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DERIVED *H*-MODULE ENDOMORPHISM RINGS

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Abstract. Let H be a Hopf algebra, A/B be an H-Galois extension. Let $D(A)$ and $D(B)$ be the derived categories of right A-modules and of right B-modules, respectively. An object $M^{\cdot} \in D(A)$ may be regarded as an object in $D(B)$ via the restriction functor. We discuss the relations of the derived endomorphism rings $E_A(M^{\cdot}) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{D(A)}(M^{\cdot}, M^{\cdot}[i])$ and $E_B(M^{\cdot}) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{D(B)}(M^{\cdot}, M^{\cdot}[i])$. If H is a finite-dimensional semi-simple Hopf algebra, then $E_A(M^{\cdot})$ is a graded sub-algebra of $E_B(M^{\cdot})$. In particular, if M is a usual A-module, a necessary and sufficient condition for $E_B(M)$ to be an H^*-Galois graded extension of $E_A(M)$ is obtained. As an application of the results, we show that the Koszul property is preserved under Hopf Galois graded extensions.

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1. Introduction. Let A/B be a Hopf Galois extension over a finite-dimensional Hopf algebra H. The motivation of this note is to try to understand how much homological properties, such as Koszul property and Gorenstein property, may be preserved under Hopf Galois extensions. To this aim, we need to discuss the derived endomorphism rings of certain modules. The endomorphism ring extension of an A-module has been studied by the second and third author in [**13**]. Let M be an A-module. It was proved that there is an isomorphism of algebras $\operatorname{End}_A(M \otimes_B A) \cong \operatorname{End}_B(M)\#H$. Also necessary and sufficient conditions for an endomorphism ring extension to be a Hopf Galois extension were obtained in [**13**]. We investigate whether these properties hold when the endomorphism rings are replaced by the derived endomorphism rings. To do this, we deal with the relevant derived functors and work with complexes or differential graded modules instead of usual modules over an algebra A.

Let H be a Hopf algebra with a bijective antipode, and let A be a right H-comodule algebra with co-invariant sub-algebra $B = A^{coH}$. We say that A/B is H-Galois if A is a Hopf Galois extension of B over H. Let $D(A)$ be the derived category of complexes of right A-modules, and $D(B)$ be the derived category of complexes of right B-modules. The restriction functor $D(A) \longrightarrow D(B)$ is an exact functor. For $M^{\cdot}, N^{\cdot} \in D(A)$, $M^{\cdot}$ and $N^{\cdot}$ may be regarded as objects in $D(B)$ via the restriction functor. As usual, write $\mathrm{Ext}_A^*(M^{\cdot}, N^{\cdot})$ for $\oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D(A)}(M^{\cdot}, N^{\cdot}[i])$. Endowed with the Yoneda product $\mathrm{Ext}_A^*(M^{\cdot}, M^{\cdot})$ becomes a graded algebra. Since A/B is H-Galois, there is a natural H-action on $\mathrm{Ext}_B^*(M^{\cdot}, M^{\cdot})$ so that it becomes a graded H-module algebra. In general, the graded algebra $\mathrm{Ext}_A^*(M^{\cdot}, M^{\cdot})$ can not be embedded into the graded algebra $\mathrm{Ext}_B^*(M^{\cdot}, M^{\cdot})$, which differs from the usual endomorphism rings of A-modules. However, if H is semi-simple, then $\mathrm{Ext}_A^*(M^{\cdot}, M^{\cdot})$ is the H-invariant sub-algebra of $\mathrm{Ext}_B^*(M^{\cdot}, M^{\cdot})$. The main result of this paper is the following one (Theorem 2.10).

THEOREM. *Let H be a finite-dimensional semi-simple Hopf algebra, and let A/B be an H-Galois extension. For a right A-module M,* $\mathrm{Ext}_B^*(M, M)$ *is a graded H^*-Galois extension of* $\mathrm{Ext}_A^*(M, M)$ *if and only if $M \otimes_B A \in$* $\mathrm{add}(M)$, *where* $\mathrm{add}(M)$ *is the category consisting of all the direct summands of finite direct sums of M.*

If M is a right (A, H)-Hopf module, then $\mathrm{Ext}_B^*(M, M)$ is not merely an H^*-Galois extension of $\mathrm{Ext}_A^*(M, M)$. In fact, we have the following result (Corollary 2.3), which can be regarded as an extension of [**11**, Theorem 3.2].

THEOREM. *Let H be a finite-dimensional semi-simple and co-semi-simple Hopf algebra. If M is a right (A, H)-Hopf module, then*

$$\mathrm{Ext}_B^*(M, M) \cong \mathrm{Ext}_A^*(M, M)\#H^*.$$

From this theorem, one can easily show that the Artin-Schelter Gorenstein property is preserved under Hopf Galois extensions when the Hopf algebra is semi-simple and co-semi-simple. With a bit more effort we can also show that the Koszul property is preserved under Hopf Galois graded extensions (Theorem 2.11).

THEOREM. *Let H be a finite-dimensional semi-simple and co-semi-simple Hopf algebra, let $A = \oplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A_i is finite-dimensional for all $i \geq 0$, and let $B = A^{coH}$. Assume that A/B is an H-Galois graded extension. If B is an N-Koszul algebra, then A is an N-Koszul algebra.*

In particular, for a cocommutative Hopf algebra A, the graded algebra $gr(A)$ associated to the coradical filtration is a Koszul algebra if A has finitely many group-like elements and the dimension of $P(A)$, the vector space of primitive elements, is finite.

The main tools used in this note are differential graded algebras and differential graded modules. Let us recall some notations and properties of differential graded (dg, for short) algebras and dg modules which will be used later. By a dg algebra we mean a graded algebra $R = \oplus_{n\in\mathbb{Z}} R^n$ with a differential d of degree 1 which is also a derivation of R. Similarly we have dg modules. We sometimes view a usual algebra A as a dg algebra concentrated in degree 0 with a trivial differential. Then a complex of A-modules can be regarded as a dg A-module. Let R be a dg algebra, and let X_R and Y_R be right dg R-modules. Write $\mathrm{Hom}_R(X, Y)$ for $\oplus_{i\in\mathbb{Z}} \mathrm{Hom}_R^i(X, Y)$, where $\mathrm{Hom}_R^i(X, Y)$ is the set of all graded R-module morphisms of degree i. $\mathrm{Hom}_R(X, Y)$ is a complex of vector space. The differential on $\mathrm{Hom}_R(X, Y)$ is given by $\delta(f) = d_Y \circ f - (-1)^{|f|} f \circ d_X$, where $|f|$ is the degree of f. Also $\mathrm{Hom}_R(X, X)$ is a dg algebra and $\mathrm{Hom}_R(X, Y)$ is a right dg $\mathrm{Hom}_R(X, X)$-module. Let H be a Hopf algebra. We say that R is a *dg left H-module algebra* if R is a left H-module algebra and the H-action respects the grading of R and is compatible with the differential of R. In this case, the smash product $R\#H$ is a dg algebra and the cohomology algebra $H^{\cdot}(R\#H) \cong H^{\cdot}(R)\#H$. Notation and terminology used in this note is the same as in [**1**, **4**, **8**]. For example, we use $\mathrm{RHom}_R(-, -)$ to denote the right derived functor of $\mathrm{Hom}_R(-, -)$, and $- \otimes^L_R -$ to denote the left derived functor of the tensor functor $- \otimes_R -$ over a dg algebra R.

We work over a fixed field k. All the algebras and modules are defined over k. Unadorned $\otimes$ means $\otimes_k$, and Hom means Hom_k.

2. Derived endomorphism rings. Throughout H will be always a Hopf algebra with a bijective antipode. Let A/B be an H-Galois extension with the canonical map $\beta : A \otimes_B A \longrightarrow A \otimes H$. For any $h \in H$, we write $\sum X_i^h \otimes Y_i^h$ for $\beta^{-1}(1 \otimes h)$. For right A-modules M and N, there is a natural left H-module action on $\mathrm{Hom}_B(M, N)$:

For $h \in H$, $m \in M$ and $f \in \mathrm{Hom}_B(M, N)$,

$$(h \cdot f)(m) = \sum f\left(mX_i^{Sh}\right) Y_i^{Sh}.$$

Similarly, if $M^{\cdot}$ and $N^{\cdot}$ are complexes of right A-modules, then $\mathrm{Hom}_B(M^{\cdot}, N^{\cdot})$ is a complex of left H-modules. Hence for any $M^{\cdot} \in D^-(A)$ and $N^{\cdot} \in D(A)$, $\mathrm{RHom}_B(M^{\cdot}, N^{\cdot})$ is a complex of left H-modules.

If M is a right A-module, then there is an isomorphism of right B-modules and right H-comodules:

$$M \otimes_B A \longrightarrow M \otimes H, \qquad m \otimes a \mapsto \sum ma_0 \otimes a_1,$$

with the inverse given by

$$M \otimes H \longrightarrow M \otimes_B A, \qquad m \otimes h \mapsto \sum mX_i^h \otimes Y_i^h.$$

These isomorphisms may be generalized to complexes. If $M^{\cdot}$ is a complex of A-modules, then we have an isomorphism of complexes: $M^{\cdot} \otimes_B A \longrightarrow M^{\cdot} \otimes H$.

THEOREM 2.1. *Let A/B be an H-Galois extension, and $M^{\cdot}$ an object in $D^-(A)$. If H is finite-dimensional, then there is an isomorphism of graded algebras*

$$\mathrm{Ext}^*_A(M^{\cdot} \otimes^L_B A, M^{\cdot} \otimes^L_B A) \longrightarrow \mathrm{Ext}^*_B(M^{\cdot}, M^{\cdot})\#H.$$

Proof. Let $P^{\cdot}$ be a bounded above complex of projective A-modules which is quasi-isomorphic to the complex $M^{\cdot}$. Then $M^{\cdot} \otimes^L_B A \cong P^{\cdot} \otimes_B A$. Of course $P^{\cdot} \otimes_B A$ is homotopically projective (see [**8**, Chapter 8]). Hence

$$\begin{aligned}
\mathrm{RHom}_A(M^{\cdot} \otimes^L_B A, M^{\cdot} \otimes^L_B A) &\cong \mathrm{RHom}_A(P^{\cdot} \otimes_B A, P^{\cdot} \otimes_B A)\\
&= \mathrm{Hom}_A(P^{\cdot} \otimes_B A, P^{\cdot} \otimes_B A)\\
&\cong \mathrm{Hom}_B(P^{\cdot}, P^{\cdot} \otimes_B A)\\
&\cong \mathrm{Hom}_B(P^{\cdot}, P^{\cdot} \otimes H)\\
&\cong \mathrm{Hom}_B(P^{\cdot}, P^{\cdot}) \otimes H\\
&= \mathrm{RHom}_B(M^{\cdot}, M^{\cdot}) \otimes H.
\end{aligned}$$

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Therefore, as graded spaces, we have $\mathrm{Ext}_A^*(M^\cdot \otimes_B^L A, M^\cdot \otimes_B^L A) \cong \mathrm{Ext}_B^*(M^\cdot, M^\cdot)\#H$. It remains to show that this isomorphism is an isomorphism of graded algebras.

From the above isomorphisms, we get an isomorphism of complexes

$$\varphi : \mathrm{Hom}_B(P^\cdot, P^\cdot) \otimes H \longrightarrow \mathrm{Hom}_A(P^\cdot \otimes_B A, P^\cdot \otimes_B A), \tag{1}$$

such that for $f \in \mathrm{Hom}_B(P^\cdot, P^\cdot)$, $h \in H$, $p \in P$ and $a \in A$,

$$\varphi(f \otimes h)(p \otimes_B a) = \sum f(p)X_i^h \otimes_B Y_i^h a.$$

We have to show that φ is an isomorphism of dg algebras from $\mathrm{Hom}_B(P^\cdot, P^\cdot)\#H$ to $\mathrm{Hom}_A(P^\cdot \otimes_B A, P^\cdot \otimes_B A)$. Since φ is compatible with the differentials, it suffices to show that it is an algebra morphism. The proof is essentially the same as that of [**13**, Theorem 2.3], so we omit it. Now by taking the cohomology algebra of the dg algebras $\mathrm{Hom}_B(P^\cdot, P^\cdot)\#H$ and $\mathrm{Hom}_A(P^\cdot \otimes_B A, P^\cdot \otimes_B A)$ we get the desired result. □

Let $M^\cdot$ be a complex of A-modules. We have $\mathrm{Hom}_A(M^\cdot, M^\cdot) \cong \mathrm{Hom}_B(M^\cdot, M^\cdot)^H$ as dg algebras, where $\mathrm{Hom}_B(M^\cdot, M^\cdot)^H$ is the H-invariant dg sub-algebra of $\mathrm{Hom}_B(M^\cdot, M^\cdot)$. If H is a finite-dimensional semi-simple Hopf algebra, then it is not difficult to see that $\mathrm{Ext}_A^*(M^\cdot, M^\cdot) \cong \mathrm{Ext}_B^*(M^\cdot, M^\cdot)^H$ as graded algebras. In other words, the graded algebra $\mathrm{Ext}_B^*(M^\cdot, M^\cdot)$ is an H^*-extension of $\mathrm{Ext}_A^*(M^\cdot, M^\cdot)$. We ask when this H^*-extension is an H^*-Galois extension.

Let $M^\cdot$ be a complex of right (A, H)-Hopf modules. Then $\mathrm{Hom}_A(M^\cdot, M^\cdot)$ is a dg left H^*-module algebra with the left H^*-action given by, for $\alpha \in H^*$ and $f \in \mathrm{Hom}_A(M^\cdot, M^\cdot)$,

$$(\alpha \cdot f)(m) = \sum \alpha_{(1)} f(S(\alpha_{(2)})m),$$

which induces a left H^*-module algebra structure on $\mathrm{Ext}_A^*(M^\cdot, M^\cdot)$.

THEOREM 2.2. *Let H be a finite-dimensional semi-simple and co-semi-simple Hopf algebra. Let A/B be an H-Galois extension. If $M^\cdot$ is a bounded above complex of right (A, H)-Hopf modules, then there is an isomorphism of graded algebras*

$$\mathrm{Ext}_B^*(M^\cdot, M^\cdot) \cong \mathrm{Ext}_A^*(M^\cdot, M^\cdot)\#H^*.$$

Proof. The complex $M^\cdot$ of right (A, H)-Hopf modules may be regarded as a complex of right $A\#H^*$-modules in a natural way. By [**2**, Theorem 2.2] the functor $\mathrm{Hom}_B(A, -) : \text{Mod-}B \longrightarrow \text{Mod-}A\#H^*$ is an equivalence of abelian categories with inverse functor $- \otimes_{A\#H^*} A$. Hence we have

$$\begin{aligned} &\mathrm{Ext}_{A\#H^*}^*(M^\cdot \otimes H^*, M^\cdot \otimes H^*) \\ &\cong \mathrm{Ext}_B^*((M^\cdot \otimes H^*) \otimes_{A\#H^*} A, (M^\cdot \otimes H^*) \otimes_{A\#H^*} A) \\ &\cong \mathrm{Ext}_B^*(M^\cdot, M^\cdot). \end{aligned}$$

The last isomorphism holds because $(M^\cdot \otimes H^*) \otimes_{A\#H^*} A \cong (M^\cdot \otimes_A (A\#H^*)) \otimes_{A\#H^*} A \cong M^\cdot$ as a complex of right B-modules. On the other hand, since as complexes of right $A\#H^*$-modules

$$M^\cdot \otimes H^* \cong M^\cdot \otimes_A^L (A\#H^*),$$

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we have

$$\begin{aligned}&\mathrm{Ext}^*_{A\#H^*}(M^\cdot \otimes H^*, M^\cdot \otimes H^*)\\ \cong\ &\mathrm{Ext}^*_{A\#H^*}(M^\cdot \otimes^L_A (A\#H^*), M^\cdot \otimes^L_A (A\#H^*))\\ \cong\ &\mathrm{Ext}^*_A(M^\cdot, M^\cdot)\#H^*.\end{aligned}$$

The last isomorphism follows from Theorem 2.1 since $A\#H^*$ is an H^*-Galois extension of A. Now we obtain the desired isomorphism

$$\mathrm{Ext}^*_B(M^\cdot, M^\cdot) \cong \mathrm{Ext}^*_A(M^\cdot, M^\cdot)\#H^*.$$

□

From the above theorem, we obtain the following corollary which may be regarded as a natural generalization of [**11**, Theorem 3.2].

COROLLARY 2.3. *Let H be a finite-dimensional semi-simple and co-semi-simple Hopf algebra, and let M be a right (A, H)-Hopf module. Then*

$$\mathrm{Ext}^*_B(M, M) \cong \mathrm{Ext}^*_A(M, M)\#H^*.$$

For an A-module M, the algebra $\mathrm{Ext}^*_B(M^\cdot, M^\cdot)$ may not be a smash product of $\mathrm{Ext}^*_A(M^\cdot, M^\cdot)$ with H^*. However, if the Hopf algebra H is unimodular, then we will see that in certain cases the graded algebra $\mathrm{Ext}^*_B(M^\cdot, M^\cdot)$ is an H^*-Galois graded extension of $\mathrm{Ext}^*_A(M^\cdot, M^\cdot)$. Firstly, we make explicit the exact meaning of a Galois graded extension.

Let E be a $\mathbb{Z}$-graded algebra, H a Hopf algebra. We say that E is a graded right H-comodule algebra if E is a right H-comodule algebra and the H-coaction on E respects the grading of E, that is; for $x \in E_n$, $\rho(x) \in E_n \otimes H$. Let $D = E^{coH}$. Then D is a graded sub-algebra of E. E/D is called an *H-Galois graded extension* if the canonical map [**10**]

$$E \otimes_D E \longrightarrow E \otimes H, \quad x \otimes x' \mapsto \sum xx'_{(0)} \otimes x'_{(1)}$$

is bijective.

We have the following observation.

LEMMA 2.4. *Let E/D be an H-Galois graded extension. If $E = \oplus_{n\geq 0} E_n$ is positively graded, then E_0/D_0 is an H-Galois extension.*

For graded right E-modules V and W, let $\underline{\mathrm{Hom}}_E(V, W) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_E(V, W)$, where $\mathrm{Hom}^i_E(V, W)$ is the set of all the graded E-module morphisms of degree i, and $\underline{\mathrm{End}}(V_E)$ denotes the sum of all the graded endomorphisms. Then $\underline{\mathrm{Hom}}_E(V, W)$ is a graded vector space and $\underline{\mathrm{End}}(V_E)$ is a graded algebra. We use $\underline{\mathrm{Ext}}^*_E(-, -)$ to denote the derived functor of $\underline{\mathrm{Hom}}_E(-, -)$. Note that $\underline{\mathrm{Ext}}^n_E(V, W)$ is a graded vector space for all $n \in \mathbb{Z}$, where the grading is induced from the gradings of V and W. Hence $\underline{\mathrm{Ext}}^*_E(V, W)$ is a bigraded vector space. To avoid possible confusions, following [**6**], we say an element x in $\underline{\mathrm{Ext}}^n_E(V, W)$ has *ext-degree n* when we ignore the grading induced from the gradings of V and W.

For H-Galois graded extensions, we have the following result which corresponds to [**2**, Theorem 1.2]. The proof is exactly the same as that of [**2**, Theorem 1.2] except that we should keep in mind that everything needs to preserve the gradings.

LEMMA 2.5. *Let H be a finite-dimensional Hopf algebra. The following are equivalent*:

(i) *E/D is an H-Galois graded extension.*

(ii) (a) *The natural map $E\#H^* \longrightarrow \underline{\mathrm{End}}(E_D)$ is an isomorphism of graded algebras.*
(b) *E_D is a finitely generated projective graded D-module.*

(iii) *As a left graded $E\#H^*$-module, E is a generator of the category of graded $E\#H^*$-modules.*

REMARK 2.6. If the H-Galois extension A/B in Theorem 2.1 (resp. 2.2) is H-Galois graded extension and $M^\cdot$ is a bounded above complex of graded A-modules (resp. of graded right (A, H)-Hopf modules), then the result still holds when the functor Ext is replaced by $\underline{\mathrm{Ext}}$. Moreover, one can check that the isomorphisms in the theorems are bigraded morphisms. In particular, the isomorphism in Corollary 2.3 has the following form, which we will use later,

$$\underline{\mathrm{Ext}}_B^*(M, M) \cong \underline{\mathrm{Ext}}_A^*(M, M)\#H^*,$$

where the bigrading of the right hand is induced from the bigrading of $\underline{\mathrm{Ext}}_A^*(M, M)$.

For later use, we need to generalize some results in [**3**, **13**].

LEMMA 2.7. *Let H be a finite-dimensional Hopf algebra, and let A/B be an H-Galois extension. If $M^\cdot \in D^-(A)$ and $N^\cdot$ is a complex of right B-modules, then there is an isomorphism in $D(k)$*

$$\mathrm{RHom}_B(M^\cdot, N^\cdot) \cong \mathrm{RHom}_A(M^\cdot, N^\cdot \otimes_B^L A).$$

Proof. Replace $M^\cdot$ and $N^\cdot$ by complexes $P^\cdot$ and $Q^\cdot$ of projective A-modules respectively. Since A/B is an H-Galois extension, A_B is a projective module. Hence $P^\cdot$ and $Q^\cdot$ are complexes of projective B-modules. We have $\mathrm{RHom}_B(M^\cdot, N^\cdot) = \mathrm{Hom}_B(P^\cdot, Q^\cdot)$, and $\mathrm{RHom}_A(M^\cdot, N^\cdot \otimes_B^L A) = \mathrm{Hom}_A(P^\cdot, Q^\cdot \otimes_B A)$.
Following [**3**], we define a map

$$\xi : \mathrm{Hom}_B(P^\cdot, Q^\cdot) \longrightarrow \mathrm{Hom}_A(P^\cdot, Q^\cdot \otimes_B A) \tag{2}$$

by $\xi(f)(p) = \sum f(pX_i^t) \otimes_B Y_i^t$ where t is a nonzero right integral in H. We claim that ξ is an isomorphism of complex of vector spaces. By essentially the same calculations as in the proof of [**3**, Theorem 5], one sees that ξ is bijective. What left to show is that ξ is compatible with the differentials. Now for $f \in \mathrm{Hom}_B(P^\cdot, Q^\cdot)$ and $p \in P^\cdot$, we have

$$\begin{aligned}
\xi(d(f))(p) &= \xi(d_{Q^\cdot} \circ f - (-1)^{|f|} f \circ d_{P^\cdot})(p)\\
&= \sum d_{Q^\cdot}(f(pX_i^t)) \otimes_B Y_i^t - \sum (-1)^{|f|} f(d_{P^\cdot}(pX_i^t)) \otimes_B Y_i^t\\
&= d_{Q^\cdot \otimes_B A}\left(\sum f(pX_i^t) \otimes_B Y_i^t\right) - (-1)^{|f|} \sum f(d_{P^\cdot}(p)X_i^t) \otimes_B Y_i^t\\
&= d_{Q^\cdot \otimes_B A} \circ \xi(f)(p) - (-1)^{|f|} \xi(f) \circ d_{P^\cdot}(p)\\
&= d(\xi(f))(p).
\end{aligned}$$

Hence we get the desired result. □

Let H be a finite-dimensional semi-simple Hopf algebra. By Theorem 2.1, we have $K = \mathrm{Ext}_A^*(M^\cdot \otimes_B^L A, M^\cdot \otimes_B^L A) \cong \mathrm{Ext}_B^*(M^\cdot, M^\cdot)\#H$. Write $E = \mathrm{Ext}_B^*(M^\cdot, M^\cdot)$ and $D = \mathrm{Ext}_A^*(M^\cdot, M^\cdot)$. Then E is a graded left K-module and a graded right D-module.

Since $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$ is a dg H-module algebra, it is a left dg $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})\#H$-module. Let $U = \mathrm{Hom}_B(P^{\cdot}, P^{\cdot})^H = \mathrm{Hom}_A(P^{\cdot}, P^{\cdot})$ be the fixed dg sub-algebra of $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$. Then $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$ is a dg $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})\#H$-$U$-bimodule. By taking the cohomology, we obtain that $E = H^{\cdot}\,\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$ is a graded left K- and right D-bimodule since $K \cong E\#H$ and $D \cong H^{\cdot}U$.

Proposition 2.8. *With notation as above, there is a graded bimodule isomorphism*

$$ {}_K E_D \cong {}_K\,\mathrm{Ext}^*_B(M^{\cdot}, M^{\cdot} \otimes^L_B A)_D. $$

Proof. By Lemma 2.7, $E \cong \mathrm{Ext}^*_B(M^{\cdot}, M^{\cdot} \otimes^L_B A)$ as graded spaces. It suffices to prove that the isomorphism is a bimodule isomorphism. Going back to the morphism defined by (2) in the proof of Lemma 2.7, one sees that ξ is a dg right $\mathrm{Hom}_A(P^{\cdot}, P^{\cdot})$-module morphism. We show that $\xi : \mathrm{Hom}_B(P^{\cdot}, P^{\cdot}) \longrightarrow \mathrm{Hom}_A(P^{\cdot}, P^{\cdot} \otimes_B A)$ is a left dg $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})\#H$ module morphism. Note that the left dg $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})\#H$ module action is given by the pullback of dg algebra morphism φ defined by (1) in the proof of Theorem 2.1. Now for $g, f \in \mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$, $h \in H$ and $p \in P^{\cdot}$, we have

$$ \begin{aligned} \xi((g\#h)\cdot f)(p) &= \xi(g \circ (h \cdot f))(p) \\ &= \sum g \circ (h \cdot f)(pX_i^t) \otimes_B Y_i^t \\ &= \sum g(f(pX_i^t X_j^{Sh})Y_j^{Sh}) \otimes_B Y_i^t. \end{aligned} $$

Since A/B is an H-Galois extension and H is semi-simple, it is not hard to check (see [**13**]) $\sum X_i^t X_j^{Sh} \otimes_B Y_j^{Sh} \otimes_B Y_i^t = \sum X_i^t \otimes_B X_j^h \otimes_B Y_j^h Y_i^t$ in $A \otimes_B A \otimes_B A$. Then we obtain

$$ \begin{aligned} \xi((g\#h)\cdot f)(p) &= \sum g(f(pX_i^t)X_j^h)) \otimes_B Y_j^h Y_i^t \\ &= \varphi(g\#h) \circ \xi(f)(p) \\ &= (g\#h)\cdot \xi(f)(p). \end{aligned} $$

Therefore ξ is a dg bimodule isomorphism. By taking the cohomologies of $\mathrm{Hom}_B(P^{\cdot}, P^{\cdot})$ and $\mathrm{Hom}_A(P^{\cdot}, P^{\cdot} \otimes_B A)$ respectively we arrive at the desired graded bimodule isomorphism. □

Let $\mathcal{T}$ and $\mathcal{D}$ be triangulated categories. Let $F : \mathcal{T} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{T}$ be a pair of adjoint exact functors with F left adjoint to G. Recall that the Auslander class is the sub-category of $\mathcal{T}$:

$$ \mathcal{A}(\mathcal{T}) = \{X \in \mathcal{T} \mid \text{the adjunction map } X \longrightarrow GFX \text{ is isomorphic}\}, $$

and the Bass class is the sub-category of $\mathcal{D}$:

$$ \mathcal{B}(\mathcal{D}) = \{Y \in \mathcal{D} \mid \text{the adjunction map } FGY \longrightarrow Y \text{ is isomorphic}\}. $$

It is well known that both the Auslander class and Bass class are thick triangulated sub-categories of the respective triangulated categories, and the adjoint functors F and G induce a pair of inverse equivalences between triangulated categories $\mathcal{A}(\mathcal{T})$ and $\mathcal{B}(\mathcal{D})$.

The following lemma is straightforward.

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LEMMA 2.9. *For $M^{\cdot} \in D^{-}(A)$, if $M^{\cdot} \otimes^{L}_{B} A$ is a direct summand of finite direct sum of $M^{\cdot}$, then $\mathrm{Ext}^{*}_{A}(M^{\cdot}, M^{\cdot} \otimes^{L}_{B} A)$ is a finitely generated graded projective $\mathrm{Ext}^{*}_{A}(M^{\cdot}, M^{\cdot})$-module.*

Let $P^{\cdot} \in D^{-}(A)$ be a complex of projective A-modules. We regard A as a dg algebra concentrated in degree 0. Then $P^{\cdot}$ is a dg right A-module. Let $R = \mathrm{Hom}_{A}(P^{\cdot}, P^{\cdot})$ be the dg algebra of endomorphisms. Then $P^{\cdot}$ is a left dg R-module, and it is also a dg R-A-bimodule. Therefore we have an exact functor

$$\mathrm{RHom}_{A}(P^{\cdot}, -) : D(A) \longrightarrow D_{dg}(R),$$

where $D_{dg}(R)$ is the derived category of right dg R-modules. This functor is naturally left adjoint to the functor

$$- \otimes^{L}_{R} P^{\cdot} : D_{dg}(R) \longrightarrow D(A).$$

Clearly the dg A-module $P^{\cdot}$ lies in the Auslander class. Since the Auslander class is a thick triangulated sub-category, all the direct summands of finite direct sums of copies of $P^{\cdot}$ belong to the Auslander class.

Let M be a right A-module. Denote by $\mathrm{add}(M)$ the category of A-modules isomorphic to direct summands of finite sums of M. Recall that $E = \mathrm{Ext}^{*}_{B}(M, M)$, $D = \mathrm{Ext}^{*}_{A}(M, M)$ and $K = E\#H$.

THEOREM 2.10. *Let H be a finite-dimensional semi-simple Hopf algebra, and A/B be an H-Galois extension. For a right A-module M, the following are equivalent.*

(i) $M \otimes_{B} A \in \mathrm{add}(M)$.

(ii) $\mathrm{End}_{B}(M)/\,\mathrm{End}_{A}(M)$ *is an H^{*}-Galois extension.*

(iii) *E/D is an H^{*}-Galois graded extension.*

Moreover, if $M \otimes_{B} A \in \mathrm{add}(M)$, then D and $E\#H$ are graded Morita equivalent.

Proof. The equivalence between (i) and (ii) was proved in [**13**, Theorem 2.4]. It remains to show that (i) and (iii) are equivalent.

Assume that E/D is an H^{*}-Galois graded extension. Since $E = \mathrm{Ext}^{*}_{B}(M, M)$ is positively graded, by Lemma 2.4, $\mathrm{Ext}^{0}_{B}(M, M)$ is an H^{*}-Galois extension of $\mathrm{Ext}^{0}_{B}(M, M)$, that is; $\mathrm{End}_{B}(M)/\mathrm{End}_{A}(M)$ is an H^{*}-Galois extension. Hence $M \otimes_{B} A \in \mathrm{add}(M)$.

Conversely, assume $M \otimes_{B} A \in \mathrm{add}(M)$. By Proposition 2.8, ${}_{K}E_{D} \cong {}_{K}\,\mathrm{Ext}^{*}_{B}(M, M \otimes^{L}_{B} A)_{D}$ as graded K-D-bimodules. It follows from Lemma 2.9 that $\mathrm{Ext}^{*}_{B}(M, M \otimes^{L}_{B} A)$, and hence E, is a finitely generated graded right D-module.

Now we prove that $\underline{\mathrm{End}}(E_{D})$ is isomorphic to the graded algebra $K = E\#H$. Let $P^{\cdot} \in D^{-}(A)$ be a projective resolution of M_{A}. Viewing A as a dg algebra concentrated in degree 0, we see that $P^{\cdot}$ is a dg R-A-bimodule, where $R = \mathrm{Hom}_{A}(P^{\cdot}, P^{\cdot})$. As mentioned before, $(\mathrm{RHom}_{A}(P^{\cdot}, -), - \otimes^{L}_{R} P^{\cdot})$ is a pair of adjoint functors; and $M \cong P^{\cdot}$ lies in the Aulander class of the adjoint pair. Therefore $\mathrm{add}(M)$ is contained in the Auslander class. We have the following graded algebra isomorphism:

$$\begin{aligned}&\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D_{dg}(R)}(\mathrm{RHom}_{A}(P^{\cdot}, M \otimes_{B} A), \mathrm{RHom}_{A}(P^{\cdot}, M \otimes_{B} A)[i])\\ \cong &\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D(A)}(M \otimes_{B} A, M \otimes_{B} A[i]).\end{aligned}$$

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Since $P^{\cdot}$ is a projective resolution, $\mathrm{RHom}_A(P^{\cdot}, M \otimes_B A) \cong \mathrm{Hom}_A(P^{\cdot}, P^{\cdot} \otimes_B A)$. Since $M \otimes_B A \in \mathrm{add}(M)$, $P^{\cdot} \otimes_B A$ is homotopically equivalent to $Q^{\cdot}$. Thus there exist an integer n and some complex $\bar{Q}^{\cdot}$ such that $Q^{\cdot} \oplus \bar{Q}^{\cdot}$ is homotopically equivalent to $(P^{\cdot})^{\oplus n}$. It follows that $\mathrm{Hom}_A(P^{\cdot}, P^{\cdot} \otimes_B A) \cong \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})$. Moreover, $\mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})$ is a homotopically projective dg R-module. Now we have the following graded algebra isomorphisms:

$$
\begin{aligned}
&\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D_{dg}(R)}(\mathrm{RHom}_A(P^{\cdot}, M \otimes_B A), \mathrm{RHom}_A(P^{\cdot}, M \otimes_B A)[i]) \\
&\cong H^{\cdot}\, \mathrm{RHom}_R(\mathrm{RHom}_A(P^{\cdot}, M \otimes_B A), \mathrm{RHom}_A(P^{\cdot}, M \otimes_B A)) \\
&\cong H^{\cdot}\, \mathrm{Hom}_R(\mathrm{Hom}_A(P^{\cdot}, Q^{\cdot}), \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})) \\
&\overset{(a)}{\cong} \underline{\mathrm{Hom}}_{H^{\cdot}R}(H^{\cdot}\, \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot}), H^{\cdot}\, \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})) \\
&= \underline{\mathrm{Hom}}_D(\mathrm{Ext}_A^*(M, M \otimes_B A), \mathrm{Ext}_A^*(M, M \otimes_B A)) \\
&\cong \underline{\mathrm{End}}(E_D).
\end{aligned}
$$

The isomorphism (a) holds because $\mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})$ is a direct summand (in the homotopy category) of a free dg R-module. More precisely, in the homotopy category of right dg R-modules, we have $\mathrm{Hom}_A(P^{\cdot}, Q^{\cdot}) \oplus \mathrm{Hom}_A(P^{\cdot}, \bar{Q}^{\cdot}) \cong \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot} \oplus \bar{Q}^{\cdot}) \cong \mathrm{Hom}_A(P^{\cdot}, P^{\cdot \oplus n}) \cong R^{\oplus n}$. Let $X = \mathrm{Hom}_A(P^{\cdot}, Q^{\cdot})$ and $Y = \mathrm{Hom}_A(P^{\cdot}, \bar{Q}^{\cdot})$. We have the following commutative diagram:

$$
\begin{tikzcd}
H^{\cdot}\, \mathrm{Hom}_R(X \oplus Y, X) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{H^{\cdot}R}(H^{\cdot}(X \oplus Y), H^{\cdot}X) \arrow[d, "\cong"] \\
H^{\cdot}\, \mathrm{Hom}_R(R^{\oplus n}, X) \arrow[r, "\cong"] & \mathrm{Hom}_{H^{\cdot}R}(H^{\cdot}(R^{\oplus n}), H^{\cdot}X).
\end{tikzcd}
$$

Since all the morphisms in the diagram are natural, we obtain a natural isomorphism $H^{\cdot}\, \mathrm{Hom}_R(X \oplus Y, X) \cong \mathrm{Hom}_{H^{\cdot}R}(H^{\cdot}(X \oplus Y), H^{\cdot}X)$, which implies the isomorphism (a): $H^{\cdot}\, \mathrm{Hom}_R(X, X) \cong \mathrm{Hom}_{H^{\cdot}R}(H^{\cdot}X, H^{\cdot}X)$.

On the other hand, we have

$$
\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D(A)}(M \otimes_B A, M \otimes_B A[i]) \cong \mathrm{Ext}_A^*(M \otimes_B A, M \otimes_B A) \cong E\#H.
$$

Thus $\underline{\mathrm{End}}(E_D) \cong E\#H$ as graded algebras. Now applying Lemma 2.5, we obtain that E/D is an H^*-Galois extension.

Assume now $M \otimes_B A \in \mathrm{add}(M)$. Then E/D is an H^*-Galois extension. By Lemma 2.5(iii), ${}_{E\#H}E$ is a graded generator. Since H is semi-simple, E is a graded projective $E\#H$-module. Therefore, E is finitely generated projective generator of the category of graded $E\#H$-modules. Thus $E\#H$ is graded Morita equivalent to D (see **[5]**). □

We next want to show that Galois extensions over a semi-simple and co-semi-simple Hopf algebra preserve Koszul property. Let us recall the definition of Koszul algebras. Let $A = \oplus_{n\geq 0} A_n$ be a graded algebra such that A_i is finite-dimensional for all $i \geq 0$, A_0 is semi-simple and $A_i A_j = A_{i+j}$ for all $i, j \geq 0$. Let $N \geq 2$ be an integer. A graded A-module M is called an *N-Koszul module* (see, **[6]** for instance) if M has a graded projective resolution:

$$
\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0
$$

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such that the graded projective module P^{-n} ($n > 0$) is generated in degree $\frac{n}{2}N$ if n is even, or $\frac{n-1}{2}N + 1$ if n is odd. If the trivial graded right A-module A_0 (via the projection) is N-Koszul, then A is called an *N-Koszul algebra*.

THEOREM 2.11. *Let H be a finite-dimensional semi-simple and co-semi-simple Hopf algebra, let $A = \oplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A_i is finite-dimensional for all $i \geq 0$, and let $B = A^{coH}$. Assume that A/B is an H-Galois graded extension. If B is an N-Koszul algebra, then A is an N-Koszul algebra.*

Proof. By the assumption, B_0 is a finite-dimensional semi-simple algebra. Since A/B is an H-Galois graded extension, A_0/B_0 is an H-Galois extension by Lemma 2.4. Now that H is semi-simple implies that $A_0\#H$ is semi-simple since it is Morita equivalent with B_0. Since H is co-semi-simple, $(A_0\#H)\#H^*$ is semi-simple by Maschke's theorem. It follows from the Morita equivalence between A_0 and $(A_0\#H)\#H^*$ that A_0 is a semi-simple algebra. As a right B_0-module, $A_0 = B_0 \oplus S$ for some finite-dimensional B_0-module S. By Remark 2.6, we have an isomorphism of bigraded algebras

$$\underline{\mathrm{Ext}}^*_B(A_0, A_0) \cong \underline{\mathrm{Ext}}^*_A(A_0, A_0)\#H. \tag{3}$$

Since B is N-Koszul, the graded space $\underline{\mathrm{Ext}}^1_B(B_0, B_0)$ is concentrated in degree 1, and $\underline{\mathrm{Ext}}^2_B(B_0, B_0)$ is concentrated in degree N (see [**6**]). Therefore $\underline{\mathrm{Ext}}^1_B(B_0, S)$, ($\underline{\mathrm{Ext}}^1_B(S, B_0)$ and $\underline{\mathrm{Ext}}^1_B(S, S)$, respectively) is concentrated in degree 1, and $\underline{\mathrm{Ext}}^2_B(B_0, S)$, ($\underline{\mathrm{Ext}}^2_B(S, B_0)$ and $\underline{\mathrm{Ext}}^2_B(S, S)$ respectively) is concentrated in degree N as S is a direct summand of of a finite sum of B_0. Hence $\underline{\mathrm{Ext}}^1_B(A_0, A_0) \cong \underline{\mathrm{Ext}}^1_B(B_0, B_0) \oplus \underline{\mathrm{Ext}}^1_B(B_0, S) \oplus \underline{\mathrm{Ext}}^1_B(B_0, S) \oplus \underline{\mathrm{Ext}}^1_B(S, S)$ is concentrated in degree 1. Since we already know that A_0 is semi-simple, A must be generated in degrees 0 and 1. Similarly, we see that $\underline{\mathrm{Ext}}^2_B(A_0, A_0)$ is concentrated in degree N. Thus A must be homogeneous in the sense of [**6**].

By the isomorphism (3), the graded algebra $\underline{\mathrm{Ext}}^*_A(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2 if and only if $\underline{\mathrm{Ext}}^*_B(A_0, A_0)$ is. Next we show that $\underline{\mathrm{Ext}}^*_B(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2. For convenience, we let $E^n = \underline{\mathrm{Ext}}^n_B(B_0, B_0)$, $D^n = \underline{\mathrm{Ext}}^n_B(S, S)$, $U^n = \underline{\mathrm{Ext}}^n_B(S, B_0)$ and $V^n = \underline{\mathrm{Ext}}^n_B(B_0, S)$. Write E and D for the graded algebras $\oplus_{n\geq 0} E^n$ and $\oplus_{n\geq 0} D^n$ respectively. Similarly, let U and V be the respective graded E-D-bimodule $\oplus_{n\geq 0} U^n$ and D-E-bimodule $\oplus_{n\geq 0} V^n$. Since S as a graded left E-module is a direct summand of of a finite sum of B_0, U is generated in ext-degree 0. Similarly, V is generated in ext-degree 0 as a graded right E-module. The graded algebra $\underline{\mathrm{Ext}}^*_B(A_0, A_0)$ is isomorphic to the following matrix algebra

$$\begin{pmatrix} E & U \\ V & D \end{pmatrix}$$

where the products UV and VU are the Yoneda products of the extensions. Now if B is N-Koszul, then E is generated by E^0, E^1 and E^2. Since V, as a graded right E-module, is generated in degree 0, we have $V = V^0 E$. Hence each element of V can be written as a sum of some multiplications of elements in E^0, E^1, E^2 and V^0. Similarly, each element of U can be written as a sum of some multiplications of elements in E^0, E^1, E^2 and U^0. Since B_0 is semi-simple, we may assume that there is a semi-simple B_0-module T such that $S \oplus T \cong \oplus_{i=1}^n Q_i$ as right B_0-modules, where $Q_i = B_0$ for all $i = 1, \ldots, n$. Notice that the actions of an element in $\oplus_{i\geq 1} B_i$ on both sides of the foregoing isomorphism are trivial. Thus the isomorphism is in fact a right B-module isomorphism. Let $\iota : S \longrightarrow \oplus_{i=1}^n Q_i$ be the inclusion map, and $\pi : \oplus_{i=1}^n Q_i \longrightarrow S$ be the projection map.

Then $\pi \circ \iota = id$. For any $n \geq 0$, let $\iota^n_{ext} : \underline{\mathrm{Ext}}^n_B(S, S) \longrightarrow \underline{\mathrm{Ext}}^n_B(S, \oplus^n_{i=1} Q_i)$ be the map induced by ι, and $\pi^n_{ext} : \underline{\mathrm{Ext}}^n_B(S, \oplus^n_{i=1} Q_i) \longrightarrow \underline{\mathrm{Ext}}^n_B(S, S)$ be the map induced by π. Then $\pi^n_{ext} \circ \iota^n_{ext} = id$. For any element $x \in D^n = \underline{\mathrm{Ext}}^n_B(S, S)$, we have $y := \iota^n_{ext}(x) \in \underline{\mathrm{Ext}}^n_B(S, \oplus^n_{i=1} Q_i) = \oplus^n_{i=1} \mathrm{Ext}^n_B(S, Q_i)$. Hence y can be written as $(y_1, y_2, \ldots, y_n)$ for some $y_i \in \mathrm{Ext}^n_B(S, B_0) = U^n$ $(i = 1, \ldots, n)$. While $\pi \in \mathrm{Hom}_{B_0}(\oplus^n_{i=1} Q_i, S) = \oplus_{n\,\text{copies}} S$, π is corresponding to a sequence of elements $(s_1, \ldots, s_n)$ of S. Now $x = \pi^n_{ext} \circ \iota^n_{ext}(x) = \pi^n_{ext}(y) = \sum^n_{i=1} s_i y_i$, where the multiplication $s_i y_i$ is the Yoneda product of the extensions, that is to say, x may be written as a sum of multiplications of some elements in V^0 and some elements in U^n. Summary, we have proved that if E is generated in ext-degrees 0, 1 and 2, then the graded matrix algebra

$$\begin{pmatrix} E & U \\ V & D \end{pmatrix}$$

is also generated in degrees 0, 1 and 2, or equivalently, the graded algebra $\underline{\mathrm{Ext}}^*_B(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2. Hence the graded algebra $\underline{\mathrm{Ext}}^*_A(A_0, A_0)$ is generated in ext-degree 0, 1 and 2. Now applying [**6**, Theorem 4.1], we obtain that A is N-Koszul. $\square$

At this moment, we do not know whether the converse of the theorem above is true. However, for some special cases, the converse holds.

Let $A = \oplus_{n \geq 0} A_n$ be a positively graded Hopf algebra such that A_0 is a semi-simple and co-semi-simple Hopf algebra, A_i is finite-dimensional for all $i \geq 0$ and $A_i A_j = A_{i+j}$. Let $H = A_0$. Then the natural projection $p : A \longrightarrow H$ is a Hopf algebra map; and A becomes a graded right H-comodule algebra through the projection p. Let $B = A^{coH}$. Then B is a positively graded algebra with $B_0 \cong k$. Now we assume that A/B is a graded H-Galois extension. View A_0 as a right A-module via the projection p. We have the following corollary.

Corollary 2.12. *With the notation as above, A is a Koszul algebra if and only if the co-invariant sub-algebra B is.*

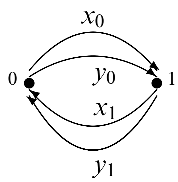
Proof. From the proof of the theorem above we see that the graded algebra $\mathrm{Ext}^*_B(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2 if and only if $\mathrm{Ext}^*_A(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2. Since $B_0 = k$, A_0 as a graded B-module is a finite sum of copies of k. Hence $\mathrm{Ext}^*_B(A_0, A_0)$ is isomorphic to a matrix algebra of the graded algebra $\mathrm{Ext}^*_B(k, k)$. Hence $\mathrm{Ext}^*_B(A_0, A_0)$ is generated in ext-degrees 0, 1 and 2 if and only if $\mathrm{Ext}^*_B(k, k)$ is. Also from the proof of the theorem above, we see that B is a homogeneous algebra. Now the result follows from [**6**, Theorem 4.1]. $\square$

Let A be a cocommutative Hopf algebra over an algebraic closed field of characteristic 0. It is well-known that $A \cong U(P(A))\#kG$ as Hopf algebras, where $P(A)$ is the Lie algebra of primitive elements of A and $U(P(A))$ is the universal enveloping algebra of $P(A)$, and G is the group of group-like elements of A. Let $gr(A)$ be the graded Hopf algebra associated to the coradical filtration of A. Then $gr(A) \cong gr(U(P(A)))\#kG$. In particular, $gr(A)$ is a Galois (graded) extension over kG. If $\dim(P(A)) < \infty$, then $gr(U(P(A)))$ is isomorphic to a polynomial algebra. Hence we have the following corollary, which results in that a cocommutative Hopf algebra is a PBW-deformations of a Koszul algebra. By the Koszul property, we may lift certain homological properties of $gr(A)$ to A, e.g. Calabi-Yau property, see [**7**].

COROLLARY 2.13. *Let A be a cocommutative Hopf algebra over an algebraic closed field of characteristic* 0 *such that G is a finite group and* $\dim(P(A)) < \infty$. *Then $gr(A)$ is a Koszul algebra.*

We end this note with an example of a cocommutative Hopf algebra which satisfies Corollary 2.12.

EXAMPLE 2.14. Let Q be the following quiver.



As an algebra, let $A = kQ/I$ where the ideal I is generated by the relations $\{x_0y_1 - y_0x_1, x_1y_0 - y_1x_0\}$. Denote by e_0 and e_1 the idempotents corresponding to the vertices. We define a coproduct and a co-unit on A so that it becomes a Hopf algebra. It is enough to define the comultiplication and the co-unit on the generators.

$$\begin{aligned}
\Delta(e_0) &= e_0 \otimes e_0 + e_1 \otimes e_1,\\
\Delta(e_1) &= e_1 \otimes e_0 + e_0 \otimes e_1,\\
\Delta(x_0) &= e_0 \otimes x_0 + e_1 \otimes x_1 + x_0 \otimes e_0 + x_1 \otimes e_1,\\
\Delta(y_0) &= e_0 \otimes y_0 + e_1 \otimes y_1 + y_0 \otimes e_0 + y_1 \otimes e_1,\\
\Delta(x_1) &= e_1 \otimes x_0 + x_1 \otimes e_0 + e_0 \otimes x_1 + x_0 \otimes e_1,\\
\Delta(y_1) &= e_1 \otimes y_0 + y_1 \otimes e_0 + e_0 \otimes y_1 + y_0 \otimes e_1,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon(e_0) = 1, \qquad \varepsilon(e_1) = 0,\\
&\varepsilon(x_0) = \varepsilon(x_1) = \varepsilon(y_0) = \varepsilon(y_1) = 0.
\end{aligned}$$

One can check that A is a cocommutative pointed bialgebra. In fact, A is also a Hopf algebra, and the antipode is given by

$$\begin{aligned}
S(e_0) &= e_0, & S(e_1) &= e_1,\\
S(x_0) &= -x_1, & S(y_0) &= -y_1,\\
S(x_1) &= -x_0, & S(y_1) &= -y_0.
\end{aligned}$$

It is clear that $H = A_0 \cong k\mathbb{Z}_2$ is a semi-simple and co-semi-simple Hopf algebra. One can also check that $B = A^{coH} \cong k[x, y]$ is a Koszul algebra. Thus A is a Koszul algebra.

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