GRAPHS OF DEGREE THREE WITH A GIVEN ABSTRACT GROUP

ROBERT FRUCHT

1. Introduction. In his well-known book on graphs [1] König proposed the following problem: "When can a given abstract group be represented as the group of the automorphisms of a (finite) graph, and if possible how can the graph be constructed?"

To understand this problem well let us begin with the precise definition of a graph and its group (of automorphisms).

A (finite) graph is a finite set of points or vertices A, B, C, . . . , and edges or arcs which join certain pairs of these vertices; i.e. for each pair of distinct vertices P, Q, there is given an adjacency number $I_{P,Q} = I_{Q,P}$ such that

$$I_{P,Q} = I_{Q,P} = \begin{cases} 0, \text{ if the graph does not contain an edge } PQ \\ n, \text{ if the graph contains } n \text{ edges joining } P \text{ with } Q. \end{cases}$$

To exclude "isolated" vertices König postulates also that each vertex is the endpoint of at least one edge; i.e. for each vertex A of the graph there is at least another vertex Q such that $I_{A,Q} \neq 0$. (We will not consider the more general graphs—called by König "Graphen im weiteren Sinne"—where an edge may have two coincident endpoints, i.e. where also adjacency numbers $I_{P,P}$ exist, some of them having the value 1.)

The group of automorphisms of a graph, or shortly the group of a graph, may then be defined as the set of all the mappings of the graph into itself, i.e. of all the permutations of the vertices and edges which preserve incidence-relations.

An equivalent, but more algebraic definition of the group of a finite graph may be given for the more restricted class of the graphs having no pairs of edges forming a closed circuit; in such graphs any two vertices are either adjacent or "neighbours"—and in this case there is just one edge joining them—or not adjacent. For this more restricted class of graphs *where the adjacency numbers only can be* 0 *or* 1, the following definition of their group may be given:

Let to every vertex P_i of the graph correspond a variable x_i; then the graph will be fully described by the *quadratic form* $\sum_{i<k} I_{P_i,P_k} x_i x_k$, and the group of the graph will be nothing else than the group of all the permutations of the x_i which leave that quadratic form unaltered.

Returning now to König's problem, it might be stated more concisely in the following terms: Given any finite group $\mathfrak{H}$ find a graph G whose group is simply isomorphic to $\mathfrak{H}$.

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It has been shown by the author [2] that this problem has always a solution by giving for any group $\mathfrak{H}$ of order $h > 1$ the construction of a graph with $2h^3 - h^2$ vertices; this number may be reduced to $h(n+1)(2n+1)$, when one knows n elements of the group $\mathfrak{H}$ which generate the whole group. For the case $h = 1$ the author gave in the same paper an example of a graph with 7 vertices; later on Kagno [3] succeeded in finding a graph with only 6 vertices which has no non-identical group.

It is obvious that for some special groups there are also graphs with fewer vertices than $h(n+1)(2n+1)$; e.g. for the symmetric group $\mathfrak{S}_N$ with $h = N!$ and $n = 2$, we have a graph with only N vertices, namely the "complete N-point" in which any two distinct vertices are joined by an edge.

But it turns out that also in general the number of vertices needed by the author in his former paper for the construction of a graph with given abstract group is rather excessive, and the main object of this paper is to give a new solution of König's problem by a graph with fewer vertices and fulfilling also the additional condition of being "regular of degree 3" or "cubical."

As to the last condition it must be remembered that *degree* of a vertex A is called the number of edges having one of their endpoints in A, i.e. the sum $\sum_P I_{A,P}$, where P runs through all the vertices of the graph; and when all the vertices of a graph are of the same degree r, König calls the graph "*regular of degree r*." Recently the graphs which are regular of degree 3 have been called "*cubical*" by Tutte [4], and we will use this shorter name too. The outstanding interest these cubical graphs deserve in the theory of graphs seemed to us to justify a study of König's problem for this special class of graphs, and the rather surprising result (Theorem 4.1) was that in spite of this new condition of regularity of degree 3 there is always a cubical graph with only $2h(n+2)$ vertices whose group is simply isomorphic to a given abstract group $\mathfrak{H}$ of order $h > 2$ and generated by n of its elements; for $h = 1$ or 2 there are cubical graphs with 12 or 10 vertices respectively (Theorems 2.3 and 2.4).

Also in the case of cubical graphs it cannot be claimed that $2h(n+2)$ vertices are always necessary, since for some special groups cubical graphs with fewer vertices are known; e.g. the "complete 4-point" for the group $\mathfrak{S}_4$ ($h = 24$, $n = 2$), Petersen's graph [5] with 10 vertices for $\mathfrak{S}_5$ ($h = 120$, $n = 2$), Kagno's graph [6] H_{17} with 6 vertices for the dihedral group of order 12, etc.

The same general principle underlying the construction of a cubical graph may also be slightly modified to give a new solution of König's problem in its primitive form (i.e. without postulating that the graph be cubical); instead of $h(n+1)(2n+1)$ the number of vertices needed for the construction will now (Theorems 3.2 and 4.2) be: $2hn$ for non-cyclic groups, and $3h$ for cyclic groups ($n = 1$) of order $h > 3$; if $h = 3$ it seems that a tenth vertex is indispensable.

Finally it must be emphasized that König's problem has been interpreted here in the sense that only *simple isomorphism* between the given group and that of the graph is required. If the given group is a permutation group $\mathfrak{P}_N$

on N symbols, there would be also the problem of finding a graph with the same number N of vertices, and whose group is *identical* with $\mathfrak{P}_N$. However this more difficult problem has not always solutions; e.g. Kagno showed [6] that there are no graphs when $\mathfrak{P}_N$ is a cyclic or an alternating group ($N > 3$), and gave for $N \leq 6$ a complete list of all the cases where a solution exists. It seems that the general case ($N \geq 7$) of this problem has not yet been treated.

2. Types; cubical graphs with groups of order 1 and 2. In this section we are only dealing with cubical graphs having no adjacency number > 1.

We begin with the introduction of the notion of the "type" (κ, λ, μ) of a vertex; this notion will prove to be useful in the investigation of the group of a cubical graph.

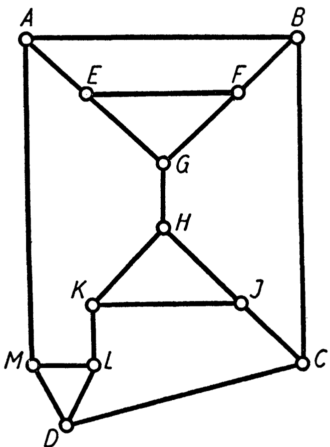

Fig. 1

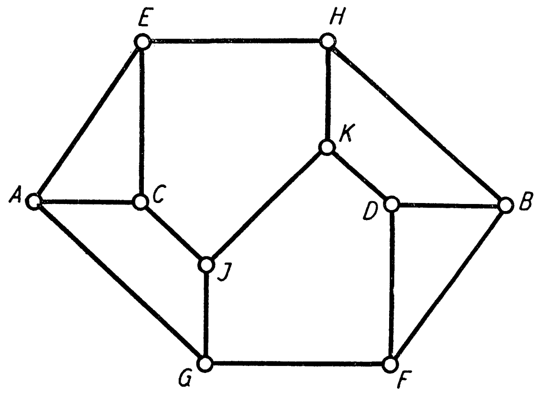

Fig. 2

In a cubical graph without adjacency numbers > 1 any vertex P has 3 distinct neighbours, say P_1, P_2, P_3. It may happen that there is no closed polygon or ν-circuit that contains the two edges PP_1 and PP_2; then let $\kappa = \infty$. (Here a closed polygon or ν-circuit is defined as a set of ν edges, A_1A_2, A_2A_3, $A_3A_4, \ldots, A_{\nu-1}A_\nu$, $A_\nu A_1$, such that no two of the ν vertices $A_1, A_2, A_3, \ldots, A_\nu$ coincide.) Otherwise there will be one or more closed polygons containing the edges PP_1 and PP_2; then let κ be the least value of ν for which such a ν-circuit exists. In an analogous manner define λ for the two edges PP_1 and PP_3, and μ for the two edges PP_2 and PP_3. Of course, since the enumeration of the 3 neighbours of P was arbitrary, we always may suppose that $\kappa \leq \lambda \leq \mu$. Then the number-triple (κ, λ, μ) will be called the *type* of the vertex P.

As an example consider the cubical graph of Figure 1 which can be drawn in the plane; it has 12 vertices (and hence 18 edges). Let us begin with determining the type of A. This vertex is endpoint of the edges AB, AE, and AM. The pair AB, AE occurs in the 4-circuit $AEFBA$ (but in no ν-circuit with $\nu<4$); the pair AE, AM occurs in the 7-circuits $AEFBCDMA$ and $AEGHK$-LMA, but in no closed polygon with fewer than 7 vertices; finally the pair

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AB, AM is contained in the 5-circuit $ABCDMA$, but in no ν-circuit with $\nu < 5$. Hence the type of the vertex A is (4, 5, 7).

In an analogous manner the following table of types for each vertex of the graph of Figure 1 was obtained:

A	(4, 5, 7)
B	(4, 5, 6)
C	(5, 5, 6)
D	(3, 5, 5)
E, F	(3, 4, 5)
G, H	(3, 6, 7)
J, K, L, M	(3, 5, 6)

The usefulness of this notion of type results from the following two theorems (whose proof is obvious):

THEOREM 2.1. *A necessary (but not sufficient) condition that a vertex P of a cubical graph may be taken into another vertex Q by a permutation belonging to the group of the graph is that P and Q are of the same type.*

THEOREM 2.2. *If for some* 3 *numbers* $\kappa \leq \lambda \leq \mu$ *there is in a cubical graph just* one *vertex P of the type* (κ, λ, μ), *then P is left fixed by each permutation belonging to the group of the graph.*

As an application of these theorems we are now going to prove the

THEOREM 2.3. *The graph given by Figure* 1 *has a group of order* 1.

Proof. Let τ be any mapping of the graph into itself. According to Theorem 2.2 the vertices A, B, C, and D are left fixed by τ. The Theorem 2.1 does not exclude the possibility that the vertex E could be changed into F by τ, but it is easy to see that such an interchange of E and F is not possible, as E is a neighbour of the invariant vertex A, but F is not. Hence also E is left fixed by τ, and F too. This last vertex has the neighbours B, E, and G, but since B and E are left fixed by τ, also G is. Of the same type as G is only the vertex H, but as we already know that G remains invariant, also H must be left fixed by τ. As to the 4 vertices of type (3, 5, 6) it is easy to see that J is left unchanged by τ as the only common neighbour of C and H; K as the only common neighbour of H and J; L as the only common neighbour of D and K; finally M remains fixed, because all the other vertices of the graph are left unchanged by τ. Thus τ does not change any vertex, and as τ was any mapping of the graph into itself, it has been proved that the group of this graph consists only of the identity.

As another example of the use of types let us consider the cubical graph of Figure 2; it has 10 vertices (and 15 edges). We will now prove the

THEOREM 2.4. *The graph given by Figure* 2 *has a group of order* 2.

Proof. Here we have the following types:

A, B, C, D	(3, 4, 5)
E, F	(3, 5, 6)
G, H	(4, 5, 6)
J, K	(4, 5, 5)

It is obvious that this graph admits (beside the identity) the following permutation of order 2:

$$\sigma = \begin{pmatrix} A & B & C & D & E & F & G & H & J & K \\ B & A & D & C & F & E & H & G & K & J \end{pmatrix}.$$

Let τ be any mapping of the graph into itself; we must show that τ is either σ or the identity. Let us distinguish two cases:

Case 1: τ leaves the vertex G unchanged. Of the same type as G is only the vertex H which in this case must be left fixed too. The 3 neighbours of the fixed-point G, namely A, F, J, are of distinct types; hence they cannot be permuted among themselves by τ, but must be left fixed too. The same consideration for the neighbours of H shows that the same is true for B, E, K. Finally C must be left fixed by τ, as it is the only common neighbour of the fixed-points A and E; and D remains fixed, as it has "no other choice." Hence in this case τ is the identity.

Case 2: τ interchanges G and H. (According to Theorem 2.1 there are no other possibilities.) Then consider the "product" $\tau\sigma$, i.e. the mapping τ followed by the mapping σ defined above. Since τ takes G into H, and σ changes H into G, the product $\tau\sigma$ leaves G unchanged. Applying the considerations of case 1 to $\tau\sigma$ (instead of τ) we see that $\tau\sigma$ must be the identity; hence $\tau = \sigma^{-1} = \sigma$.

It would be easy to see that another graph with 12 vertices and group of order 1 (but different from that of Figure 1) might be obtained from that of Figure 2 by joining the mid-points of the edges AG and CJ by an edge.

3. Cubical graphs with cyclic groups.

THEOREM 3.1. *If $\mathfrak{H}$ is the cyclic group of order $h > 2$, there is a cubical graph with $6h$ vertices whose group is simply isomorphic to $\mathfrak{H}$.* (For the case $h = 2$ see Theorem 2.4).

Proof. Let $h > 2$. The quadratic form in $6h$ variables $a_i, b_i, c_i, d_i, e_i, f_i$ $(i = 1, 2, \ldots, h)$:

$$\sum_{i=1}^{h} (a_i b_i + a_i e_i + a_i f_i + b_i c_i + c_i d_i + c_i f_i + e_i f_i)$$
$$+ \sum_{j=1}^{h-1} (b_j e_{j+1} + d_j d_{j+1}) + b_h e_1 + d_1 d_h$$

defines (see introduction) a cubical graph with $6h$ vertices (which will be indicated by the same letters a_i, b_i, etc.); it has to be shown that its group is simply isomorphic to the cyclic group of order h.

In Figure 3 this graph is given for the case $h = 5$; in this case the types of the vertices are (always for $i = 1, 2, 3, 4, 5$):

a_i and f_i	(3, 4, 5)
b_i	(4, 7, 9)
c_i	(4, 7, 7)
d_i	(5, 7, 7)
e_i	(3, 7, 8).

In the general case of an h whatsoever the types would be the same except that of the vertices d_i which would be $(h, 7, 7)$ or $(7, 7, h)$ if $h \leq 11$, and $(7, 7, 11)$, if $h > 11$.

It is obvious that any cyclic permutation of all the suffices does not change the graph; we must therefore only show that any mapping τ of the graph into itself is either the identity or some cyclic permutation of all the suffices, i.e. some power of the permutation ω which changes each a_i into a_{i+1} (except a_h which is taken into a_1), each b_i into b_{i+1} (except b_h which is changed into b_1), etc.

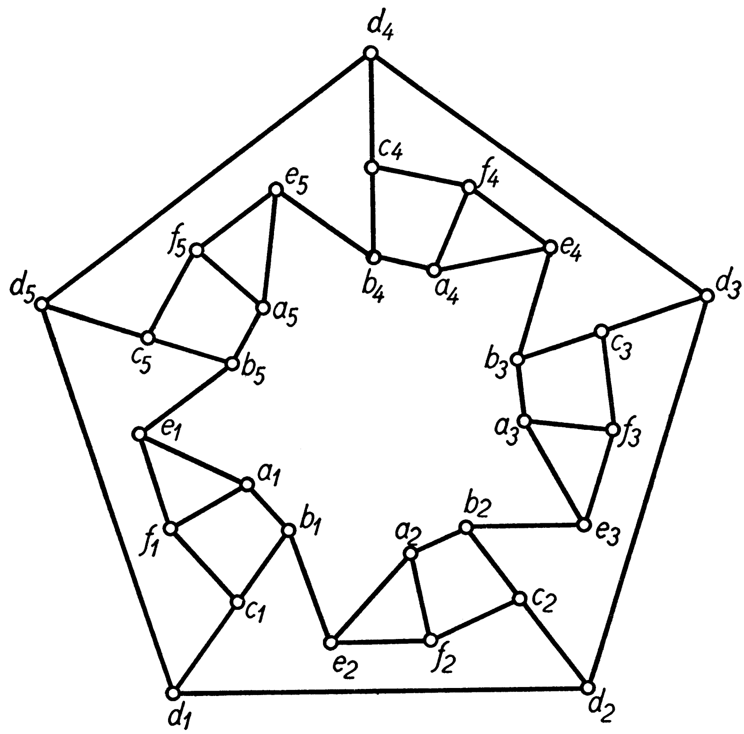

Fig. 3

In the proof we may distinguish two cases:

Case 1: τ leaves b_1 unchanged. Since the neighbours of b_1, namely a_1, c_1, e_2, are of distinct types, they cannot be permuted among themselves by τ, and are left fixed. The same argument holds for the neighbours of a_1, leading to the conclusion that also e_1 and f_1 remain unaltered. Hence, since two of the 3 neighbours of the fixed-point c_1, viz. b_1 and f_1, do not change, also the third, d_1, must be left unchanged by τ; and the same is true for b_h, the third neighbour of e_1 (beside a_1 and f_1). Now the same chain of considerations may be repeated for the vertices with subscript h, leading to the conclusion that they remain all unchanged by τ, and b_{h-1} too. Then the same considerations may be repeated for the vertices with subscript $h - 1$, etc., until the final conclusion is reached that no vertex is changed by τ; thus τ is the identity in this case.

Case 2: If τ changes b_1 into some other vertex, according to Theorem 2.1 (and the list of types given above) that other vertex must be a b_i too ($i = 2, 3, 4, \ldots, h$), say b_j. But there is also the mapping ω^{j-1} which takes b_1 into b_j (as ω changes b_i into b_{i+1}); hence the product $\tau(\omega^{j-1})^{-1}$ leaves b_1 unchanged and is therefore the identity (see Case 1); that means that $\tau = \omega^{j-1}$.

Having thus proved the Theorem 3.1, it may be remarked that a graph with fewer vertices than $6h$ can be found, when we remove the restriction that the graph is to be cubical:

THEOREM 3.2. *If $\mathfrak{H}$ is the cyclic group of order $h > 3$, there is a graph with $3h$ vertices whose group is simply isomorphic to $\mathfrak{H}$.*

If $h = 2$, there is of course the graph consisting only of two vertices joined by one edge. The case $h = 3$ seems to be exceptional, as I did not succeed in finding a graph with 9 vertices (or fewer); one with 10 vertices is given by Figure 4.

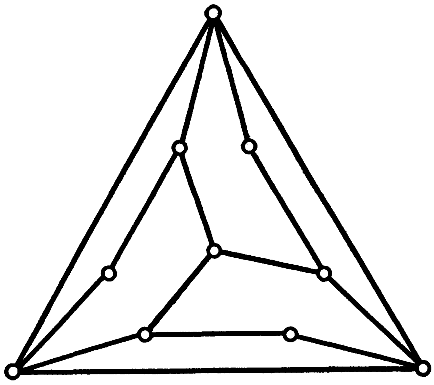

Fig. 4

Proof of Theorem 3.2. Let x_i, y_i, z_i ($i = 1, 2, \ldots, h$) be $3h$ variables, and consider the graph defined by the quadratic form:

$$\sum_{i=1}^{h} (x_i y_i + y_i z_i) + \sum_{j=1}^{h-1} (x_j x_{j+1} + z_j x_{j+1}) + x_1 x_h + x_1 z_h + \sum_{k<l} z_k z_l;$$

for $h = 5$ see Figure 5. In this graph each vertex x_i is of degree 4, each vertex y_i is of degree 2, and each vertex z_i is of degree $h + 1$. The proof that the group of this graph is simply isomorphic to the cyclic group of order h (if $h > 3$) is very easy.

Case 1: If any mapping τ of the graph into itself leaves x_1 unchanged, it must leave fixed also y_1, the only neighbour of x_1 having degree 2; hence τ leaves unchanged also z_1, the other neighbour of y_1. Now z_1 has only one neighbour of degree 4, namely x_2, which must be left fixed too, etc. Going on in this way it is easily seen that τ is in this case the identity.

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Case 2: If τ changes x_1 into another x_j (and there are no other possibilities if $h > 3$), then the product $\tau(\gamma^{j-1})^{-1}$ will be the identity, when γ is the substitution changing each x_i into x_{i+1}, each y_i into y_{i+1}, each z_i into z_{i+1} (being understood that for $i = h$ the subscript $h + 1$ is to be replaced by 1).

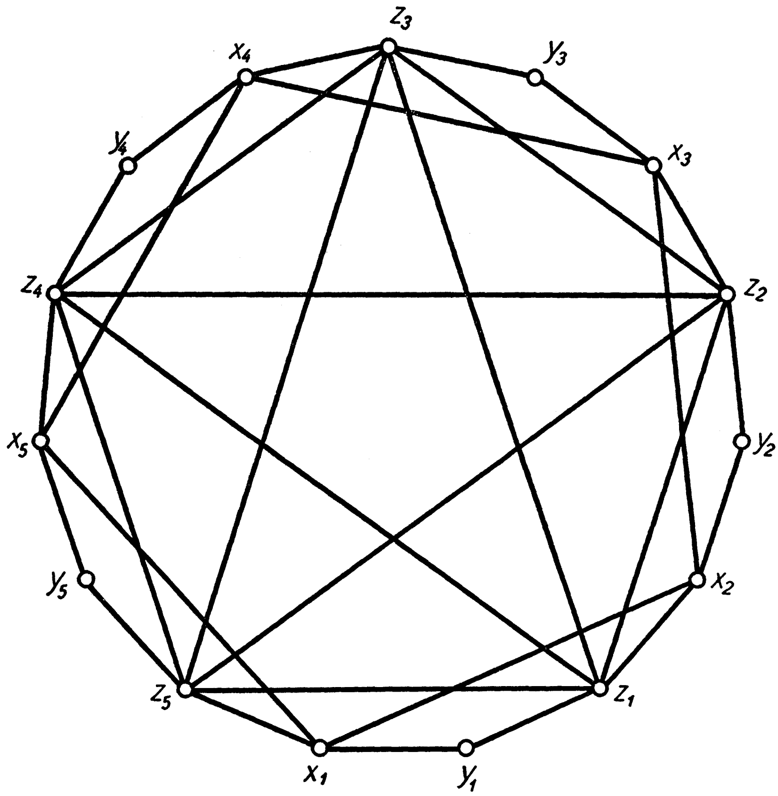

Fig. 5

As a final remark on Theorem 3.2 we wish to emphasize that there may be graphs with still fewer vertices than $3h$ (at least for certain values of $h > 3$). This is readily verified when the order of the cyclic group is not a prime nor a prime power. E.g. let $h = p^\alpha q^\beta$ (p and q being distinct primes); since $\mathfrak{H}$, the cyclic group of order h, is then the direct product of a cyclic group of order p^α and another cyclic group of order q^β, and since these groups have (according to Theorem 3.2) graphs with $3p^\alpha$ and $3q^\beta$ vertices respectively, these two graphs together form a (not connected) graph with only $3(p^\alpha + q^\beta)$ instead of $3h = 3p^\alpha q^\beta$ vertices and belonging to the same group $\mathfrak{H}$. (By adding $9p^\alpha q^\beta$ edges between each vertex of the one component and each of the other, also a connected graph with the same number of vertices and the same abstract group could be obtained.)

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4. Cubical graphs with non-cyclic groups.

THEOREM 4.1. *If $\mathfrak{H}$ is any finite group of order $h \geqq 3$ which may be generated by n of its elements, then it is always possible to find a cubical graph with $2(n + 2)h$ vertices that has a group simply isomorphic to $\mathfrak{H}$.*

Proof. If $n = 1$, the group is cyclic, and the Theorem 4.1 is only a restatement of Theorem 3.1.

If $n > 1$, let the elements of the given abstract group $\mathfrak{H}$ be enumerated as follows: let H_h be the unit of the group, and $H_1, H_2, H_3, \ldots, H_n$ the n elements generating the group; $H_{n+1}, H_{n+2}, \ldots, H_{h-1}$ will be the other elements of $\mathfrak{H}$.

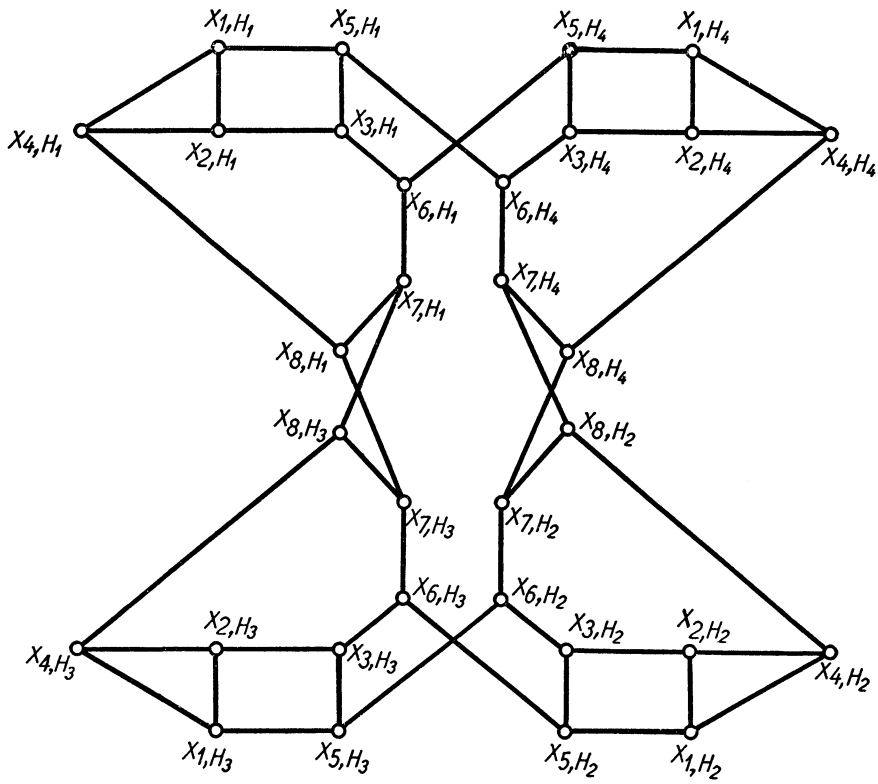

Fig. 6

To obtain a cubical graph with a group simply isomorphic to $\mathfrak{H}$ let us introduce $(2n + 4)h$ variables x_{i, H_k} (where the first subscript is a number $i = 1, 2, \ldots, 2n + 4$, and the second is an element H_k of the group, $k = 1, 2, \ldots, h$). Let us use the abbreviation Q_{ij} for the "scalar product":

$$Q_{ij} = \sum_{k=1}^{h} x_{i, H_k} x_{j, H_k}.$$

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Then consider the quadratic form

$$Q = Q_{12} + Q_{14} + Q_{15} + Q_{23} + Q_{24} + Q_{35} + Q_{36} + Q_{4,\,2n+4} + Q_{67} + Q_{78} + Q_{89} + \cdots + Q_{2n+3,\,2n+4} + S,$$

where S stands as an abbreviation for

$$S = \sum_{k=1}^{h} (x_{5,H_k} x_{6,H_1H_k} + x_{7,H_k} x_{8,H_2H_k} + \cdots + x_{2n+3,H_k} x_{2n+4,H_nH_k}).$$

It is easy to see that each variable x_{i,H_k} appears in Q just 3 times; hence the quadratic form Q defines a cubical graph G with $(2n + 4)h$ vertices. We give it in Figure 6 for the example of the direct product of two cyclic groups of order 2 ($h = 4$, $n = 2$), i.e. the group with the following multiplication table:

	H_1	H_2	H_3	H_4
H_1	H_4	H_3	H_2	H_1
H_2	H_3	H_4	H_1	H_2
H_3	H_2	H_1	H_4	H_3
H_4	H_1	H_2	H_3	H_4

(Of course for the construction of the graph only the first two lines are needed, as $n = 2$).

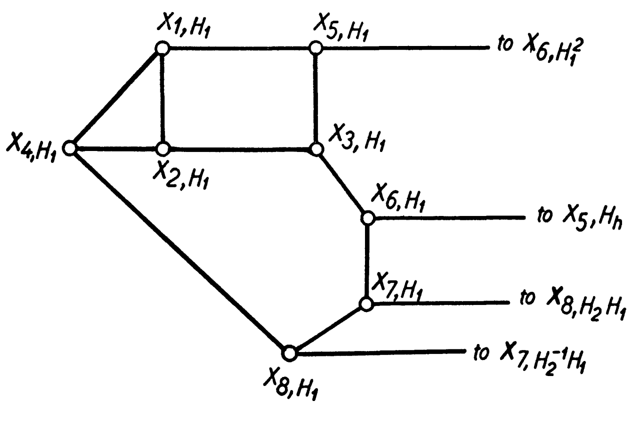

Fig. 7

For the more general case of any group generated by $n = 2$ elements we give in Figure 7 only a "corner" of the graph G, namely the vertices corresponding to the variables x_{i,H_1} (with the second subscript equal to H_1).

The proof that the graph G defined by the quadratic form Q has a group simply isomorphic to the given abstract group $\mathfrak{H}$ may be divided into two steps: firstly it has to be shown that G admits h different mappings into itself which constitute a group isomorphic to $\mathfrak{H}$; in the second place it must be proved that there are no other mappings of G into itself.

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First step: It must be shown that there are h different permutations σ_1, $\sigma_2, \ldots \sigma_h$ of the variables x_{i,H_k} constituting a group simply isomorphic to $\mathfrak{H}$ and leaving the quadratic form Q unaltered. This is not difficult, as these permutations may be given explicitly by the formula

$$\sigma_m = \begin{pmatrix} x_{i,H_k} \\ x_{i,H_kH_m} \end{pmatrix}, \qquad (i = 1, 2, \ldots, 2n+4;\ k = 1, 2, \ldots, h)$$

where m runs through $1, 2, \ldots h$. It is obvious that $\sigma_1, \sigma_2, \ldots, \sigma_h$ constitute a permutation group simply isomorphic to $\mathfrak{H}$, and it is also readily shown that each σ_m leaves the quadratic form Q unaltered. Indeed it is easy to verify that each σ_m leaves unchanged each of the $(2n+6)$ quadratic forms Q_{ij} which appear in the formula given above for Q, since σ_m changes Q_{ij} into

$$\sum_{k=1}^{h} x_{i,H_kH_m} x_{j,H_kH_m} = \sum_{k'=1}^{h} x_{i,H_{k'}} x_{j,H_{k'}} = Q_{ij}$$

as with H_k also $H_kH_m = H_{k'}$ runs through the whole group $\mathfrak{H}$. Similarly it can be shown that also each of the sums $\sum_{k=1}^{h} x_{5,H_k} x_{6,H_1H_k}$, $\sum_{k=1}^{h} x_{7,H_k} x_{8,H_2H_k} \cdots$ (whose sum equals S) is left unchanged by σ_m. Let us verify this e.g. for the second sum: $\sum_{k=1}^{h} x_{7,H_k} x_{8,H_2H_k}$. It is changed by σ_m into $\sum_{k=1}^{h} x_{7,H_kH_m} x_{8,H_2H_kH_m}$, i.e. it remains unchanged, as with H_k also the product H_kH_m runs through the whole group.

Second step: It remains to be shown that *any* mapping τ of the graph G into itself coincides with one of the h permutations $\sigma_1, \sigma_2, \ldots, \sigma_h$.

For the sake of simplicity we will give this part of the proof only for the case $n = 2$, but the reader will easily see that it might be likewise given for any $n > 2$; however it would be still more tedious than for $n = 2$.

With the aid of Figure 7 (and of Figure 6, when H_1 or H_2 are elements of order 2 in the group $\mathfrak{H}$) the following types are found for the vertices x_{i,H_1} of G:

x_{1,H_1} and x_{2,H_1}	(3, 4, 5)
x_{3,H_1}	(4, 6, 7) if H_1 is not of order 2 (4, 6, 6) if H_1 is of order 2
x_{4,H_1}	(3, 6, 7)
x_{5,H_1}	(4, *, *) if H_1 is not of order 2 (4, 6, 8) if H_1 is of order 2
x_{6,H_1}	(6, *, *) if H_1 is not of order 2 (6, 6, 9) if H_1 is of order 2
x_{7,H_1}	(6, *, *) if H_2 is not of order 2 (4, 6, 8) if H_2 is of order 2
x_{8,H_1}	(6, *, *) if H_2 is not of order 2 (4, 6, 8) if H_2 is of order 2

and the same values would hold for the other vertices x_{i,H_k} $(k = 2, 3, \ldots, h)$. (The stars stand for higher values than those given for generating elements of order 2.)

Since the only vertices of type (3, 6, 7) are $x_{4,H_1}, x_{4,H_2}, \ldots, x_{4,H_h}$, these can be only permuted among themselves by any mapping τ of the graph into itself. We must now distinguish two cases:

Case 1: τ leaves x_{4,H_1} unchanged. In this case we wish to show that $\tau = \sigma_h$ (identity). To this purpose let us consider in the first place the vertex x_{8,H_1}. (If $n > 2$ we should begin with x_{2n+4,H_1}). It must remain fixed too, as the other two neighbours of x_{4,H_1} (viz. $x_{1,H1}$ and x_{2,H_1}) have other types. Can these two neighbours of x_{4,H_1} be interchanged by τ? It is true that they have the same type (3, 4, 5); but x_{2,H_1} has a neighbour of type (4, 6, 6) or (4, 6, 7)—namely x_{3,H_1}—and x_{1,H_1} has no such neighbour. Hence x_{1,H_1} and x_{2,H_1} cannot be interchanged by τ, and therefore must be left fixed. The same will then be true of their "third neighbours" x_{3,H_1} and x_{5,H_1}, and of x_{6,H_1} as the third neighbour of x_{3,H_1}. Finally x_{7,H_1} must remain unchanged as the only common neighbour of the fixed-points x_{6,H_1} and x_{8,H_1}. (If $n > 2$ this part of the proof would be still longer, because it ought to be shown analogously that the whole "chain" $x_{6,H_1}, x_{7,H_1}, x_{8,H_1}, \ldots, x_{2n+4,H_1}$ remains unchanged).

Hence all the vertices x_{i,H_1} (with the second subscript H_1) are left fixed by τ. But x_{6,H_1} has the third neighbour x_{5,H_h} which must remain unchanged too, and by a reasoning similar to that given above it will follow that also all the vertices x_{i,H_h} (with the second subscript H_h) are left fixed by τ; the same will hold for all the vertices x_{i,H_2H_1}, since x_{7,H_1} has the third neighbour x_{8,H_2H_1}, etc. Going on in this manner until all the vertices of G are recognized as fixed, it will be possible to show that (in the case considered here) τ is the identity.

Case 2: τ maps x_{4,H_1} into some other x_{4,H_r} $(r \neq 1)$. Then the multiplication table of the group $\mathfrak{H}$ will always allow us to find a subscript s such that

$$H_s = H_1^{-1}H_r;$$

then the permutation

$$\sigma_s = \begin{pmatrix} x_{i,H_k} \\ x_{i,H_kH_s} \end{pmatrix}$$

will take x_{4,H_1} into $x_{4,H_1H_s} = x_{4,H_r}$, and σ_s^{-1}, the inverse permutation of σ_s, will change x_{4,H_r} into x_{4,H_1}. Hence the product $\tau\sigma_s^{-1}$ will be a mapping of G into itself which leaves x_{4,H_1} unchanged, and (according to the result obtained in the foregoing case 1) must be the identity: $\tau\sigma_s^{-1} = \sigma_h$. Hence $\tau = \sigma_s$.

Having thus finished the proof of Theorem 4.1, it should be remarked that in the case of some special groups fewer than $(2n + 4)h$ vertices may be needed to obtain a cubical graph whose group is simply isomorphic to a given abstract group $\mathfrak{H}$; e.g. for the direct product of two groups of order 2 our theorem gave us the cubical graph of Figure 6 with 32 vertices, but to the same abstract group belongs also the cubical graph of Figure 8 with only 8 vertices. Other examples were already mentioned in the introduction.

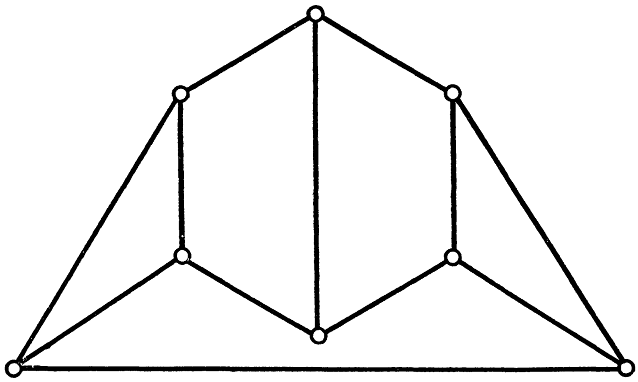

Fig. 8

A general reduction of the number of necessary vertices is however possible when we remove the condition that the graph is to be cubical.

THEOREM 4.2. *If $\mathfrak{H}$ is a non-cyclic group of order h and generated by n of its elements, one can find a graph with $2nh$ vertices whose group is simply isomorphic to $\mathfrak{H}$.* (For cyclic groups see Theorem 3.2).

This graph may be given by the following quadratic form in $2nh$ variables $x_{i,H_k} (i = 1, 2, \ldots, 2n;\ k = 1, 2, \ldots, h)$:

$$\sum_{k=1}^{h} (x_{1,H_k}x_{2,H_k} + x_{2,H_k}x_{3,H_k} + \ldots + x_{2n-1,H_k}x_{2n,H_k})$$

$$+ \sum_{k=1}^{h} (x_{1,H_k}x_{2n,H_k} + x_{2,H_k}x_{2n,H_k} + \ldots + x_{2n-2,H_k}x_{2n,H_k})$$

$$+ \sum_{\kappa<\lambda} x_{1,H_\kappa}x_{1,H_\lambda} + \sum_{k=1}^{h} (x_{1,H_k}x_{2,H_1H_k} + x_{3,H_k}x_{4,H_2H_k} + \ldots + x_{2n-1,H_k}x_{2n,H_nH_k}).$$

(The meaning of the H_k is the same as above.)

We omit the proof of Theorem 4.2, as it is similar to that of Theorem 4.1 and can be readily supplied by the reader.

As a final remark to the theorems of this section we wish to emphasize that the number of vertices we needed for the construction of a (cubical or general) graph with given abstract group of order h depends not only on h, but also on the number n of elements needed to generate the group. If we wish, however, to obtain an upper bound for the number of necessary vertices depending only on the order h of the given group, we might proceed in the following way: Let $h = p^\alpha q^\beta r^\gamma \ldots$ (where $p, q, r, \ldots$ are distinct primes); then it can easily be proved that it is always possible to generate a group of order h with $n \leqq \alpha + \beta + \gamma + \ldots$ elements; hence

$$2^n \leqq 2^\alpha \times 2^\beta \times 2^\gamma \times \ldots \leqq p^\alpha q^\beta r^\gamma \ldots = h,$$

and

$$n \leqq \log h / \log 2.$$

(The base of the logarithms does not matter of course; it needs only to be the same in numerator and denominator). Hence:

THEOREM 4.3. *To any abstract group of order $h > 1$ belongs a cubical graph with at most* $[2h(2 + \log h/\log 2)]$ *vertices.*

In the same way it would be easy to prove the following theorem for non-cubical graphs:

THEOREM 4.4. *To any abstract group of order $h > 3$ belongs a graph with at most* $[2h \log h/\log 2]$ *vertices.*

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Universidad Técnica F. Santa Maria
Valparaiso, Chile

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