal with* $l(v) = 2$. *Then*

$$F = F_{11} \cup F_{12} \cup F_{21} \cup F_{22} \cup M_1 \cup M_2 \cup L_1 \cup L_2 \cup L_{12}$$

where

$$K \cap F = M_1 \cup M_2,\ L_{12} \subset \langle M_1, M_2 \rangle,\ (L_1 \cup L_2) \cap C = \emptyset$$

and the F_{ij}*'s are open, disjoint regions described in* 4.3 *such that*

i) F_{11}, F_{12} *and a non-empty* F_{22} *contain elliptic points,*

ii) *every* $r \in F_{21}$ *such that* $l(r) = 0$ *is hyperbolic and*

iii) $F_{22} = \emptyset$ *if and only if* $L_1 = L_2$.

We refer to Figure 2 for a representation of F. The surface in P^3 defined by

$$x_0(x_1^2 + x_2^2) + x_3(x_0^2 + x_1x_2) = 0$$

satisfies 4.6 with $M_1 \equiv x_0 = x_1 = 0$, $M_2 \equiv x_0 = x_2 = 0$, $L_{12} \equiv x_0 = x_3 = 0$, $L_1 \equiv x_3 = -2x_0 = 2^{1/2}(x_1 - x_2)$, $L_2 \equiv x_3 = -2x_0 = 2^{1/2}(x_2 - x_1)$ and $K \equiv x_0^2 + x_1x_2 = 0$.

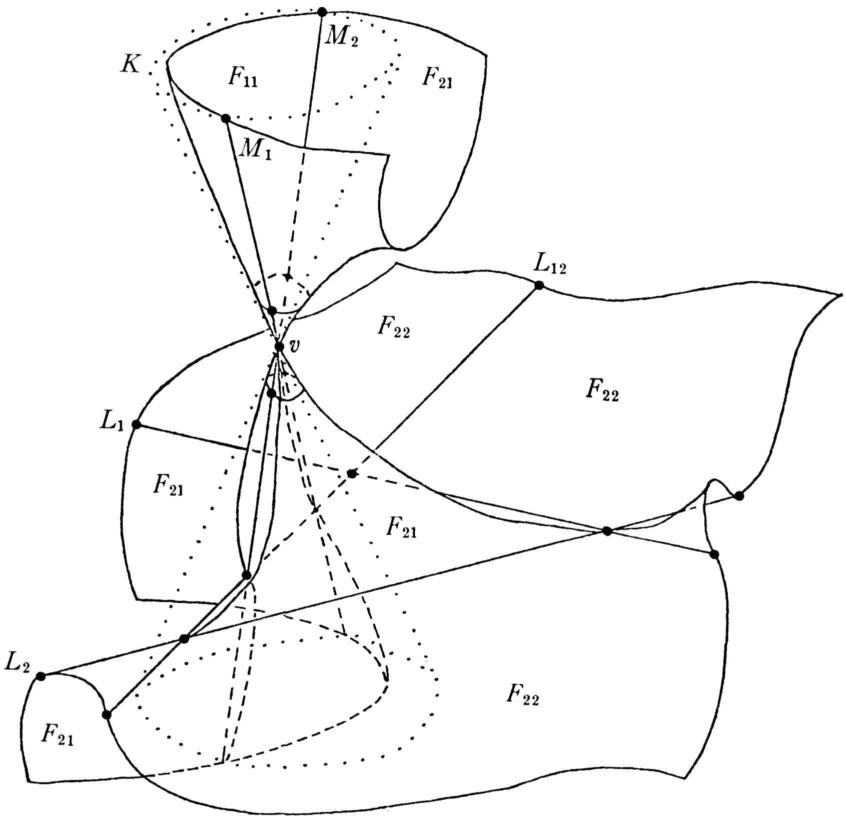

FIGURE 2

5. F with $l(v) = 4$ and $l(F) = 11$.

5.0 Let F be C-nodal with the C-node v, $l(F) = l(v) + 7 = 11$. Let $K \cap F = \bigcup M_i$, $i \in \mathscr{S}_4$. Then (cf. the proof of 2.4(3)) the other lines of F are L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34} and L_0. Since $L_0 \cap K = \emptyset$, $L_{ij} \subset \langle M_i, M_j \rangle$ implies that $L_0 \cap L_{ij}$ is a point q_{ij}; $i \neq j$ in $\mathscr{S}_4$.

We assume that the line $\langle M_1, M_3 \rangle \cap \langle M_2, M_4 \rangle \subset \text{int}(K) \cup \{v\}$. As K is a cone of order two, this implies that $\langle M_1, M_3 \rangle$ separates K into two disjoint regions, one of which contains $M_2 \backslash \{v\}$ and the other $M_4 \backslash \{v\}$. More simply, M_1, M_3 *separates* M_2, M_4 in K. Then

$$\{p_0\} = L_{13} \cap L_{24} \subset \text{int}(K) \quad \text{and}$$

$$(L_{12} \cap L_{34}) \cup (L_{14} \cap L_{23}) \subset \text{ext}(K).$$

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the closed half-spaces of P^3 determined by $\langle M_1, M_3 \rangle$

and $\langle M_2, M_4\rangle$. Then

$$\langle M_1, M_3\rangle \cap \langle M_2, M_4\rangle \subset \text{int } (K) \cup \{v\}$$

implies that say

$$L_{12} \cap L_{34} \subset \mathscr{P}_1 \quad \text{and} \quad L_{14} \cap L_{23} \subset \mathscr{P}_2.$$

Then

1. $\mathscr{P}_1 \cap (L_{14} \cup L_{23}) = \overline{\text{int } (K)} \cap (L_{14} \cup L_{23})$ and
2. $\mathscr{P}_2 \cap (L_{12} \cup L_{34}) = \overline{\text{int } (K)} \cap (L_{12} \cup L_{34})$.

Finally $L_0 \subset \text{ext } (K)$ yields that

3. $\{q_{12}, q_{34}\} \subset \mathscr{P}_1$ and $\{q_{14}, q_{23}\} \subset \mathscr{P}_2$.

Let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be the closed half-spaces of P^3 determined by $\langle L_0, v\rangle$ and $\langle L_0, L_{13}, L_{24}\rangle$. We assume that $\langle L_0, L_{14}, L_{23}\rangle \subset \mathscr{Q}_1$. Then M_1, M_3 separates M_2, M_4 in K and the continuity of the plane sections of F through say M_1 imply that $M_1 \cap L_{13}$, $\{v\}$ separates $M_1 \cap L_{12}$, $M_1 \cap L_{14}$ and thus $\langle L_0, L_{12}, L_{34}\rangle \subset \mathscr{Q}_2$. Clearly $\langle L_{14}, L_{23}\rangle$ [$\langle L_{12}, L_{34}\rangle$] decomposes $\mathscr{Q}_1$ [$\mathscr{Q}_2$] into two closed "quarter-spaces", say $\mathscr{Q}_{11}$ and $\mathscr{Q}_{12}$ [$\mathscr{Q}_{21}$ and $\mathscr{Q}_{22}$]. We assume that $\mathscr{Q}_{11} \cap \mathscr{Q}_{22} = \langle L_0, v\rangle$ and hence $\mathscr{Q}_{12} \cap \mathscr{Q}_{21} = \langle L_{13}, L_{24}\rangle$.

Finally let $\mathscr{P}_{ijk} = \mathscr{P}_i \cap \mathscr{Q}_{ij}$, $\{i, j, k\} \subseteq \mathscr{S}_2$. Then

$$P^3 = \overline{\text{int } (K)} \cup \overline{\text{ext } (K)} = \cup \mathscr{P}_{ijk}$$

implies that

$$F = (F \cap \overline{\text{ext } (K)}) \cup (\cup (F \cap \overline{\text{int } (K)} \cap \mathscr{P}_{ijk})), \{i, j, k\} \subseteq \mathscr{S}_2.$$

5.1 Let $\beta \subset \mathscr{P}_i$, $l(\beta) = 0$ and $i \in \mathscr{S}_2$. Then $\langle v, p_0\rangle \subset \beta$ and from 2.1, v is the double point of

$$\beta \cap F = \mathscr{L}_\beta \cup \mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta}.$$

Since $\pi(p_0) \cap F = L_{13} \cup L_{24} \cup L_0$ and $L_0 \subset \text{ext } (K)$, $\beta \cap \pi(p_0)$ meets both int $(K) \cap F$ and ext $(K) \cap F$. Thus

$$\mathscr{L}_\beta \subset \overline{\text{int } (K)} \quad \text{and} \quad \mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta} \subset \overline{\text{ext } (K)}$$

as in 4.1.4.

As $\langle L_0, v\rangle \cap \mathscr{L}_\beta = \{v\}$ and $\langle L_{13}, L_{24}\rangle \cap \mathscr{L}_\beta = \{p_0\}$, this implies that either $\mathscr{L}_\beta \subset \mathscr{Q}_1$ and $\mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta} \subset \mathscr{Q}_2$ or $\mathscr{L}_\beta \subset \mathscr{Q}_2$ and $\mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta} \subset \mathscr{Q}_1$. Then continuity of $\beta \cap F$ for $\beta \subset \mathscr{P}_i$ clearly yields that either $\mathscr{L}_\beta \subset \mathscr{Q}_1$ for all such $\beta \subset \mathscr{P}_i$ or $\mathscr{L}_\beta \subset \mathscr{Q}_2$ for all such $\beta \subset \mathrm{P}_i$.

5.2 LEMMA. 1. *Let* $\beta \cap F = \mathscr{L}_\beta \cup \mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta}$, $\langle v, p_0\rangle \subset \beta$. *Then* $\beta \subset \mathscr{P}_1$ *if and only if* $\mathscr{L} \subset \mathscr{Q}_1$.

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2. *Let* $\beta_1 = \langle v, p_0, L_{12} \cap L_{34}\rangle$ *and* $\beta_2 = \langle v, p_0, L_{14} \cap L_{23}\rangle$. *Then* v *is the double point of* $\beta_i \cap F$ *and* $\beta_i \cap F \subset \mathscr{Q}_i \cap \mathscr{Q}_{jj}$, $\{i, j\} = \mathscr{S}_2$.

Proof. 1. Since $\mathscr{L}_\beta \subset \overline{\text{int }(K)}$, $\langle L_{14}, L_{23}\rangle \subset \mathscr{Q}_1$ and $\langle L_{12}, L_{34}\rangle \subset \mathscr{Q}_2$, the result follows from 5.0.1 and 5.0.2.

2. Clearly v is the double point of $\beta_1 \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and $\{q\} = L_{12} \cap L_{34} \subset \mathscr{P}_1$ implies that $\beta_1 \subset \mathscr{P}_1, \mathscr{L} \subset \mathscr{Q}_1$ and $q \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \mathscr{Q}_2$. Since $\pi(q) = \langle L_{12}, L_{34}, L_0\rangle$, either $q \in L_0$ and $q = q_{12} = q_{34}$ is the inflection point of $\mathscr{A}_1 \cup \mathscr{A}_2$ or $q \notin L_0$ and $\beta_1 \cap L_0 \subset \pi(q)$. In either case, bd $(\mathscr{Q}_{22}) = \langle L_0, v\rangle \cup \langle L_{12}, L_{34}\rangle$ and $\mathscr{A}_1 \cup \mathscr{A}_2$ a curve of order three readily yield that $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \mathscr{Q}_{22}$.

By a similar argument we obtain that $\beta_2 \cap F \subset \mathscr{Q}_2 \cup \mathscr{Q}_{11}$.

5.3 Lemma. *Let* $L_0 \subset \alpha \neq \langle L_0, v\rangle$, $l(\alpha) = 1$.

1. $\alpha \cap F$ *consists of* L_0 *and a curve* S_α^1 *of order two.*
2. *If* $\lim \alpha = \langle L_{ij}, L_{kl}\rangle$ $[\langle L_0, v\rangle]$, *then*

$$\lim S_\alpha^1 = L_{ij} \cup L_{kl}[\{v\}]; \quad \mathscr{S}_4 = \{i, j, k, l\}.$$

3. *If* $\alpha \subset \mathscr{Q}_{11} \cup \mathscr{Q}_{22}$, *then* $\alpha \cap \langle v, p_0\rangle \subset i(S_\alpha^1)$.
4. *If* $\alpha \subset \mathscr{Q}_{12} \cup \mathscr{Q}_{21}$, *then* $\{q_{13}, q_{24}\} \subset i(S_\alpha^1)$.
5. *If* $\alpha \subset \mathscr{Q}_{12}$ $[\mathscr{Q}_{21}]$, *then* $i(S_\alpha^1)$ *contains* $L_0 \cap \mathscr{P}_1$ $[L_0 \cap \mathscr{P}_2]$.

Proof. 1. This is immediate since $L_0 \cap K = \emptyset$ and $\alpha \cap M_i \neq \emptyset$ for $i \in \mathscr{S}_4$. We note that $\alpha \cap M_i \subset S_\alpha^1$, $q_{ij} \notin S_\alpha^1$, $\alpha \cap M_1$ and $\alpha \cap M_3$ separate $\alpha \cap \langle v, p_0\rangle$ and $\{q_{13}\}$ and $\alpha \cap M_2$, $\alpha \cap M_4$ separates $\alpha \cap \langle v, p_0\rangle$, $\{q_{24}\}$.

2. Since

$$\langle L_{ij}, L_{kl}\rangle \cap F = L_{ij} \cup L_{kl} \cup L_0,$$

$$\langle L_0, v\rangle \cap F = L_0 \cup \{v\}, v \notin L_0$$

and $v \in M_i$ for $i \in \mathscr{S}_4$, the result follows by 1.5.4 and 1.5.5 a).

3. Let $\alpha \subset \mathscr{Q}_{ii}$. Since $v \notin L_0$, 2 implies that $L_0 \cap S^1 = \emptyset$ for α sufficiently close to $\langle L_0, v\rangle$. Since S_α^1 depends continuously on $\alpha \subset \mathscr{Q}_{ii}$ and $q_{13} \notin S_\alpha^1$, we obtain that $q_{13} \in L_0 \subset e(S_\alpha^1)$ for $\alpha \subset \mathscr{Q}_{ii}$. From the proof of 1, it follows that

$$\alpha \cap \langle v, p_0\rangle \subset i(S_\alpha^1).$$

4. Let $\alpha \in \mathscr{Q}_{ij}$, $\{i, j\} = \mathscr{S}_2$. From 5.2.2, v is the double point of $\beta_j \cap F \subset \mathscr{Q}_j \cup \mathscr{Q}_{ii}$. Hence $\alpha \in \mathscr{Q}_j \cup \mathscr{Q}_{ii}$ implies that

$$\alpha \cap (\beta_j \cap F) = \beta_j \cap L_0.$$

But

$$\alpha \cap (\beta_j \cap F) = \beta_j \cap (\alpha \cap F) = \beta_j \cap (L_0 \cup S_\alpha^1)$$

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yields that $|(\beta_j \cap \alpha) \cap S_\alpha^1| \leqq 1$. Thus

$$\langle v, p_0\rangle \cap \alpha \subset \beta_j \cap \alpha \subset e(S_\alpha^1) \quad \text{and} \quad \{q_{13}, q_{24}\} \subset i(S_\alpha^1).$$

5. Since

$$\text{bd }(\mathcal{Q}_{12}) = \langle L_{13}, L_{24}\rangle \cup \langle L_{14}, L_{23}\rangle \quad \text{and}$$
$$\text{bd }(\mathcal{Q}_{21}) = \langle L_{13}, L_{24}\rangle \cup \langle L_{12}, L_{34}\rangle,$$

the result follows by 4, 2 and 5.0.3.

5.4 Let $i + j \equiv 1 \pmod 2$, $\mathcal{S}_4 = \{i, j, k, l\}$. Then L_{ij} is met by M_i, M_j, L_{kl} and L_0 in $\overline{\text{ext }(K)}$. Let $\mathcal{R}_{ij}$ and $\mathcal{R}_{ij}^*$ be the closed half-spaces of P^3 determined by $\langle M_i, M_j\rangle$ and $\langle L_0, L_{kl}\rangle$. We assume that $\pi(q') \subset \mathcal{R}_{ij}$ for some $q' \in L_{ij} \cap \text{int }(K)$.

5.5 Lemma. *Let $L_{ij} \subset \gamma$, $l(\gamma) = 1$, $i + j \equiv 1 \pmod 2$ and $\mathcal{S}_4 = \{i, j, k, l\}$.*

1. *$\gamma \cap F$ consists of L_{ij} and a curve S_γ^1 of order two.*
2. *If $\lim \gamma = \langle M_i, M_j\rangle\ [\langle L_0, L_{kl}\rangle]$, then*

$$\lim S_\gamma^1 = M_i \cup M_j[L_0 \cup L_{kl}].$$

3. *If $\gamma \subset \mathcal{R}_{ij}$, then*

$$L_{ij} \cap (M_i \cup M_j \cup L_{kl} \cup L_0) \subset e(S_\gamma^1).$$

Proof. 1. This is immediate since $\gamma \cap M_k \neq \emptyset \neq \gamma \cap M_l$.

2. cf. the proof of 5.3.2.

3. Let $\bar\gamma = \pi(\bar q)$, $\bar q \in L_{ij} \cap \text{int }(K)$. From 5.0, $l(\bar q) = 1$ and thus $l(\bar\gamma) = 1$, $\bar\gamma \cap F = L_{ij} \cup S_{\bar\gamma}^1$, $\bar q \in S_{\bar\gamma}^1$ and $S_{\bar\gamma}^1 \cap (M_i \cup M_j) = \emptyset$. Since $S_{\bar\gamma}^1 \cap K = \bar\gamma \cup (M_k \cup M_l)$ and $S_{\bar\gamma}^1$ and $\bar\gamma \cap K$ are both curves of order two, we obtain that either

$$L_{ij} \cap S_{\bar\gamma}^1 = \{\bar q\} \quad \text{or} \quad |(L_{ij} \cap \text{int }(K)) \cap S_{\bar\gamma}^1| = 2.$$

It is easy to check that both cases occur and hence we assume that $L_{ij} \cap S_{\bar\gamma}^1 = \{\bar q\}$. Then

$$L_{ij} \subset \overline{\overset{0}{e}(S_{\bar\gamma}^1)}.$$

Let $\gamma = \langle S_\gamma^1\rangle$ range between $\bar\gamma$ and $\langle M_i, M_j\rangle$. Then S_γ^1 depending continuously on γ,

$$L_{ij} \cap S_{\bar\gamma}^1 = \{\bar q\} \subset \text{int }(K),$$
$$L_{ij} \cap \text{ext }(K) \subset e(S_{\bar\gamma}^1)$$

and 2 imply that

$$|(L_{ij} \cap \text{int }(K)) \cap S_\gamma^1| = 2$$

(thus $\gamma = \pi(q)$ for $q \in L_{ij} \cap \text{int}\ (K) \cap S_\gamma^1$) and

$$L_{ij} \cap \text{ext}\ (K) \subset e(S_\gamma^1).$$

Finally $\pi(q') \subset \mathscr{R}_{ij}$ for some $q' \in L_{ij} \cap \text{int}\ (K)$ and $\text{bd}\ (\mathscr{R}_{ij}) = \langle M_i, M_j\rangle \cup \langle L_0, L_{kl}\rangle$ imply that $\gamma \subset \mathscr{R}_{ij}$.

Let $\gamma = \langle S_\gamma^1\rangle$ range between $\bar{\gamma}$ and $\langle L_0, L_{kl}\rangle$. Then the preceding and $\bar{\gamma} \subset \mathscr{R}_{ij}$ imply that $\gamma \subset \mathscr{R}_{ij}$, $\gamma \neq \pi(q)$ for any $q \in L_{ij} \cap \text{int}\ (K)$ and thus

$$L_{ij} \cap \text{int}\ (K) \cap S_\gamma^1 = \emptyset.$$

Then S_γ^1 depending continuously on γ and 2 readily imply that either

$$L_{ij} \cap S_\gamma^1 = \emptyset \quad \text{or} \quad L_{ij} \cap S_\gamma^1 \subset L_{ij}^*,$$

the open segment of $L_{ij} \cap \text{ext}\ (K)$ bounded by $L_0 \cap L_{ij}$ and $L_{kl} \cap L_{ij}$, and $L_{ij} \backslash L_{ij}^* \subset e(S_\gamma^1)$. Clearly, $l(q) = 1$ for each $q \in L_{ij}^*$ and thus 3.

5.6 We recall that

$$F = (F \cap \overline{\text{ext}\ (K)}) \cup (\cup\ (F \cap \overline{\text{int}\ (K)} \cap \mathscr{P}_{ijk}))$$

where $\mathscr{P}_{ijk} = \mathscr{P}_i \cap \mathscr{Q}_{jk}$ and $\{i, j, k\} \subseteq \mathscr{S}_2$. In this subsection we analyse $F \cap \overline{\text{int}\ (K)}$ and in 5.7, $F \cap \overline{\text{ext}\ (K)}$.

Let $\beta \subset \mathscr{P}_i$, $l(\beta) = 0$ and $\mathscr{S}_2 = \{i, j\}$. From 5.1 and 5.2, v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$,

$$\mathscr{L} \subset \overline{\text{int}\ (K)} \cap \mathscr{P}_i \cap \mathscr{Q}_i \quad \text{and}$$
$$\mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{ext}\ (K)} \cap \mathscr{P}_i \cap \mathscr{Q}_j.$$

Thus we obtain that

1. $l(r) > 0$ for $r \in F \cap \overline{\text{int}\ (K)} \cap (\mathscr{P}_{121} \cup \mathscr{P}_{122} \cup \mathscr{P}_{211} \cup \mathscr{P}_{212})$.

Let $i = 1$. Then $\mathscr{L} = \beta \cap \text{int}\ (K) \cap \mathscr{P}_1 \cap \mathscr{Q}_1$, 5.0.1 and 5.0.2 imply that

$$\mathscr{L} \cap (L_{12} \cup L_{34}) = \emptyset \quad \text{and} \quad \mathscr{L} \cap L_{14} \neq \emptyset \neq \mathscr{L} \cap L_{23}.$$

As $\mathscr{Q}_1 = \mathscr{Q}_{11} \cup \mathscr{Q}_{12}$ and $\mathscr{Q}_{11} \cap \mathscr{Q}_{12} = \langle L_{14}, L_{23}\rangle$, $v \in \mathscr{Q}_{11}$ and $p_0 \in \mathscr{Q}_{12}$ yield that (cf. 1.4)

$$\mathscr{L} = \mathscr{L}(v, \mathscr{L} \cap L_{14}, p_0) \cup \mathscr{L}(v, \mathscr{L} \cap L_{23}, p_0)$$
$$= (\mathscr{Q}_{11} \cap \mathscr{L}) \cup (\mathscr{Q}_{12} \cap \mathscr{L})$$

where

$$\mathscr{Q}_{11} \cap \mathscr{L} = \mathscr{L}(\mathscr{L} \cap L_{14}, v, \mathscr{L} \cap L_{23}) \quad \text{and}$$
$$\mathscr{Q}_{12} \cap \mathscr{L} = \mathscr{L}(\mathscr{L} \cap L_{14}, p_0, \mathscr{L} \cap L_{23}).$$

Since $\text{bd}\ (\mathscr{P}_1) = \langle M_1, M_3\rangle \cup \langle M_2, M_4\rangle$ and $L_{14} \cap L_{23} \subset \text{ext}\ (K) \cap \mathscr{P}_2$, the preceding readily implies that

2. $F \cap \overline{\text{int }(K)} \cap \mathscr{P}_{111} = \bar{G}_{14} \cup \bar{G}_{23}$

where G_{14} and G_{23} are non-empty, open triangular regions such that $l(r) = 0$ for $r \in G_{14} \cup G_{23}$, $\bar{G}_{14} \cap \bar{G}_{23} = \{v\}$ and say

$$\text{bd }(G_{14}) \subset M_1 \cup M_4 \cup L_{14} \quad \text{and} \quad \text{bd }(G_{23}) \subset M_2 \cup M_3 \cup L_{23}.$$

The preceding argument is symmetric in $\mathscr{P}_1$ and $\mathscr{P}_2$ and thus,

3. $F \cap \overline{\text{int }(K)} \cap \mathscr{P}_{222} = \bar{G}_{12} \cup \bar{G}_{34}$

where G_{12} and G_{34} are non-empty, open triangular regions such that $l(r) = 0$ for $r \in G_{12} \cup G_{34}$, $\bar{G}_{12} \cap \bar{G}_{34} = \{v\}$ and say

$$\text{bd }(G_{12}) \subset M_1 \cup M_2 \cup L_{12} \quad \text{and} \quad \text{bd }(G_{34}) \subset M_3 \cup M_4 \cup L_{34}.$$

We note that there are similar decompositions for both $F \cap \overline{\text{int }(K)} \cap \mathscr{P}_{112}$ and $F \cap \overline{\text{int }(K)} \cap \mathscr{P}_{221}$. Since we do not need them, we simply let

4. $F \cap \overline{\text{int }(K)} \cap (\mathscr{P}_{112} \cup \mathscr{P}_{221}) = \tilde{F}$.

5.7 As in 5.0, we obtain that L_0, L_{12} and L_{34} [L_0, L_{14} and L_{23}] are either concurrent or determine (cf. Figure 3) an open triangular region G_1 [G_2] in ext (K). We note that $G_i \subset \mathscr{P}_i$, G_i satisfies 1.5.11 and hence $G_i \cap E \neq \emptyset$; $i \in \mathscr{S}_2$.

Let $G_1 = \emptyset$ [$G_2 = \emptyset$] if L_0, L_{12} and L_{34} [L_0, L_{14} and L_{23}] are concurrent and, in any case, put

$$F^* = (\overline{\text{ext }(K)} \cap F) \backslash (G_1 \cup G_2).$$

From the proof of 5.2.2, we recall that

$$\beta_i \cap (\overline{\text{ext }(K)} \cap F) \subset \mathscr{Q}_{jj}, \quad \{i, j\} = \mathscr{S}_2.$$

Hence $\beta_i \cap G_i \subset \beta_i \cap \overline{\text{ext }(K)} \cap F$ and $l(r) = 0$ for $r \in G_i$ imply that $G_1 \subset \mathscr{Q}_{22}$ and $G_2 \subset \mathscr{Q}_{11}$.

5.8 THEOREM. *$F \cap \overline{\text{int }(K)} = \bar{G}_{12} \cup \bar{G}_{14} \cup \bar{G}_{23} \cup \bar{G}_{34} \cup \tilde{F}$ where*
1. *$G_{ij} \cap E \neq \emptyset$ and $v \in \overline{G_{ij} \cap E}$ for each G_{ij} and*
2. *every $r \in \tilde{F}$ such that $l(r) = 0$ is hyperbolic.*

Proof. 1. cf. the proof of 4.4.
2. Let $r_0 \in \tilde{F}$, $l(r_0) = 0$. By 5.6.4,

$$r_0 \in \mathscr{P}_{iij} = \mathscr{P}_i \cap \mathscr{Q}_{ij}, \mathscr{S}_2 = \{i, j\}.$$

Then v is the double point of $\langle v, p_0, r_0 \rangle \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and

$$\{p_0, r_0\} \subset \mathscr{L} \subset \overline{\text{int }(K)} \cap \mathscr{P}_i \cap \mathscr{Q}_i.$$

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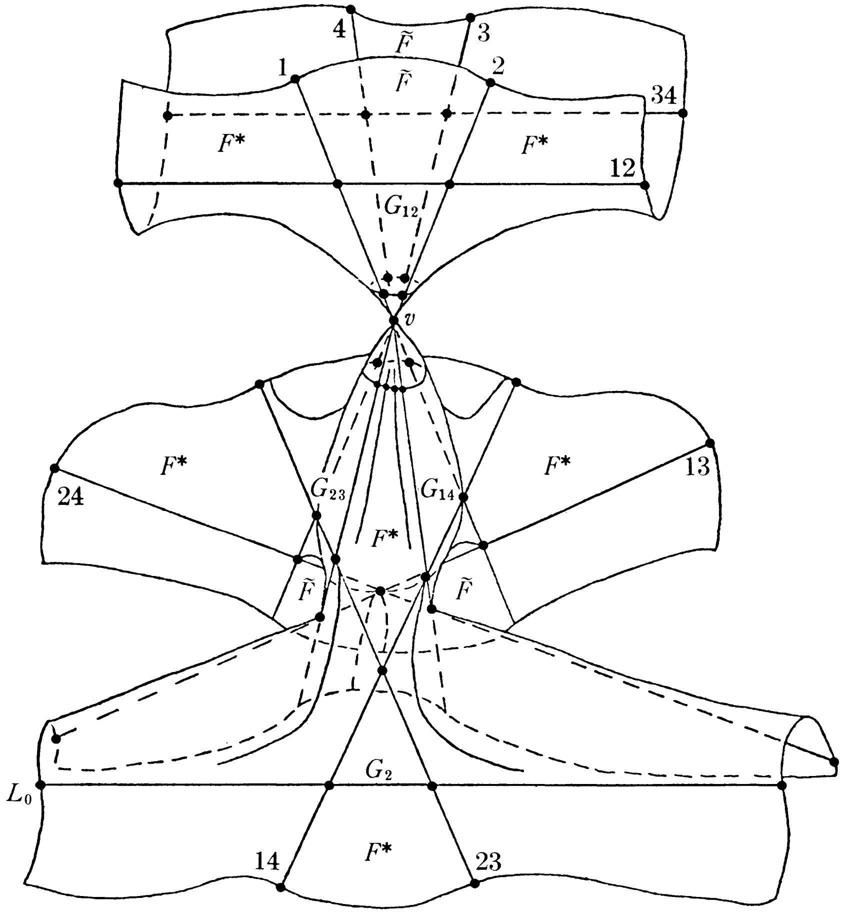

FIGURE 3

Let $\langle v, p_0 \rangle \subset \beta \subset \mathscr{P}_j$, $l(\beta) = 0$. Again v is the double point of $\beta \cap F = \mathscr{L}_\beta \cup \mathscr{A}_{1,\beta} \cup \mathscr{A}_{2,\beta}$ but

$$\mathscr{L}_\beta \subset \overline{\text{int }(K)} \cap \mathscr{P}_j \cap \mathscr{Q}_j.$$

Since $\{v, p_0\} = \mathscr{L} \cap \mathscr{L}_\beta$,

$$\mathscr{Q}_i \cap \mathscr{Q}_j = \langle L_0, v \rangle \cup \langle L_0, p_0 \rangle$$

implies that $e(\mathscr{L}) \cap e(\mathscr{L}_\beta) = \emptyset$ and thus $\mathscr{L}$ and $\mathscr{L}_\beta$ are incompatible by 1.5.8.

Since $\{p_0, r_0\} \subset \mathscr{L} \cap \mathscr{Q}_{ij}$, it is clear that $l(r') = 0$ for each r' in the interior of the subarc $\mathscr{A} \subset \mathscr{L} \cap \mathscr{Q}_{ij}$ bounded by p_0 and r_0. Let

$$\beta' = \langle v, \beta \cap L_0, r' \rangle, r' \in \mathscr{A} \backslash \{p_0\}.$$

Since $\beta \cap L_0 \subset \text{ext } (K)$ and $r' \in \text{int } (K)$, $l(r') = 0$ and 2.1 yield that v is the double point of $\beta' \cap F = \mathscr{L}' \cup \mathscr{A}_1' \cup \mathscr{A}_2'$ for each r'. As $\lim r' = p_0$ implies that $\lim \beta' = \beta$, 1.5.4 and 1.5.5 a) yield that $\lim \mathscr{L}' = \mathscr{L}_\beta$. Then $\mathscr{L}_\beta \subset \text{int } (K)$, $\mathscr{L}_\beta$ and $\mathscr{A}$ p_0-incompatible and 1.5.7 imply that $\mathscr{L}' \subset \overline{\text{int } (K)}$ and $\mathscr{L}'$ and $\mathscr{A}$ are r'-incompatible for each $r' \in \mathscr{A} \backslash \{p_0\}$. Hence $r_0 \in \mathscr{A} \subset H$ by 1.5.9 ii).

5.9 THEOREM. *Let $r \in F^*$ such that $l(r) = 0$. Then r is hyperbolic.*

Proof. Since $P^3 = \mathscr{P}_1 \cup \mathscr{P}_2$, we assume that $r \in \mathscr{P}_1$ say. Then

$$\beta = \langle v, p_0, r \rangle \subset \mathscr{P}_1,$$

v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and

$$r \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{ext } (K)} \cap \mathscr{Q}_2.$$

From 5.7, we note that

$$\beta \cap G_1 \subset \mathscr{A}_1 \cup \mathscr{A}_2 \quad \text{and} \quad \beta \cap G_2 = \emptyset.$$

Let $\alpha = \langle L_0, r \rangle$. By 5.3.1,

$$\alpha \cap F = L_0 \cup S_\alpha^1$$

where $r \in S_\alpha^1$ and either $\alpha \subset \mathscr{Q}_{22}$ or $\alpha \subset \mathscr{Q}_{21}$.

i) $\alpha \subset \mathscr{Q}_{22}$.

Then $\alpha \cap \langle v, p_0 \rangle \subset i(S_\alpha^1)$ by 5.3.3. Clearly

$$\alpha \cap \langle v, p_0 \rangle \subset e(\mathscr{A}_1) \cup e(\mathscr{A}_2)$$

and thus if $r \in \text{int } (\mathscr{A}_i)$ for $i \in \mathscr{S}_2$, then S_α^1 and $\mathscr{A}_i$ satisfy 1.5.8 and $r \in H$.

Let r^* be the inflection point of $\mathscr{A}_1 \cup \mathscr{A}_2$. Then

$$\mathscr{A}_1 \cap \mathscr{A}_2 = \{v, r^*\}.$$

If $\beta = \beta_1$ and $G_1 = \emptyset$, then $r^* = q_{12} = q_{34}$ (cf. the proof of 5.2.2) and $r^* \neq r$. Let $\beta \neq \beta_1$ or $G_1 \neq \emptyset$. Then either

$$|\langle L_0, L_{12}, L_{34} \rangle \cap (\mathscr{A}_1 \cup \mathscr{A}_2)| = 3$$

or

$$|\langle L_0, L_{12}, L_{34} \rangle = (\mathscr{A}_1 \cup \mathscr{A}_2)| = 2$$

and

$$\langle L_0, L_{12}, L_{34} \rangle = \pi(r') \quad \text{for some } r' \in \mathscr{A}_1 \cup \mathscr{A}_2.$$

As $\mathscr{A}_1$ and $\mathscr{A}_2$ are curves of order two, this implies that both $\mathscr{A}_1$ and $\mathscr{A}_2$

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meet $\langle L_0, L_{12}, L_{34}\rangle$. It is easy to check that $r^* \in \mathscr{A}_1 \cap \mathscr{A}_2$ yields that either $r^* \in \mathscr{Q}_{21}$ or $r^* \in G_1 \subset \mathscr{Q}_{22}$. Thus $r^* \neq r$.

ii) $\alpha \subset \mathscr{Q}_{21}$.

Then

$$\{q_{14}, q_{23}\} \subset L_0 \cap \mathscr{P}_2 \subset i(S_\alpha{}^1)$$

by 5.0.3 and 5.3.5. Let $\gamma_1 = \langle L_{14}, r\rangle$ and $\gamma_2 = \langle L_{23}, r\rangle$. Then

$$\gamma_1 \cap F = L_{14} \cap S_{14}{}^1 \quad \text{and} \quad \gamma_2 \cap F = L_{23} \cup S_{23}{}^1$$

where $r \in S_{14}{}^1 \cap S_{23}{}^1$ by 5.5.1. We claim that either $\gamma_1 \subset \mathscr{R}_{14}$ or $\gamma_2 \subset \mathscr{R}_{23}$ and thus either $q_{14} \in e(S_{14}{}^1)$ or $q_{23} \in e(S_{23}{}^1)$ by 5.5.3. Then 1.5.8 yields that $r \in H$.

Let $\beta \cap L_0 = \{r_0\}$ and $\beta \cap L_{ij} = \{r_{ij}\}$, $i \neq j$ in $\mathscr{S}_4$. Then

$$\{r_{14}, r_{23}\} \subset \mathscr{L} \subset \overline{\text{int }(K)} \quad \text{and} \quad \{r_0, r_{12}, r_{34}\} \subset \mathscr{A}_1 \cup \mathscr{A}_2$$

by 5.0.1 and 5.0.2. From 5.4, $\beta \cap \mathscr{R}_{14}$ $[\beta \cap \mathscr{R}_{23}]$ is the closed half-plane of β, determined by $\langle r_{14}, v\rangle$ and $\langle r_{14}, r_{23}, r_0\rangle$ $[\langle r_{23}, v\rangle$ and $\langle r_{23}, r_{14}, r_0\rangle]$, containing $\beta \cap \pi(r_{14})$ $[\beta \cap \pi(r_{23})]$. Since $\{r_{14}, r_{23}\} \subset \mathscr{L}$, 2.2 yields that both $\beta \cap \pi(r_{14})$ and $\beta \cap \pi(r_{23})$ meet $\mathscr{A}_1 \cup \mathscr{A}_2$ and thus $\mathscr{R}_{14} \cap (\mathscr{A}_1 \cup \mathscr{A}_2)$ and $\mathscr{R}_{23} \cap (\mathscr{A}_1 \cup \mathscr{A}_2)$ are subarcs of $\mathscr{A}_1 \cup \mathscr{A}_2$ bounded by v and r_0. Then either

$$\mathscr{R}_{14} \cap (\mathscr{A}_1 \cup \mathscr{A}_2) = \mathscr{R}_{24} \cap (\mathscr{A}_1 \cup \mathscr{A}_2) \quad \text{or}$$
$$r \in \mathscr{A}_1 \cup \mathscr{A}_2 = (\mathscr{R}_{14} \cap (\mathscr{A}_1 \cup \mathscr{A}_2)) \cup (\mathscr{R}_{23} \cap (\mathscr{A}_1 \cup \mathscr{A}_2)) \subset \mathscr{R}_{14} \cup \mathscr{R}_{23}.$$

Since each point of $(\mathscr{A}_1 \cup \mathscr{A}_2)\backslash\{v\}$ lies on the tangent of exactly one point of $\mathscr{L}\backslash\{v\}$, $r_0 \in \beta \cap \pi(p_0)$ and $p_0 \in \mathscr{L}$ imply that a subarc of $\mathscr{A}_1 \cup \mathscr{A}_2$, bounded by v and r_0, is met by the tangents of exactly one subarc of $\mathscr{L}$, bounded by v and p_0. From 5.6,

$$\mathscr{L} = \mathscr{L}(v, r_{14}, p_0) \cup \mathscr{L}(v, r_{23}, p_0)$$

and hence

$$\mathscr{R}_{14} \cap (\mathscr{A}_1 \cup \mathscr{A}_2) \cap \mathscr{R}_{23} = \{v, r_0\}.$$

The preceding argument is symmetric in $\mathscr{P}_1$ and $\mathscr{P}_2$.

5.10 SUMMARY. *Let F be a C-nodal surface satisfying* 5.0. *Then*

$$F = \bar{G}_{12} \cup \bar{G}_{14} \cup \bar{G}_{23} \cup \bar{G}_{34} \cup \bar{G}_1 \cup \bar{G}_2 \cup \tilde{F} \cup F^*$$

where G_{ij}, G_λ, $\tilde{F}$ and F^* are described in 5.6 *and* 5.7, *every $r \in \tilde{F} \cup F^*$ such that $l(r) = 0$ is hyperbolic, $v \in \overline{G_{ij} \cap E}$ and $G_\lambda \cap E \neq \emptyset$ if $G_\lambda \neq \emptyset$.*

The surface in P^3 defined by

$$x_0(x_1{}^2 - x_2{}^2) + x_3(x_0{}^2 + x_1x_2) = 0$$

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satisfies 5.10 with $M_1 \equiv x_0 = x_1 = 0$, $M_2 \equiv x_0 = x_2 = 0$, $M_3 \equiv x_0 - x_1 = x_1 + x_2 = 0$, $M_4 \equiv x_0 + x_1 = x_1 + x_2 = 0$, $L_{12} \equiv x_0 = x_3 = 0$, $L_{13} \equiv x_0 - x_1 = x_1 - x_2 + x_3 = 0$, $L_{14} \equiv x_0 + x_1 = x_1 - x_2 - x_3 = 0$, $L_{23} \equiv x_0 + x_2 = x_1 - x_2 - x_3 = 0$, $L_{24} \equiv x_0 - x_2 = x_1 - x_2 + x_3 = 0$, $L_{34} \equiv x_1 + x_2 = x_3 = 0$, $L_0 \equiv x_1 - x_2 = x_3 = 0$ and $K \equiv x_0^2 + x_1x_2 = 0$.

6. F with $l(v) = 6$ and $l(F) = 21$. Let $\lambda \equiv i + j \pmod 6$, $i \neq j$ in $\mathscr{S}_6$. Then $\lambda \in \mathscr{S}_6$ and $\mathscr{S}_6 = \{i, i + 1, \ldots, i + 5\}$. For the sake of generality, we also assume that $\mathscr{S}_6 = \{i, j, k, l, m, n\}$.

6.0 Let F be C-nodal with the C-node v, $l(F) = l(v) + 15 = 21$. Let $K \cap F = \cup M_i$, $i \in \mathscr{S}_6$. The other fifteen lines of F are L_{ij}, $i \neq j$ in $\mathscr{S}_6$, with the properties listed in the proof of 2.4 (4). We note that $L_{ij} \subset \alpha$ and $l(\alpha) = 3$ imply that α is $\langle M_i, M_j\rangle$, $\langle L_{kl}, L_{mn}\rangle$, $\langle L_{km}, L_{ln}\rangle$ or $\langle L_{kn}, L_{lm}\rangle$.

We label the lines of F through v cyclically; that is, M_i, M_{i+2} separates (cf. 5.0) M_{i+1} from each of M_{i+3}, M_{i+4} and M_{i+5}, $i \in \mathscr{S}_6$. Then

1. no line of F meets int $(K) \cap L_{i,i+1}$,
2. exactly $L_{i+1,i+3}$, $L_{i+1,i+4}$ and $L_{i+1,i+5}$ meet int $(K) \cap L_{i,i+2}$ and
3. exactly $L_{i+1,i+4}$, $L_{i+1,i+5}$, $L_{i+2,i+4}$ and $L_{i+2,i+5}$ meet int $(K) \cap L_{i,i+3}$.

6.1 In this subsection, we determine the configuration of the twenty-one lines of F.

Let α be a plane through M_i, $i \in \mathscr{S}_6$. From 2.1, either $\alpha \cap K = M_i$ or $\alpha = \pi(p)$ for some $p \in M_i\backslash\{v\}$. Since $\pi(p)$ depends continuously on $p \in M_i\backslash\{v\}$ and the lines of F through v are labelled cyclically, we obtain that $M_i\backslash\{v\}$ meets $L_{i,i+\lambda}$ in the sequence

1. $L_{i,i+1}$, $L_{i,i+2}$, $L_{i,i+3}$, $L_{i,i+4}$, $L_{i,i+5}$;

that is, $M_i \cap L_{i,\lambda}$, $M_i \cap L_{i,\lambda+2}$ separates $M_i \cap L_{i,\lambda+1}$, $\{v\}$.

We can determine (as in 5.0) for any L_{ij}, the separation of the planes α through L_{ij} with $l(\alpha) = 3$. For example, $(i, j) = (1, 4)$ implies that $\langle M_1, M_4\rangle$, $\langle L_{25}, L_{36}\rangle$ separates $\langle L_{23}, L_{56}\rangle$, $\langle L_{26}, L_{35}\rangle$.

Finally we wish to determine the sequence in which L_{ij} meets the lines of F. Since

$$L_{i,i+4} = L_{i+4,(i+4)+2} \quad \text{and} \quad L_{i,i+5} = L_{i+5,(i+5)+1},$$

we need only consider the intersection points of $L_{i,i+1}$, $L_{i,i+2}$ and $L_{i,i+3}$, $i \in \mathscr{S}_6$. We note that it is not always possible to determine a precise sequence and in such cases we indicate the uncertainty by ( ). From 6.0, we obtain that lines of F meet

2. $L_{i,i+1}$ in the sequence

$$M_i, M_{i+1}, L_{i+2,i+3}, L_{i+2,i+4}, (L_{i+3,i+4}, L_{i+2,i+5}), L_{i+3,i+5}, L_{i+4,i+5};$$

3. $L_{i,i+2}$ in the sequence

$$M_i, L_{i+1,i+5}, L_{i+1,i+4}, L_{i+1,i+3}, M_{i+2}, L_{i+3,i+4}, L_{i+3,i+5}, L_{i+4,i+5};$$

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4. $L_{i,i+3}$ in the sequence

$$M_i,\ L_{i+1,i+5},\ (L_{i+1,i+4},\ L_{i+2,i+5}),\ L_{i+2,i+4},\ M_{i+3},\ (L_{i+1,i+2},\ L_{i+4,i+5}).$$

We observe that as each uncertainty involves only a pair of points, it does not affect the configuration; cf. Figure 4.

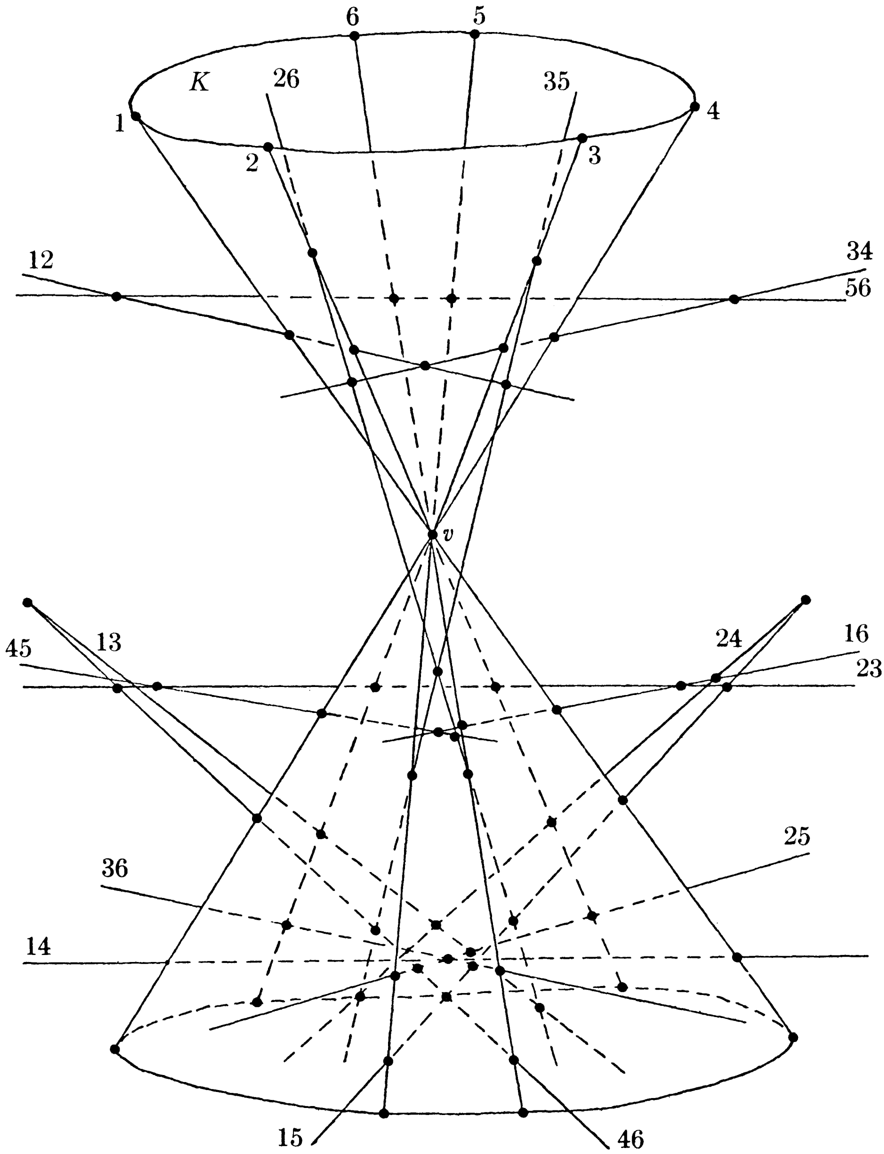

FIGURE 4

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6.2 By 6.0.3 and 6.1.4, L_{14}, L_{25} and L_{36} are either concurrent or determine an open triangular region G_0 in int $(K) \cap F$. It is easy to check that a non-empty G_0 satisfies 1.5.11 and hence contains elliptic points.

If $G_0 = \emptyset$, let $\Delta = L_{14} \cap L_{25} \cap L_{36} = \{p_0\}$. If $G_0 \neq \emptyset$, let $\Delta = \text{bd}\ (G_0)$ and $p_0 \in G_0$. Let $\langle v, p_0 \rangle \subset \beta$, $l(\beta) = 0$. Then $p_0 \in \text{int}\ (K)$ and 2.1 yield that v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$.

If $\Delta = \{p_0\}$, then clearly p_0 is the inflection point of $\beta \cap F$, $p_0 \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}\ (K)}$ (cf. 3.0) and $\mathscr{L} \subset \overline{\text{ext}\ (K)}$. If $\Delta = \text{bd}\ (G_0)$, then

$$p_0 \in G_0 \quad \text{and} \quad \Delta \subset \text{int}\ (K) \cap \langle L_{14}, L_{25}, L_{36} \rangle$$

imply that either $|\beta \cap \Delta| = 3$ or $\beta \cap \Delta = \{p, q\}$ where $p \neq q$ and $\langle L_{14}, L_{25}, L_{36} \rangle$ is either $\pi(p)$ or $\pi(q)$. Since $\mathscr{L}$ is of order two, either case implies that

$$p_0 \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}\ (K)} \quad \text{and} \quad \mathscr{L} \subset \overline{\text{ext}\ (K)}.$$

In view of the preceding and for the sake of simplicity, we assume in our arguments that $\Delta = \{p_0\}$.

6.3 LEMMA. *Let $\langle v, p_0 \rangle \subset \beta$, $l(\beta) = 0$. Then v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$, $p_0 \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}\ (K)}$ and $\mathscr{L} \subset \overline{\text{ext}\ (K)}$.*

6.4 Let $\mathscr{P}_i$ be the closed half-space of P^3 determined by $\langle M_{i+1}, M_{i+4} \rangle$ and $\langle M_{i+2}, M_{i+5} \rangle$ such that

$$\langle M_i M_{i+3} \rangle \cap \mathscr{P}_i = \langle v, p_0 \rangle, \quad i \in \mathscr{S}_3.$$

Then $P^3 = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$ and int $(\mathscr{P}_i) \cap$ int $(\mathscr{P}_j) = \emptyset$ for $i \neq j$.

Let $\mathscr{R}_i$ and $\mathscr{R}_i^*$ be the closed half-spaces of P^3 determined by

$$\alpha_0 = \langle L_{14}, L_{25}, L_{36} \rangle \quad \text{and} \quad \langle M_i, M_{i+3} \rangle, i \in \mathscr{S}_3.$$

Let $\beta \subset \mathscr{P}_i$, $l(\beta) = 0$ and $i \in \mathscr{S}_3$. Then v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$, p_0 is the inflection point of $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int}\ (K)}$ and $\mathscr{L} \subset \overline{\text{ext}\ (K)}$ by 6.3. Since $\langle v, p_0 \rangle$ supports both $\mathscr{L}$ and $\mathscr{A}_1 \cup \mathscr{A}_2$ at v, $\langle v, p_0 \rangle$ cuts $\mathscr{A}_1 \cup \mathscr{A}_2$ at p_0 (cf. [**1**], 1.3.1) and $\alpha_0 \cap \mathscr{L} = \emptyset$, we obtain that either i) $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \mathscr{R}_j$ and $\mathscr{L} \subset \mathscr{R}_j^*$ or ii) $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \mathscr{R}_j^*$ and $\mathscr{L} \subset \mathscr{R}_j$; $j \in \mathscr{S}_3$. The continuity of $\beta \cap F$ for $\beta \subset \mathscr{P}_i$ implies that either i) holds for all such β or ii) holds for all such β. Let

1. $\mathscr{A}_1 \cup \mathscr{A}_2 \subset \mathscr{R}_i \quad \text{and} \quad \mathscr{L} \subset \mathscr{R}_i^*, i \in \mathscr{S}_3.$

Then by the preceding,

2. $\mathscr{P}_i \cap \text{int}\ (K) \cap F \subset \mathscr{R}_i \quad \text{and} \quad \mathscr{P}_i \cap \text{ext}\ (K) \cap F \subset \mathscr{R}_i^*, i \in \mathscr{S}_3.$

We now examine the relationship among $\mathscr{P}_i$, $\mathscr{R}_j$ and $\mathscr{R}_j^*$, $j \in \mathscr{S}_3 \backslash \{i\}$. Let $L \subset \alpha_0$ such that $L \cap K = \emptyset$ and $|L \cap F| = 3$. Then $l(\langle L, v \rangle) = 0$, v is the isolated point of $\langle L, v \rangle \cap F = F^1 \cup \{v\}$ and $F^1 \subset \text{ext}\ (K)$. We

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note that

$$F^1 = (\mathscr{P}_1 \cap F^1) \cup (\mathscr{P}_2 \cap F^1) \cup (\mathscr{P}_3 \cap F^1),$$
$$\mathscr{P}_i \cap \mathscr{P}_j \cap F^1 \subset L$$

and L cuts F^1 at each point of intersection. By 2, $\mathscr{P}_j \cap F^1 \subset \mathscr{R}_j^*$ and thus

$$\mathscr{P}_i \cap F^1 \subset \mathscr{R}_j.$$

Then by ii),

3. $\mathscr{P}_i \cap \text{int}\,(K) \cap F \subset \mathscr{R}_j^*$ and

$\mathscr{P}_i \cap \text{ext}\,(K) \cap F \subset \mathscr{R}_j, \quad j \in \mathscr{S}_3 \backslash \{i\}.$

As in 6.1, we note that α_0, $\langle M_i, M_{i+3}\rangle$ separates $\langle L_{i+1,i+2}, L_{i+4,i+5}\rangle$, $\langle L_{i+1,i+5}, L_{i+2,i+4}\rangle$. From the definition of $\mathscr{P}_i$, we observe that

4. $\mathscr{P}_i \cap (L_{i+1,i+2} \cup L_{i+4,i+5}) = \overline{\text{int}\,(K)} \cap (L_{i+1,i+2} \cup L_{i+4,i+5}).$

Then $\beta \cap L_{ij} \neq \emptyset$ and 2 imply that

5. $\langle L_{i+1,i+2}, L_{i+4,i+5}\rangle \subset \mathscr{R}_i$ and

$\langle L_{i+1,i+5}, L_{i+2,i+4}\rangle \subset \mathscr{R}_i^*, \quad i \in \mathscr{S}_3.$

Let $\mathscr{R}_{i1}$ and $\mathscr{R}_{i2}$ [$\mathscr{R}_{i1}^*$ and $\mathscr{R}_{i2}^*$] be the closed quarter-spaces of $\mathscr{R}_i[\mathscr{R}_i^*]$ determined by $\langle L_{i+1,i+2}, L_{i+4,i+5}\rangle$ [$\langle L_{i+1,i+5}, L_{i+2,i+4}\rangle$], $i \in \mathscr{S}_3$. We assume that

$$\mathscr{R}_{i1} \cap \mathscr{R}_{i1}^* = \langle M_i, M_{i+3}\rangle \quad \text{and} \quad \mathscr{R}_{i2} \cap \mathscr{R}_{i2}^* = \alpha_0, \quad i \in \mathscr{S}_3.$$

Then 6.1.1 implies that iii) int $(\mathscr{R}_{i1}) \cap M_j$ is an open line-segment, bounded by v and $M_j \cap L_{j,j+1}$, and not intersected by any other line of F; $j \in \mathscr{S}_6 \backslash \{i, i+3\}$ and $i \in \mathscr{S}_3$.

Finally, let $\mathscr{Q}_j$ and $\mathscr{Q}_j^*$ be the closed half-spaces of P^3 determined by $\langle M_j, M_{j+1}\rangle$ and $\langle M_j, M_{j+5}\rangle$ such that

$$\mathscr{Q}_j \cap (M_{j+2} \cup M_{j+3} \cup M_{j+4}) = \{v\}, \quad j \in \mathscr{S}_6.$$

6.5 LEMMA. *Let* $L_{i,i+3} \subset \alpha$, $l(\alpha) = 1$ *and* $i \in \mathscr{S}_3$.
1. $\alpha \cap F$ *consists of* $L_{i,i+3}$ *and a curve* S_α^1 *of order two.*
2. *If* $\lim \alpha = \langle L_{jk}, L_{lm}\rangle$ [$\langle M_i, M_{i+3}\rangle$], *then* $\lim S_\alpha^1 = L_{jk} \cup L_{lm}$ [$M_i \cup M_{i+3}$]; $\mathscr{S}_6 = \{i, i+3, j, k, l, m\}$.
3. *If* $\alpha \subset \mathscr{R}_{i1}$, *then* $L_{i,i+3} \cap \overline{\text{int}\,(K)} \subset i(S_\alpha^1)$.
4. *If* $\alpha \subset \mathscr{R}_{i2}$, *then* $L_{i,i+3} \cap \overline{\text{int}\,(K)} \subset e(S_\alpha^1)$.
5. *If* $\alpha \subset \mathscr{R}_{i2}^*$, *then* $L_{i,i+3} \cap \overline{\text{ext}\,(K)} \subset i(S_\alpha^1)$ *and* $p_0 \in e(S_\alpha^1)$.
6. *If* $\alpha \subset \mathscr{R}_{i1}^*$, *then* $L_{i,i+3} \cap \overline{\text{ext}\,(K)} \subset e(S_\alpha^1)$ *and* $p_0 \in i(S_\alpha^1)$.

Proof. 1 and 2 are immediate. It is easy to check that $S_\alpha^1 \cap K = \alpha \cap (M_{i+1} \cup M_{i+2} \cup M_{i+4} \cup M_{i+5})$, 2, and 6.1.4 imply 3 to 6.

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6.6 LEMMA. *Let $M_j \subset \gamma \subset \mathcal{Q}_j$, $l(\gamma) = 1$ and $j \in \mathcal{S}_6$. Then $\gamma \cap F$ consists of M_j and a curve S_γ^1 of order two, $v \in M_j \cap S_\gamma^1$ and*

$$M_j \cap L_{jk} \subset e(S_\gamma^1), \quad k \in \mathcal{S}_6 \backslash \{j\}.$$

Proof. Clearly $\gamma \cap K = M_j$ or $\gamma \cap K = M_j \cup N$ where $N \cap K = \{v\}$ and thus $\gamma \cap F = M_j \cup S_\gamma^1$ from 2.1. We note that $\gamma \cap K = M_j$ yields that $M_j \cap S_\gamma^1 = \{v\}$ and $M_j \subset e(S_\gamma^1)$.

Since K is a (n.n.d.) cone, there exist planes γ_1 and γ_2 in $\mathcal{Q}_j$ such that the closest subspace $\mathcal{Q}_j' \subset \mathcal{Q}_j$, bounded by γ_1 and γ_2, contains all γ with $\gamma \cap K = M_j$. If $\gamma_1 = \gamma_2$, then $\mathcal{Q}_j' = \gamma_1$.

Let γ tend to $\gamma_1[\gamma_2]$ in $\mathcal{Q}_j \backslash \mathcal{Q}_j'$. Then

$$\lim S_\gamma^1 = S_{\gamma_1}^1[S_{\gamma_2}^1] \quad \text{and} \quad \lim (M_j \cap S_\gamma^1) = \{v\}.$$

Thus $M_j \backslash \{v\} \subset e(S_{\gamma_1}^1) \cap e(S_{\gamma_2}^1)$ implies that $M_j \cap L_{jk} \subset e(S_\gamma^1)$ for γ sufficiently close to $\gamma_1[\gamma_2]$ and $k \in \mathcal{S}_6 \backslash \{j\}$. Since S_γ^1 depends continuously on γ, $M_j \cap L_{jk} \not\subset S_\gamma^1$ implies the lemma.

6.7 Let $i \in \mathcal{S}_3$ and $\mathcal{S}_i^* = \mathcal{S}_6 \backslash \{i, i + 3\}$. From 6.4.2,

$$\mathcal{P}_i \cap \text{int } (K) \cap F \subset \mathcal{R}_i = \mathcal{R}_{i1} \cup \mathcal{R}_{i2}$$

and thus

$$\overline{\mathcal{P}_i \cap \text{int } (K) \cap F} = F_i' \cup F_i$$

where

$$F_i' = \mathcal{P}_i \cap \mathcal{R}_{i1} \cap \overline{\text{int } (K)} \cap F \quad \text{and}$$
$$F_i = \mathcal{P}_i \cap \mathcal{R}_{i2} \cap \overline{\text{int } (K)} \cap F.$$

Clearly both F_i' and F_i are non-empty. From 6.1 and 6.4,

$$\begin{aligned} \text{bd } (F_i) = {} & (\mathcal{P}_i \cap (L_{i+1,i+2} \cup L_{i+4,i+5})) \\ & \cup (\overline{\text{int } (K)} \cap (L_{i+1,i+4} \cup L_{i+2,i+5})) \\ & \cup (\mathcal{R}_{i2} \cap (\cup M_j)), \quad j \in \mathcal{S}_i^*, \end{aligned}$$

and

$$\text{bd } (F_i') = (\mathcal{P}_i \cap (L_{i+1,i+2} \cup L_{i+4,i+5})) \cup (\mathcal{R}_{i1} \cap (\cup M_j)), \quad j \in \mathcal{S}_i^*.$$

From 6.0.1 and 6.4. iii), the six line segments in bd (F_i') determine two triangles with the common point v such that

$$\text{int } (F_i') = G_i \cup G_i',$$

G_i and G_i' are open triangular regions, say

$$\text{bd } (G_i) = (\mathcal{P}_i \cap L_{i+1,i+2}) \cup (\mathcal{R}_{i1} \cap (M_{i+1} \cup M_{i+2})),$$
$$\text{bd } (G_i') = (\mathcal{P}_i \cap L_{i+4,i+5}) \cup (\mathcal{R}_{i1} \cap (M_{i+4} \cup M_{i+5})) \quad \text{and} \quad \overline{G_i} \cap \overline{G_i'} = \{v\}.$$

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6.8 THEOREM. *For* $i \in \mathscr{S}_3$,

$$\overline{\mathscr{P}_i \cap \text{int }(K)} \cap F = \overline{G_i} \cup \overline{G_i'} \cup F_i$$

where

1. $G_i \cap E \neq \emptyset \neq G_i' \cap E$ *and* $v \in (\overline{G_i \cap E}) \cap (\overline{G_i' \cap E})$ *and*
2. *every* $r \in F_i$ *such that* $l(r) = 0$ *is hyperbolic.*

Proof. 1. Let $r \in G_i \cup G_i'$ and $\langle v, r\rangle \subset \delta$. Clearly both G_i and G_i' satisfy 1.5.11 and thus $l(r) = l(\delta) = 0$. We choose δ so that

$$\delta \cap (L_{i+1,i+2} \cup L_{i+4,i+5}) \subset \text{ext }(K).$$

Then $r \in \text{int }(K)$ and 2.1 imply that v is the double point of $\delta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and bd $(G_i) \cup$ bd $(G_i') \subset \overline{\text{int }(K)}$ implies that

$$\delta \cap (\text{bd }(G_i) \cup \text{bd }(G_i')) = \{v\}.$$

Since every line of δ meets $\mathscr{A}_1 \cup \mathscr{A}_2$, we obtain that

$$r \in \mathscr{L} \subset G_i \cup G_i' \cup \{v\}.$$

We now argue as in the proof of 4.4.

2. Let $r \in F_i$, $l(r) = 0$. Then $r \in \mathscr{R}_{i2}$ and 6.5.4 imply that $p_0 \in e(S_\alpha^1)$, $\alpha = \langle L_{i,i+3}, r\rangle$. Let $\beta = \langle v, p_0, r\rangle$. Then $l(r) = 0$ and 6.3 yield that v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and $r \in \mathscr{A}_1 \cup \mathscr{A}_2$. Since $\mathscr{A}_1 \cap \mathscr{A}_2 = \{v, p_0\}$, $r \in \mathscr{A}_1$ say. Let $\mathscr{A}_1' \subset \mathscr{A}_1$ be a subarc such that $r \in \text{int }(\mathscr{A}_1')$, $p_0 \in \mathscr{A}_1'$ and $S_\alpha^1 \cap \mathscr{A}_1' = \{r\}$. As $\mathscr{A}_1$ is of order two,

$$p_0 \in i(\mathscr{A}_1 \backslash \{p_0\}) \subset i(\mathscr{A}_1').$$

It is immediate that $r \in \overline{e(\mathscr{A}_1') \cap e(S_\alpha^1)}$ and thus $r \cap H$ by 1.5.8.

We refer to Figure 5 for a representation of

$$\overline{\text{int }(K)} \cap F = \cup\, (\overline{G_i} \cup \overline{G_i'} \cup F_i), \quad i \in \mathscr{S}_3.$$

6.9 Henceforth, we assume that $i \in \mathscr{S}_3 = \{i, j, k\}$, $j \equiv i + 1 \pmod 3$ and $k \equiv i + 2 \pmod 3$. As in 6.2, 6.0.3 and 6.1.4 imply that $L_{i,i+3}$, $L_{i+1,i+2}$ and $L_{i+4,i+5}$ are either concurrent or determine an open, triangular region $G_i^* \subset \text{ext }(K) \cap F$ satisfying 1.5.11. In the former case, we set $G_i^* = \emptyset$.

From 6.4, $L_{i,i+3} \subset \mathscr{P}_j \cap \mathscr{P}_k$ and thus $G_i^* \subset \mathscr{P}_j \cup \mathscr{P}_k$ and $G_i^* \cap \mathscr{P}_i = \emptyset$ since

$$\langle L_{i+1,i+2}, L_{i+4,i+5}\rangle \subset \mathscr{R}_i \backslash \mathscr{R}_i^*, G_i^* \subset \mathscr{R}_i \quad \text{and} \quad G_i^* \cap \mathscr{R}_i^* = \emptyset.$$

Finally

$$\mathscr{P}_i \cap \text{ext }(K) \cap F \subset \mathscr{R}_i^* = \mathscr{R}_{i1}^* \cup \mathscr{R}_{i2}^* \quad \text{and} \quad \alpha_0 \subset \mathscr{R}_{i2}^* \backslash \mathscr{R}_{i1}^*$$

imply that $G_j^* \cup G_k^* \subset \mathscr{R}_{i2}^*$.

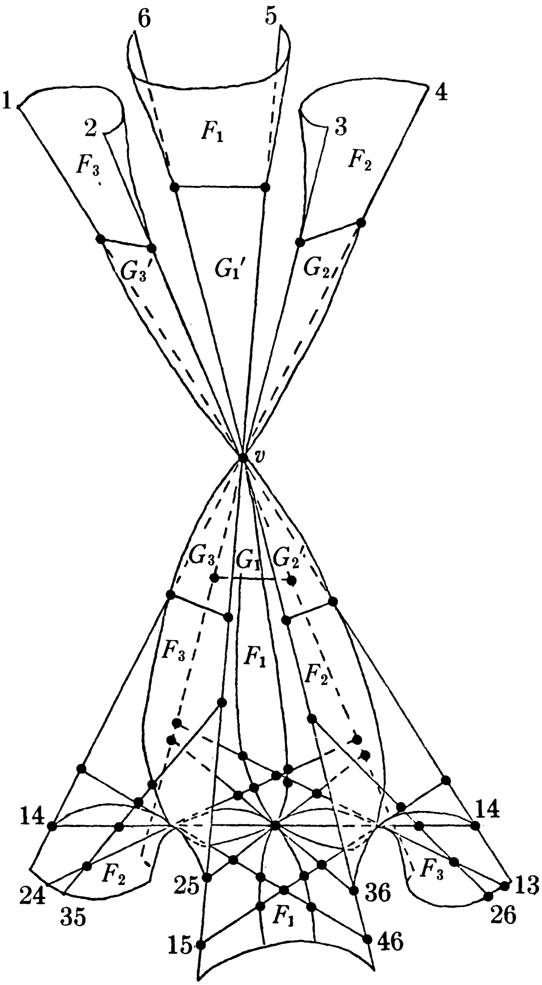

FIGURE 5

From 6.4, 2 and 3,

$$\mathscr{P}_i \cap \text{ext } (K) \cap F \subset \mathscr{R}_i^* \cap \mathscr{R}_j \cap \mathscr{R}_k.$$

Let

$$F_i^* = \mathscr{P}_i \cap (\mathscr{R}_{i1}^* \cup \mathscr{R}_{j1} \cup \mathscr{R}_{k1} \cup \mathscr{Q}_i \cup \mathscr{Q}_{i+3}) \cap \overline{\text{ext } (K)} \cap F$$

and

$$\tilde{F}_i = \mathscr{P}_i \cap (\mathscr{R}_{i2}^* \cap \mathscr{R}_{j2} \cap \mathscr{R}_{k2} \cap \mathscr{Q}_i^* \cap \mathscr{Q}_{i+3}^*) \cap \overline{\text{ext } (K)} \cap F.$$

Then

$$\overline{\mathscr{P}_i \cap \text{ext }(K)} \cap F = F_i^* \cup \tilde{F}_i \quad \text{and}$$
$$\overline{\text{ext }(K)} \cap F = \cup (F_i^* \cup \tilde{F}_i), \quad i \in \mathscr{S}_3.$$

6.10 THEOREM. *For* $i \in \mathscr{S}_3$, *every* $r \in F_i^*$ *such that* $l(r) = 0$ *is hyperbolic and* $(G_j^* \cup G_k^*) \cap F_i^* = \emptyset$.

Proof. Let $r \in F_i^*$, $l(r) = 0$ and $\beta = \langle v, p_0, r\rangle$. Then v is the double point of $\beta \cap F = \mathscr{L} \cup \mathscr{A}_1 \cup \mathscr{A}_2$, $p_0 \in \mathscr{A}_1 \cup \mathscr{A}_2 \subset \overline{\text{int }(K)}$, $r \in \mathscr{L} \subset \overline{\text{ext }(K)}$ and $p_0 \in e(\mathscr{L})$.

If $r \in \mathscr{R}_{i1}^* \cup \mathscr{R}_{j1} \cup \mathscr{R}_{k1}$, then

$$p_0 \in i(S^1(L_{i,i+3}, r)) \cup i(S^1(L_{j,j+3}, r)) \cup i(S^1(L_{k,k+3}, r))$$

by 6.5, 5 and 3. As in the proof of 6.8.2, $p_0 \in e(\mathscr{L})$ implies that $r \in H$. If $r \notin \mathscr{R}_{i1}^* \cup \mathscr{R}_{j1} \cup \mathscr{R}_{k1}$, then

$$r \in \mathscr{R}_{i2}^* \cap (\mathscr{Q}_i \cup \mathscr{Q}_{i+3}).$$

From 6.6.5 and 6.6,

$$L_{i,i+3} \cap (M_i \cup M_{i+3}) \subset i(S^1(L_{i,i+3}, r))$$

and

$$M_i \cap L_{i,i+3} \subset e(S^1(M_i, r)) \quad \text{or} \quad M_{i+3} \cap L_{i,i+3} \subset e(S^1(M_{i+3}, r)).$$

Then

$$r \notin \overline{e(S^1(L_{i,i+3}, r)) \cap e(S^1(M_i, r))}$$

or

$$r \notin \overline{e(S^1(L_{i,i+3}, r)) \cap e(S^1(M_{i+3}, r))},$$

and $r \in H$ by 1.5.8.

Since $G_j^* \cup G_k^* \neq \emptyset$ implies that $(G_j^* \cup G_k^*) \cap E \neq \emptyset$,

$$(G_j^* \cup G_k^*) \cap F_i^* = \emptyset.$$

6.11 Let $r \in \mathscr{P}_i \cap \text{ext }(K) \cap F$, $l(r) = 0$ and $\beta = \langle v, p_0, r\rangle$. Then v is the double point of $\beta \cap F = \mathscr{L}_\beta \cup \mathscr{A}_1 \cup \mathscr{A}_2$ and

$$r \in \mathscr{L}_\beta = \beta \cap \mathscr{R}_i^* \cap \mathscr{R}_j \cap \mathscr{R}_k \cap \overline{\text{ext }(K)} \cap F.$$

Let $\beta \cap L_{mn}$ be the point r_{mn}, $\{m, n\} \subset \mathscr{S}_6$. Then $\mathscr{L}_\beta \subset \mathscr{R}_i^* \cap \mathscr{R}_j \cap \mathscr{R}_k$ and 6.4 imply that

$$r_{i+1,i+5}, r_{i+2,i+4}, r_{i+2,i+3} (= r_{j+1,j+2}), r_{i,i+5}, r_{i+3,i+4}, r_{i,i+1}$$

are the only r_{mn} in $\mathscr{L}_\beta$ and

1. $\{p_0\} = \langle r_{i+1,i+5}, r_{i+2,i+4}\rangle \cap \langle r_{i,i+5}, r_{i+2,i+3}\rangle \cap \langle r_{i,i+1}, r_{i+3,i+4}\rangle$.

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We note that $r_{mn} \neq r_{\lambda\mu}$ if $\{m, n\} \cap \{\lambda, \mu\} \neq \emptyset$ and from the definition of $\mathscr{P}_i$, there exists a $\beta^* \subset \mathscr{P}_i$ such that $l(\beta^*) = 0$ and

$$r_{i+1,i+5} = r_{i+2,i+4} = r^*.$$

Let $r_{mn} \neq r_{\lambda\mu}$ in $\mathscr{L}_\beta$. We denote by $\mathscr{L}_\beta(m, n; \lambda, \mu)$ the subarc of $\mathscr{L}_\beta \backslash \{v\}$ bounded by r_{mn} and $r_{\lambda\mu}$. Hence $v \notin \mathscr{L}_\beta(m, n; \lambda, \mu)$. From 1 and $r^* = r_{i+1,i+5} = r_{i+2,i+4}$, we obtain that

2. $r^* \in \mathscr{L}_{\beta^*}(i, i+5; i+2, i+3) \cap \mathscr{L}_{\beta^*}(i, i+1; i+3, i+4)$.

Let $r \in \tilde{F}_i$. Then

$$r \in \mathscr{R}_{i2}^* \cap \mathscr{R}_{j2} \cap \mathscr{R}_{k2} \cap \mathscr{Q}_i^* \cap \mathscr{Q}_{i+3}^*.$$

Since bd $(\mathscr{R}_{i2}^*) = \alpha_0 \cup \langle L_{i+1,i+5}, L_{i+2,i+4} \rangle$, $r \in \mathscr{R}_{i2}^*$ implies that

3. $r \in \mathscr{L}_\beta(i+1, i+5; i+2, i+4)$.

Similarly $r \in \mathscr{R}_{j2} \cap \mathscr{R}_{k2}$ implies that

4. $r \in \mathscr{L}_\beta(i, i+5; i+2, i+3) \cap \mathscr{L}_\beta(i, i+1; i+3, i+4)$.

From the definition of $\mathscr{Q}_i^*$ and $\mathscr{P}_i$, we obtain that $\beta \cap K \subset \mathscr{Q}_i^*$, $\mathscr{Q}_i^* \cap \mathscr{L}_\beta$ is the subarc of $\mathscr{L}_\beta$, bounded by $r_{i,i+1}$ and $r_{i,i+5}$, containing v and

$$\mathscr{Q}_i \cap \mathscr{L}_\beta = \mathscr{L}_\beta(i, i+1; i, i+5).$$

Similarly

$$\mathscr{Q}_{i+3} \cap \mathscr{L}_\beta = \mathscr{L}_\beta(i+2, i+3; i+3, i+4).$$

Hence

5. $r \notin \mathscr{L}_\beta(i, i+1; i, i+5) \cup \mathscr{L}_\beta(i+2, i+3; i+3, i+4)$.

Finally, the cyclical labelling in 6.0 implies that

6. $r_{i+1,i+5} \in \mathscr{L}_\beta(i, i+1; i, i+5)$ and $r_{i+2,i+4} \in \mathscr{L}_\beta(i+2, i+3; i+3, i+4)$.

The preceding readily yields that

$$r \in \mathscr{L}_\beta(i, i+5; i+2, i+3) \subset \mathscr{L}_\beta(i+1, i+5; i+2, i+4) \subset \mathscr{L}_\beta(i, i+1; i+3; i+4)$$

or

$$r \in \mathscr{L}_\beta(i, i+1; i+3, i+4) \subset \mathscr{L}_\beta(i+1, i+5; i+2, i+4) \subset \mathscr{L}_\beta(i, i+5; i+2, i+3).$$

More precisely, r_{mn} and $r \in \tilde{F}_i$ are contained in $\mathscr{L}_\beta \backslash \{v\} \subset \mathscr{P}_i$ in the sequence

7. $r_{i,i+1}, r_{i+1,i+5}, r_{i,i+5}, r, r_{i+2,i+3}, r_{i+2,i+4}, r_{i+3,i+4}$

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or

8. $r_{i,i+5}, r_{i+1,i+5}, r_{i,i+1}, r, r_{i+3,i+4}, r_{i+2,i+4}, r_{i+2,i+3}$.

6.12 THEOREM. $\mathscr{P}_i \cap (G_j^* \cup G_k^*) = \{r \in \tilde{F}_i | l(r) = 0\}, \mathscr{S}_3 = \{i, j, k\}$.

Proof. From 6.9 and 6.10,

$$\mathscr{P}_i \cap (G_j^* \cup G_k^*) \subseteq \{r \in \tilde{F}_i | l(r) = 0\}.$$

Let $r' \in \tilde{F}_i$, $l(r') = 0$ and $\beta' = \langle v, p_0, r' \rangle$. Then v is the double point of $\beta' \cap F = \mathscr{L}' \cup \mathscr{A}_1' \cup \mathscr{A}_2'$ and the sequence in $\mathscr{L}' \backslash \{v\}$ is say 6.11.7:

a) $r_{i,i+1}, r_{i+1,i+5}, r_{i,i+5}, r', r_{i+2,i+3}, r_{i+2,i+4}, r_{i+3,i+4}$.

If $\mathscr{P}_i \cap G_j^* = \emptyset$, then

$$L_{i,i+5} \cap L_{i+2,i+3} \not\subset \text{int } (\mathscr{P}_i) \quad \text{and}$$
$$r_{i,i+5} \neq r_{i+2,i+3} \text{ for all } \beta \subset \mathscr{P}_i \text{ such that } l(\beta) = 0.$$

Since $\{r_{i,i+5}, r_{i+2,i+3}\} \subset \mathscr{L}'(i + 1, i + 5; i + 2, i + 4)$, 6.11.2 and the continuity of $\mathscr{L}_\beta$ for $\beta \subset \mathscr{P}_i$ imply that this is a contradiction.

Let $\bar{r} \in \mathscr{P}_i \cap G_j^*$ and $\bar{\beta} = \langle v, p_0, \bar{r} \rangle$. Again, v is the double point of $\bar{\beta} \cap F = \mathscr{L}_{\bar{\beta}} \cup \mathscr{A}_1 \cup \mathscr{A}_2$. Then $l(r) = 0$ for

$$r \in G_j^*, G_j^* \subset \tilde{F}_i \quad \text{and}$$
$$\text{bd } (G_j^*) \subset L_{i+1,i+4} \cup L_{i,i+5} \cup L_{i+2,i+3}$$

imply that the sequence in $\mathscr{L}_{\bar{\beta}} \backslash \{v\}$ is 6.11.7:

b) $r_{i,i+1}, r_{i+1,i+5}, r_{i,i+5}, \bar{r}, r_{i+2,i+3}, r_{i+2,i+4}, r_{i+3,i+4}$.

By the continuity of $\mathscr{L}_\beta$ for $\beta \subset \mathscr{P}_i$, a) and b) imply that $r' \in G_j^*$.

From 6.12,

$$\bar{G}_1^* \cup \bar{G}_2^* \cup \bar{G}_3^* \subseteq \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3.$$

It is immediate that $\tilde{F}_i \subset F_i^*$ if int $(\tilde{F}_i) = \emptyset$ and thus

$$\overline{\text{ext } (K)} \cap F = \cup (\bar{G}_i^* \cup F_i^*), \quad i \in \mathscr{S}_3.$$

6.13 SUMMARY. *Let F be a C-nodal surface satisfying* 6.0. *Then*

$$F = (\cup (\bar{G}_i \cup \bar{G}_i' \cup \bar{G}_i^* \cup F_i \cup F_i^*)) \cup G_0, \quad i \in \mathscr{S}_3,$$

where

1. *$G_0, G_i, G_i', G_i^*, F_i$ and F_i^* are defined in* 6.2, 6.7 *and* 6.9,
2. *every $r \in F_i \cup F_i^*$ such that $l(r) = 0$ is hyperbolic,*
3. *$v \in \overline{G \cap E}$ if $G \neq \emptyset$ and $G = G_i$ or G_i', and*
4. *$G \cap E \neq \emptyset$ if $G \neq \emptyset$ and $G = G_0$ or G_i^*.*

We refer to Figure 6 for a representation of a C-nodal surface with twenty-one lines. The surface in P^3 defined by

$$x_0(4x_1 + x_2)(x_1 + x_2) + x_3(x_0^2 + x_1x_2) = 0$$

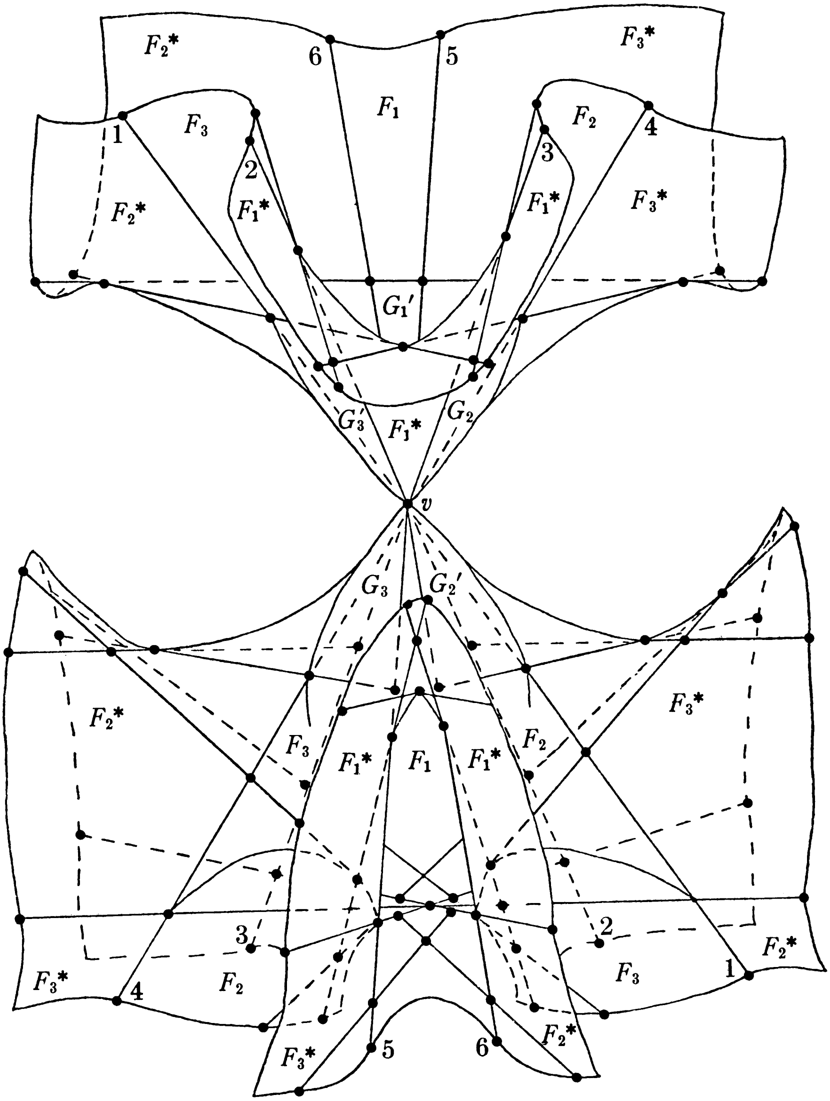

FIGURE 6

satisfies 6.13 with $M_1 \equiv x_0 = x_1 = 0$, $M_2 \equiv x_0 = x_2 = 0$, $M_3 \equiv x_1 + x_2 = x_0 + x_1 = 0$, $M_4 \equiv x_1 + x_2 = x_0 + x_2 = 0$, $M_5 \equiv 4x_1 + x_2 = x_0 + 2x_1 = 0$, $M_6 \equiv 4x_1 + x_2 = x_0 - 2x_1 = 0$, $L_{12} \equiv x_0 = x_3 = 0$, $L_{13} \equiv x_0 + x_1 = 4x_1 + x_2 - x_3 = 0$, $L_{14} \equiv x_0 - x_1 = 4x_1 + x_2 + x_3 = 0$, $L_{15} \equiv x_0 + 2x_1 = 2x_1 + 2x_2 - x_3 = 0$, $L_{16} \equiv x_0 - 2x_1 = 2x_1 + 2x_2$

$+ x_3 = 0$, $L_{23} \equiv x_0 - x_2 = 4x_1 + x_2 + x_3 = 0$, $L_{24} \equiv x_0 + x_2 = 4x_1 + x_2 - x_3 = 0$, $L_{25} \equiv 2x_0 - x_2 = 2x_1 + 2x_2 + x_3 = 0$, $L_{26} \equiv 2x_0 + x_2 = 2x_1 + 2x_2 - x_3 = 0$, $L_{34} \equiv x_1 + x_2 = x_3 = 0$, $L_{35} \equiv x_3 + 9x_0 = 3x_0 + 2x_1 - x_2 = 0$, $L_{36} \equiv x_0 + x_3 = x_0 + 2x_1 + x_2 = 0$, $L_{45} \equiv x_0 + x_3 = x_0 - 2x_1 - x_2 = 0$, $L_{46} \equiv x_3 + 9x_0 = 3x_0 - 2x_1 + x_2 = 0$, $L_{56} \equiv 4x_1 + x_2 = x_3 = 0$ and $K \equiv x_0^2 + x_1x_2 = 0$.

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