

OPERATOR-VALUED MULTIPLIER THEOREMS CHARACTERIZING HILBERT SPACES

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Abstract

We show that the operator-valued Marcinkiewicz and Mihlin Fourier multiplier theorem are valid if and only if the underlying Banach space is isomorphic to a Hilbert space.

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1. Introduction

Mihlin's multiplier theorem is of great importance in analysis. It says that a bounded function $m \in C^1(\mathbb{R} \setminus \{0\})$ such that $tm'(t)$ is bounded, defines an $L^p(\mathbb{R})$ -multiplier for $1 < p < \infty$. In the context of partial differential equations vector-valued spaces $L^p(\mathbb{R}; X)$ occur in a natural way, where X is a Banach space. Thus the function m should take its values in $\mathcal{L}(X)$. Our aim is to show that Mihlin's multiplier theorem does hold for such operator-valued functions if and only if X is isomorphic to a Hilbert space.

The phenomenon that operator-valued versions of certain classical multiplier theorems are only valid in Hilbert spaces was first observed by Pisier (unpublished) as a consequence of Kwapien's deep characterization of Hilbert spaces. More recently, new versions of operator-valued multiplier theorems turned out to be most useful in the theory of evolution equations (see the references and comments below) and it

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seems to us that it is important to elaborate in some detail why the classical result merely holds on Hilbert spaces.

In another context, it helps to impose a Mikhlin’s condition of order k

$$(1) \quad m \in C^k(\mathbb{R} \setminus \{0\}; \mathcal{L}(X)), \quad \sup_{t \in \mathbb{R} \setminus \{0\}, 0 \leq l \leq k} \|t^l m^{(l)}(t)\| < \infty.$$

In fact, Amann [1] discovered that if m satisfies (1) with $k = 2$, then m is a multiplier for Besov spaces and in particular for the space $C^\theta(\mathbb{R}; X)$, $0 < \theta < 1$ (see also [2] and [11]). We show here that imposing higher order Mikhlin’s conditions does not help in the context of operator-valued L^p -multipliers.

We also consider the groups \mathbb{T} and \mathbb{Z} instead of \mathbb{R} . In fact, the case \mathbb{T} corresponds to Marcinkiewicz’s classical theorem and its operator-valued version is already treated in [3] for the order-1-case.

Now we would like to comment on the new vector-valued multiplier theorems which were found recently. It were Berkson-Gillespie [4] who introduced the notion of R -boundedness (after implicit use of Bourgain [6]). They use R as an abbreviation for Riesz, but in many subsequent papers people seem think rather of Rademacher or ‘Randomized’ because the definition involves Rademacher functions. A multiplier theorem of Marcinkiewicz type was established by Clément-de Pagter-Sukochev-Witvliet [8] for multipliers of the form $m(t)I$ (I is the identity operator) clarifying the role of R -boundedness. Then Weis [18] established Mikhlin’s theorem for operator-valued functions (without restriction) replacing boundedness by the stronger condition of R -boundedness. Then in [3] the corresponding periodic theorem (that is, Marcinkiewicz’s theorem) was proved on the basis of results in [8]. Štrkalj and Weis [17] gave an R -version of the variational version of the Marcinkiewicz theorem. Further important contributions were given by Clément-Prüss [9], Denk-Hieber-Prüss [10], and Girardi-Weis [11].

2. Periodic multipliers

Let us first recall some notions. Let X be a Banach space. Denote by r_j the j -th Rademacher function on $[0, 1]$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$. Let Y be another Banach space. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y . If $X = Y$ we will denote $\mathcal{L}(X, Y)$ simply by $\mathcal{L}(X)$. A family $\mathbf{T} \subset \mathcal{L}(X, Y)$ is called R -bounded if for some $q \in [1, \infty)$ there exists a constant $c_q \geq 0$ such that

$$(2) \quad \left\| \sum_{j=1}^n r_j \otimes T_j x_j \right\|_{L^q(0,1;Y)} \leq c_q \left\| \sum_{j=1}^n r_j \otimes x_j \right\|_{L^q(0,1;X)}$$

for all $T_1, \dots, T_n \in \mathbf{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. By Kahane’s inequality [14, Theorem 1.e.13] if such constant c_q exists for some $q \in [1, \infty)$, then it also exists for each $q \in [1, \infty)$.

It is known that R -boundedness is strictly stronger than boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. More precisely, each bounded subset in $\mathcal{L}(X, Y)$ is R -bounded if and only if X is of cotype 2 and Y is of type 2 (see [3, Proposition 1.13]). In particular, by a result of Kwapien [14, pages 73–74], each bounded subset in $\mathcal{L}(X)$ is R -bounded if and only if X is isomorphic to a Hilbert space.

For $1 \leq p < \infty$, consider the Banach space $L^p(0, 2\pi; X)$ with norm $\|f\|_p := (\int_0^{2\pi} \|f(t)\|^p dt)^{1/p}$. For $f \in L^p(0, 2\pi; X)$ we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt,$$

the k -th Fourier coefficient of f , where $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, $x \in X$ we let $e_k(t) = e^{ikt}$ and $(e_k \otimes x)(t) = e_k(t)x$ ($t \in \mathbb{R}$). A function $f \in L^p(0, 2\pi; X)$ is called a *trigonometric polynomial* if f is given by $f = \sum_{k \in \mathbb{Z}} e_k \otimes x_k$, where $x_k \in X$ is 0 for all but finitely many $k \in \mathbb{Z}$.

Let $M = (M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ be a sequence and let $1 \leq p, q < \infty$. We say that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p - L^q -Fourier multiplier* if there exists a constant $C > 0$ such that

$$\left\| \sum_{k \in \mathbb{Z}} e_k \otimes M_k x_k \right\|_{L^q(0, 2\pi; Y)} \leq C \left\| \sum_{k \in \mathbb{Z}} e_k \otimes x_k \right\|_{L^p(0, 2\pi; X)}$$

for all X -valued trigonometric polynomials $\sum_{k \in \mathbb{Z}} e_k \otimes x_k$. In this case, there exists a unique operator $M \in \mathcal{L}(L^p(0, 2\pi; X), L^q(0, 2\pi; Y))$ such that $(Mf)\hat{}(k) = M_k \hat{f}(k)$ for $k \in \mathbb{Z}$ [3]. When $p = q$, we say simply that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p -Fourier multiplier*. For $k \in \mathbb{Z}$, we let $(\Delta^1 M)(k) = M_{k+1} - M_k$ and $(\Delta^m M)(k) = (\Delta^1(\Delta^{m-1} M))(k)$ for $m \geq 2$. Notice that $\Delta^m M$ is a discrete analogue of the m -th derivative of M .

The classical Marcinkiewicz Fourier multiplier theorem has been extended to the operator-valued case in the following way: let X and Y be UMD spaces and let $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$; if both $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R -bounded, then $(M_k)_{k \in \mathbb{Z}}$ defines a periodic L^p -Fourier multiplier for each $1 < p < \infty$ [3]. Indeed, $(M_k)_{k \in \mathbb{Z}}$ is a periodic L^p - L^q -Fourier multiplier whenever $1 \leq q \leq p < \infty$.

We will need the following inequality of Pisier [15]. Let $1 \leq p < \infty$ and let $\Lambda = \{n_k : k \in N\} \subset \mathbb{Z}$ be a Sidon subset [16, page 120]. Then there exists $C > 0$ such that for any Banach space X and for any finite sequence $(y_k)_{1 \leq k \leq N}$ of X , we have

$$(3) \quad C^{-1} \left\| \sum_k r_k \otimes y_k \right\|_2 \leq \left\| \sum_k e_{n_k} \otimes y_k \right\|_p \leq C \left\| \sum_k r_k \otimes y_k \right\|_2.$$

Note that if $\lambda > 1$, then any subset $\{n_k : k \in \mathbb{N}\}$ satisfying $n_{k+1}/n_k \geq \lambda$ ($k \in \mathbb{N}$) is a Sidon subset of \mathbb{Z} [16, page 127].

The following result shows that one cannot replace R -boundedness in the operator-valued Marcinkiewicz theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 1. *Let X be a Banach space. Then the following assertions are equivalent:*

- (i) X is isomorphic to a Hilbert space.
- (ii) For some $1 \leq q < p < \infty$, each sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ satisfying
 - (a) $\sup_{k \in \mathbb{Z}} \|M_k\| < \infty$,
 - (b) $\sup_{k \in \mathbb{Z}} \|k^l (\Delta^l M)(k)\| < \infty$ for $l \in \mathbb{N}$,
 - (c) $M_k = 0$ for $k \leq 0$,

is a periodic L^p - L^q -Fourier multiplier.

- (iii) For all $1 < p < \infty$, each sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ satisfying

- (a) $\sup_{k \in \mathbb{Z}} \|M_k\| < \infty$,
- (b) $\sup_{k \in \mathbb{Z}} \|k(\Delta^1 M)(k)\| < \infty$,

is a periodic L^p -Fourier multiplier.

REMARK 2. For $l = 1$, the condition formulated in (iii) is the classical condition considered by Marcinkiewicz in the scalar case. For arbitrary $l \in \mathbb{N}$ we therefore speak of the *Marcinkiewicz condition of order l* . For $p = q$ and $l = 1$, Theorem 1 has been proved in [3, Proposition 1.17.]. However, a more refined choice of test functions is needed here for the general case. The motivation to consider $l > 1$ stems from the results on Fourier multipliers for spaces of Hölder continuous functions where, indeed, the Marcinkiewicz condition of order 2 suffices (see [2] and also the Concluding Remarks at the end of this article). Theorem 1 shows that this is not the case in the L^p -context even if we consider weaker multipliers by allowing $q < p$. This has also been done by Kalton-Lancien in the context of maximal regularity for Cauchy problems [13] (see also the Concluding Remarks 5 (b) below).

PROOF. (i) \Rightarrow (iii). Assume that X is isomorphic to a Hilbert space, then considering an orthonormal basis one easily verifies that each bounded subset in $\mathcal{L}(X)$ is actually R -bounded, so the result follows from the operator-valued Marcinkiewicz Fourier multiplier theorem in [3].

(iii) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (i). Assume that for some $1 \leq q < p < \infty$, each sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ satisfying $\sup_{k \in \mathbb{Z}} \|M_k\| < \infty$, $\sup_{k \in \mathbb{Z}} \|k^l (\Delta^l M)(k)\| < \infty$ for $l \in \mathbb{N}$ and $M_k = 0$ for $k \leq 0$, is a periodic L^p - L^q -Fourier multiplier. Let $N = (N_k)_{k \in \mathbb{N}} \subset \mathcal{L}(X)$ be a bounded sequence.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space. Let $\phi_1 \in \mathcal{S}(\mathbb{R})$ be such that $\text{supp}(\phi_1) \subset [2, 4]$ and $\phi_1(3) = 1$. For $n \geq 1$, we let $h_n = 2^{2n-2}$. Define $\phi_n = \phi_1(\cdot/h_n)$. Then $\text{supp}(\phi_n) \subset [2h_n, 4h_n]$ and $\phi_n(3h_n) = 1$. Let $\phi : \mathbb{R} \rightarrow \mathcal{L}(X)$ be defined by

$$\phi(t) = \begin{cases} \phi_n(t)N_n & \text{if } 2h_n \leq t \leq 4h_n \text{ for some } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = (\phi(k))_{k \in \mathbb{Z}}$. We claim that

$$(4) \quad \sup_{k \in \mathbb{Z}} \|\phi(k)\| < \infty,$$

$$(5) \quad \sup_{k \in \mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty,$$

for $l \in \mathbb{N}$. Indeed (4) is clearly true. We will only give the proof for (5) when $l = 2$, the proof for the general case is similar.

First notice that when $4h_n \leq k \leq 8h_n - 2$ for some $n \in \mathbb{N}$, or $k \leq 0$, then $(\Delta^2 M)(k) = 0$. While when $2h_n - 2 < k < 4h_n$ for some $n \in \mathbb{N}$

$$\begin{aligned} (\Delta^2 M)(k) &= (\phi_n(k+2) - 2\phi_n(k+1) + \phi_n(k))N_n \\ &= \left(\phi_1\left(\frac{k+1}{h_n} + \frac{1}{h_n}\right) - 2\phi_1\left(\frac{k+1}{h_n}\right) + \phi_1\left(\frac{k+1}{h_n} - \frac{1}{h_n}\right) \right) N_n \\ &= \frac{1}{2h_n^2}(\phi_1''(\eta_1) + \phi_1''(\eta_2))N_n \end{aligned}$$

for some $\eta_1, \eta_2 \in \mathbb{R}$. We deduce that

$$\sup_{k \in \mathbb{Z}} \|k^2(\Delta^2 M)(k)\| \leq \sup_{n \in \mathbb{N}} \frac{16h_n^2}{4h_n^2} \|N_n\| \sup_{x \in \mathbb{R}} |\phi_1''(x)| \leq 4 \sup_{n \in \mathbb{N}} \|N_n\| \sup_{x \in \mathbb{R}} |\phi_1''(x)|.$$

Thus $M = (\phi(k))_{k \in \mathbb{Z}}$ is a periodic L^p - L^q -Fourier multiplier by assumption. Hence there exists $C > 0$ such that for $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X$, we have

$$\left\| \sum_k e_k \otimes \phi(k)x_k \right\|_q \leq C \left\| \sum_k e_k \otimes x_k \right\|_p,$$

and, in particular,

$$\left\| \sum_{n \geq 1} e_{3h_n} \otimes M_k x_{3h_n} \right\|_q \leq C \left\| \sum_{n \geq 1} e_{3h_n} \otimes x_{3h_n} \right\|_p.$$

By (3), this implies that the sequence $(M_k)_{k \geq 1}$ is R -bounded. It is easy to check that if each countable subset of T is R -bounded then so is T . We deduce from this that each bounded subset in $\mathcal{L}(X)$ is actually R -bounded. By [3, Proposition 1.13], this implies that X is isomorphic to a Hilbert space. □

3. Multipliers on the line

Let X be a Banach space and consider the Banach space $L^p(\mathbb{R}; X)$ for $1 < p < \infty$. We denote by $\mathcal{D}(\mathbb{R}; X)$ the space of all X -valued C^∞ -functions with compact support. $\mathcal{S}(\mathbb{R}; X)$ will be the X -valued Schwartz space and we let $\mathcal{S}'(\mathbb{R}; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}); X)$, where $\mathcal{S}(\mathbb{R})$ denotes the \mathbb{C} -valued Schwartz space. Let Y be another Banach space. Then given $M \in L^1_{loc}(\mathbb{R}; \mathcal{L}(X, Y))$, we may define an operator $T : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X) \rightarrow \mathcal{S}'(\mathbb{R}; Y)$ by means of

$$T\phi := \mathcal{F}^{-1}M\mathcal{F}\phi \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R}; X),$$

where \mathcal{F} denotes the Fourier transform. Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$, we see that T is well defined on a dense subset of $L^p(\mathbb{R}; X)$. We say that M is an L^p -Fourier multiplier on $L^p(\mathbb{R}; X)$ if T can be extended to a bounded linear operator from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R}; Y)$.

The classical Mihklin Fourier multiplier theorem has been extended to the operator-valued case by Weis. Let X and Y be UMD spaces, $1 < p < \infty$ and let $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$. If both $\{M(x) : x \neq 0\}$ and $\{xM'(x) : x \neq 0\}$ are R -bounded, then M defines a L^p -Fourier multiplier on $L^p(\mathbb{R}; X)$ [18].

The following result shows that one cannot replace R -boundedness in the operator-valued Mihklin theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 3. *Let X be a Banach space. Then the following assertions are equivalent:*

- (i) X is isomorphic to a Hilbert space.
- (ii) For some $1 < p < \infty$, each function $M \in C^\infty(\mathbb{R}; \mathcal{L}(X))$ satisfying

- (a) $M(x) = 0$ for $x \leq 0$,
- (b) $\sup_{x \in \mathbb{R}} \|M(x)\| < \infty$,
- (c) $\sup_{x \in \mathbb{R}} (1 + |x|)^l \|M^{(l)}(x)\| < \infty$ for $l \in \mathbb{N}$,

defines an L^p -Fourier multiplier on $L^p(\mathbb{R}; X)$.

(iii) For all $1 < p < \infty$, each function $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{L}(X))$ satisfying the conditions

- (a) $\sup_{x \neq 0} \|M(x)\| < \infty$,
- (b) $\sup_{x \neq 0} \|xM'(x)\| < \infty$,

defines an L^p -Fourier multiplier on $L^p(\mathbb{R}; X)$.

PROOF. (i) \Rightarrow (iii). Assume that X is isomorphic to a Hilbert space. Then considering an orthonormal basis one easily verifies that each bounded subset in $\mathcal{L}(X)$ is

actually R -bounded, so the result follows from the operator-valued Mikhlin Fourier multiplier theorem of Weis [18].

(iii) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (i). Assume (ii) holds. Let $(M_k)_{k \geq 0} \subset \mathcal{L}(X)$ be a bounded sequence and let $\phi \in \mathcal{D}(\mathbb{R})$ satisfying $\text{supp}(\phi) \subset [1, 2]$, $\sup_{x \in \mathbb{R}} |\phi(x)| = 1$ and $\phi(3/2) = 1$. Define $M \in C^\infty(\mathbb{R}; \mathcal{L}(X))$ by

$$M(x) = \begin{cases} 0 & \text{if } x \leq 1; \\ \phi(2^{-k}x)M_k & \text{if } 2^k \leq x < 2^{k+1} \text{ for some } k \geq 0. \end{cases}$$

Then $\sup_{x \in \mathbb{R}} \|M(x)\| = \sup_{k \geq 0} \|M_k\| < \infty$ and for $l \in \mathbb{N}$,

$$\begin{aligned} & \sup_{x \in \mathbb{R}} (1 + |x|)^l \|M^{(l)}(x)\| \\ & \leq 2^m \left(\sup_{x \in \mathbb{R}} |x|^l \phi^{(l)}(x) \sup_{k \geq 0} \|M_k\| + \sup_{x \in \mathbb{R}} |\phi^{(l)}(x)| \sup_{k \geq 0} 2^{-lk} \|M_k\| \right) < \infty. \end{aligned}$$

So M is an L^p -Fourier multiplier on $L^p(\mathbb{R}; X)$ by assumption. By [9, Proposition 1] this implies that the set $\{M(x) : x \in \mathbb{R}\}$ is R -bounded. In particular, the sequence $(M_k)_{k \geq 0}$ is R -bounded. We deduce from this that each bounded subset in $\mathcal{L}(X)$ is R -bounded, by [3, Proposition 1.13] X is isomorphic to a Hilbert space. \square

4. Multipliers on \mathbb{Z}

Let X, Y be Banach spaces and consider the Banach space $\ell^p(\mathbb{Z}; X)$ for $1 < p < \infty$. Let $\mathbb{T} = \{e^{it} : 0 \leq t < 2\pi\}$ be the torus. We consider the dense subspace P of $\ell^p(\mathbb{Z}; X)$ consisting of all elements having a finite support. Then for $f = (f_n)_{n \in \mathbb{Z}} \in P$, the Fourier transform of f is a function on $[-\pi, \pi]$ defined by $(\mathcal{F}f)(t) = \sum_{n \in \mathbb{Z}} f_n e^{int}$. Let $M \in L^\infty(-\pi, \pi; \mathcal{L}(X, Y))$. Then the function $M\mathcal{F}f$ is in $L^\infty(-\pi, \pi; Y)$, where \mathcal{F}^{-1} denotes the inverse Fourier transform. We deduce that that $Tf := \mathcal{F}^{-1}(M\mathcal{F}f) \in c_0(\mathbb{Z}; Y)$ makes sense. We say that M is an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$ if the mapping T can be extended to a bounded linear operator from $\ell^p(\mathbb{Z}; X)$ to $\ell^p(\mathbb{Z}; Y)$.

The classical Mikhlin Fourier multiplier theorem on $\ell^p(\mathbb{Z})$ has been extended to the operator-valued case by Blunck. Let $1 < p < \infty$, X be a UMD space, let $M \in C^1((-\pi, 0) \cup (0, \pi); \mathcal{L}(X))$ be such that both $\{M(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ and $\{(e^{it} - 1)(e^{it} + 1)M'(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ are R -bounded. Then M is an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$ [5]. In particular, each $M \in C^1([-\pi, 0) \cup (0, \pi]; \mathcal{L}(X))$ such that both $\{M(t) : t \neq 0\}$ and $\{tM'(t) : t \neq 0\}$ are R -bounded, defines an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$. Blunck has also established the R -boundedness of

L^p -Fourier multipliers on $\ell^p(\mathbb{Z}; X)$: when M is an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$, then $\{M(t) : t \text{ is a Lebesgue point of } M\}$ is R -bounded.

The following result shows that one cannot replace the R -boundedness in Blunck's result by the boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. As the proof is similar to that of Theorem 3, we omit it.

THEOREM 4. *Let X be a Banach space. Then the following assertions are equivalent:*

- (i) X is isomorphic to a Hilbert space.
- (ii) For some $1 < p < \infty$, each function $M \in C^\infty([-\pi, \pi]; \mathcal{L}(X))$ satisfying
 - (a) $\sup_{x \in [-\pi, \pi]} \|M(x)\| < \infty$,
 - (b) $\sup_{x \in [-\pi, \pi]} |x|^l \|M^{(l)}(x)\| < \infty$ for $l \in \mathbb{N}$,
 - (c) $M(x) = 0$ for $x \leq 0$,

defines an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$.

(iii) For all $1 < p < \infty$, each function $M \in C^1([-\pi, 0) \cup (0, \pi]; \mathcal{L}(X))$ satisfying

- (a) $\sup_{x \neq 0} \|M(x)\| < \infty$,
- (b) $\sup_{x \neq 0} \|xM'(x)\| < \infty$,

defines an L^p -Fourier multiplier on $\ell^p(\mathbb{Z}; X)$.

5. Concluding remarks

(a) One can actually show by using [3, Theorem 1.3] and the same argument as in the proof of Theorem 1, that when X and Y are UMD-spaces, then the assertions (ii) and (iii) in Theorem 1 are still equivalent for sequences in $\mathcal{L}(X, Y)$. Similarly, using [18, Theorem 3.4] (respectively, [5, Theorem 1.3]) one can show that when X and Y are UMD-spaces, the assertions (ii) and (iii) in Theorem 3 (respectively, Theorem 4) are still equivalent for functions with values in $\mathcal{L}(X, Y)$. Furthermore, these assertions are equivalent to X having cotype 2 and Y having type 2. This contains our Theorem 1, Theorem 3 and Theorem 4 by a result of Kwapien [14, pages 73–74], saying that a Banach space X is isomorphic to a Hilbert space if and only if X is of cotype 2 and of type 2.

(b) A restricted version of our results follows from the recent work of Kalton and Lancien on the maximal regularity problem [12]. In particular, the counterexample constructed in [12] can be used to show that the equivalences in Theorem 1 and Theorem 3 are true within the class of UMD Banach spaces which have an unconditional basis.

(c) In contrast to the L^p -spaces case, the situation for Hölder continuous function spaces is quite different. It has been shown that the operator-valued Marcinkiewicz

(respectively, Mihlin) Fourier multiplier theorem holds true on $C_{\text{per}}^\alpha([0, 2\pi]; X)$ (respectively, $C^\alpha(\mathbb{R}; X)$) for every Banach space X and $0 < \alpha < 1$ and for each sequence $M = (M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ satisfying a second order condition:

$$\sup_{k \in \mathbb{Z}} \|M_k\| + \sup_{k \in \mathbb{Z}} \|k(\Delta^1 M)(k)\| + \sup_{k \in \mathbb{Z}} \|k^2(\Delta^2 M)(k)\| < \infty$$

(respectively, each function $M \in C^2(\mathbb{R} \setminus \{0\}; \mathcal{L}(X))$ satisfying a second order condition: $\sup_{x \neq 0} \|M(x)\| + \sup_{x \neq 0} \|xM'(x)\| + \sup_{x \neq 0} \|x^2M''(x)\| < \infty$) (see Amann [1] and [2]). Here $C_{\text{per}}^\alpha([0, 2\pi]; X)$ denotes the space of all functions in $C^\alpha(\mathbb{R}, X)$ which are 2π -periodic. If the Banach space has a non-trivial type, then even the Marcinkiewicz condition of order 1 suffices (see [2] and [11]).

(d) Periodic L^p -Fourier multipliers (respectively, L^p -Fourier multipliers on $L^p(\mathbb{R}; X)$) of the form $M = (m_k I)_{k \in \mathbb{Z}}$, where $m_k \in \mathbb{C}$ for $k \in \mathbb{Z}$ (respectively, $M = f I$, where $f \in C^1(\mathbb{R} \setminus \{0\})$) on $L^p(0, 2\pi; X)$ (respectively, on $L^p(\mathbb{R}; X)$) have been studied by Zimmermann [19], where I denotes the identity operator on X . Actually Zimmermann's results follow from the operator-valued Marcinkiewicz (respectively, Mihlin) Fourier multiplier theorem established in [3] (respectively, in [18]) as each subset $M \subset \mathcal{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R -bounded whenever $\Omega \subset \mathbb{C}$ is bounded. Zimmermann's results together with a result of Burkholder [7] show that the scalar-valued Marcinkiewicz (respectively, Mihlin) Fourier multiplier theorem holds true for $L^p(0, 2\pi; X)$ (respectively, $L^p(\mathbb{R}; X)$) for some $1 < p < \infty$ if and only if X is a UMD space. A similar result characterizing UMD spaces via a scalar-valued Fourier multiplier theorem on $\ell^p(\mathbb{Z}; X)$ can be established based on results in [4].

(e) It is remarkable that in all three cases we consider here (Theorem 1, Theorem 3 and Theorem 4), the sequence $(M_k)_{k \in \mathbb{Z}}$ (or the function M) satisfying the Marcinkiewicz condition (of order l) without being a Fourier multiplier consists of operators of rank 1 (see [3, Proposition 1.13.]).

References

- [1] H. Amann, 'Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications', *Math. Nachr.* **186** (1997), 5–56.
- [2] W. Arendt, C. Batty and S. Bu, 'Fourier multipliers for Hölder continuous functions and maximal regularity', *Studia Math.* **160** (2004), 23–51.
- [3] W. Arendt and S. Bu, 'The operator-valued Marcinkiewicz multiplier theorem and maximal regularity', *Math. Z.* **240** (2002), 311–343.
- [4] E. Berkson and T. A. Gillespie, 'Spectral decompositions and harmonic analysis on UMD-spaces', *Studia Math.* **112** (1994), 13–49.
- [5] S. Blunck, 'Maximal regularity of discrete and continuous time evolution equations', *Studia Math.* **146** (2001), 157–163.

- [6] J. Bourgain, 'Vector-valued singular integrals and the H^1 -BMO duality', in: *Probability theory and harmonic analysis* (ed. D. Burkholder) (Dekker, New York, 1986) pp. 1–19.
- [7] D. Burkholder, 'A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions', in: *Proc. of Conf. on Harmonic Analysis in Honor of Antoni Zygmund, Chicago 1981* (Wadsworth Publishers, Belmont, CA, 1983) pp. 270–286.
- [8] Ph. Clément, B. de Pagter, F. A. Sukochev and M. Witvliet, 'Schauder decomposition and multiplier theorems', *Studia Math.* **138** (2000), 135–163.
- [9] Ph. Clément and J. Prüss, 'An operator-valued transference principle and maximal regularity on vector-valued L_p -spaces', in: *Evolution equations and their applications in physics and life sciences* (eds. Lumer and L. Weis), Lecture Notes in Pure and Appl. Math. 215 (Dekker, New York, 2000) pp. 67–87.
- [10] R. Denk, M. Hieber and J. Prüss, 'R-boundedness, Fourier multipliers and problems of elliptic and parabolic type', *Mem. Amer. Math. Soc.* 166 (Amer. Math. Soc., Providence, 2003), p. 788.
- [11] M. Girardi and L. Weis, 'Operator-valued Fourier multiplier theorems on Besov spaces', *Math. Nachr.* **251** (2003), 34–51.
- [12] N. J. Kalton and G. Lancien, 'A solution of the L^p -maximal regularity', *Math. Z.* **235** (2000), 559–568.
- [13] ———, ' L^p -maximal regularity on Banach spaces with a Schauder basis', *Arch. Math. (Basel)* **78** (2002), 397–408.
- [14] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II* (Springer, Berlin, 1979).
- [15] G. Pisier, *Les inégalités de Khintchine-Kahane d'après C. Borell*, Séminaire sur la géométrie des espaces de Banach 7 (Ecole Polytechnique Paris, 1977).
- [16] W. Rudin, *Fourier analysis on groups* (Wiley, New York, 1990).
- [17] Z. Štrkalj and L. Weis, 'On operator-valued Fourier multiplier theorems', preprint, 1999.
- [18] L. Weis, 'Operator-valued Fourier multiplier theorems and maximal L_p -regularity', *Math. Ann.* **319** (2001), 735–758.
- [19] F. Zimmermann, 'On vector-valued Fourier multiplier theorems', *Studia Math.* **93** (1989), 201–222.

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