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Fundamental Solutions of Kohn Sub-Laplacians on Anisotropic Heisenberg Groups and H-type Groups

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Abstract. We prove that the fundamental solutions of Kohn sub-Laplacians $\Delta + i\alpha\partial_t$ on the anisotropic Heisenberg groups are tempered distributions and have meromorphic continuation in α with simple poles. We compute the residues and find the partial fundamental solutions at the poles. We also find formulas for the fundamental solutions for some matrix-valued Kohn type sub-Laplacians on H-type groups.

1 Introduction

The purpose of this paper is to study explicit formulas for the fundamental solutions of certain sub-Laplacians on anisotropic (or non-isotropic) Heisenberg groups and on nilpotent groups of H-type. Let $\mathfrak{n} = V \oplus \mathfrak{t}$ be a step-two nilpotent algebra equipped with an Euclidean inner product $(\cdot, \cdot)$. We identify $\mathfrak{n}$ with its Lie group N. Take an orthonormal basis $X_1, \dots, X_p$ of V, viewed as left-invariant differential operators on N. The Kohn sub-Laplacian is then

$$\triangle_0 = -\sum_{i=1}^{p} X_i^2.$$

The fundamental solution of the sub-Laplacian $\triangle_0$ on general nilpotent groups has been studied extensively, see *e.g.*, [1,6,9] and references therein. In a recent article [3], Chang and Tie considered the anisotropic Heisenberg group $N = \mathbb{C}^n \oplus \mathbb{R}$ and found an explicit formal formula for the fundamental solution of the sub-Laplacian $\mathcal{L}_\alpha = \triangle_0 + i\alpha T$, where $T = \frac{\partial}{\partial t}$ acting on the last variable; see also [2], where some similar sub-elliptic operators on the Heisenberg group and some Hermite operators on $\mathbb{R}^n$ are studied. This operator actually appears when we study the boundary CR complex; more precisely the sub-Laplacian associated with the boundary CR operator $\bar\partial_b$,

$$\Box = \bar\partial_b \bar\partial_b^* + \bar\partial_b^* \bar\partial_b,$$

acting on the $(0, q)$ forms on the Heisenberg group, can be expressed in terms of the operator $\mathcal{L}_\alpha$. See [10].

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The operator $\mathcal{L}_\alpha$ has fundamental solution for generic values of α. However, when α is in certain discrete set Λ, then $\mathcal{L}_\alpha$ has kernel and thus there arises the question of finding the corresponding partial fundamental solution and the integral kernel for the projection onto the kernel of $\mathcal{L}_\alpha$. See [10] for the case of the Heisenberg group with $\alpha = 4n$, where the integral kernel becomes the Cauchy–Szegö kernel for the Hardy space of the Siegel domain. In this paper we find the partial fundamental solution in the setting of anisotropic Heisenberg groups. Since many formal computations in finding the fundamental solution (see *e.g.,* [3]) are done using Fourier transform, it is desirable to prove that the fundamental solution K_α is a tempered distribution. Indeed we prove that this is the case and that K_α has a meromorphic continuation as tempered distribution to all $\alpha \in \mathbb{C}$ with simple poles at the singular set Λ (when properly normalized). The partial fundamental solution for $\alpha_0 \in \Lambda$ becomes the constant term $K^{(0)}_{\alpha_0}$ in the Laurent expansion of K_α near α_0, while as the integral kernel onto the null space of $\mathcal{L}_\alpha$ is $-iTK^{(-1)}_{\alpha_0}$ with $K^{(-1)}_{\alpha_0}$ being the residue (see Theorem 2.2). The formulas for the partial fundamental solution and the integral kernel are in terms of integrals or sum of certain simple functions. In particular, in the case of Heisenberg groups, those formulas become rather simple, and the integral kernel for the null space has also been obtained earlier by Strichartz [12] using slightly different methods.

We consider a similar problem for a nilpotent group of H-type with Lie algebra $\mathfrak{n} = V \oplus \mathfrak{t}$. We define a certain operator d_H acting on the horizontal differential forms on $\mathfrak{n}$ and consider the similar operator $\square_H = d_H d_H^* + d_H^* d_H$. We express $\square_H$ in terms of the action of $\mathfrak{t}$ (more precisely, the dual $\mathfrak{t}^*$) on the horizontal forms. For one-forms this question can be formulated in terms of Clifford module action of $\mathfrak{t}$ on V^*, and $\square_H = d_H d_H^* + d_H^* d_H$ can be written as $-\sum E_i^2 + 2\rho(\partial_2)$, where $\rho(\partial_2)$ is a Dirac operator. We consider thus the general case of a Clifford bundle, and we find the fundamental solution of $\mathcal{L}_\alpha = -\sum E_i^2 + \alpha\rho(\partial_2)$. The space of horizontal differential forms and the horizontal vector fields play an important role in the theory of quasi-regular mappings [6] and are of considerable interest. It would be interesting to find the fundamental solutions for forms of higher degrees. To our knowledge, even in the case of the Heisenberg groups, the fundamental solutions for the sub-Laplacians $\square_H + \alpha\rho(\partial_2)$ on general (p, q)-forms are still not known except for special values of p and q. We hope that our result will also shed some light on that question.

The paper is organized as follows. The main results are stated in Theorems 2.1 through 3.3. Section 2 is devoted to computations of the partial fundamental solutions on anisotropic Heisenberg groups. The fundamental solution for $\mathcal{L}_\alpha$ on Clifford module-valued functions is computed in Section 3.

2 Anisotropic Heisenberg Groups

2.1 Meromorphic Continuation of the Fundamental Solution of $\mathcal{L}_\alpha$

Let $a = (a_1, a_2, \dots, a_n)$, $a_1, a_2, \dots, a_n > 0$. We equip $\mathbb{C}^n$ with the anisotropic (or non-isotropic) Hermitian inner product

$$\langle z, w\rangle_a = \sum_{j=1}^{n} a_j z_j \bar{w}_j.$$

Let $N = N(a) = \mathbb{C}^n + \mathbb{R}$ be the corresponding anisotropic Heisenberg group with the product

$$(z,t) \circ (w,s) = (z+w, t+s+2\mathrm{Im}\langle z,w\rangle_a).$$

We consider the following left-invariant sub-Laplacian on N,

$$\mathcal{L}_\alpha = -2\sum_{j=1}^{n}(\bar{Z}_j Z_j + Z_j\bar{Z}_j) - i\alpha T$$

where

$$Z_j = \frac{\partial}{\partial z_j} - ia_j\bar{z}_j\frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} + ia_j z_j\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

are the left-invariant differential operators on N generated by the right translation with respect to the coordinates $(z,\bar{z},t)$. Vectors $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$, will also be identified with $(x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{R}^{2n}$.

The fundamental solutions K_α of $\mathcal{L}_\alpha$ at $0 \in N$, *i.e.*, $\mathcal{L}_\alpha K_\alpha = \delta_0$ were studied by Chang and Tie [3]; see also [10] for the isotropic case $a_j = 1$ for all j. They found the integral formula

$$K_\alpha(z,t) = \frac{(n-1)!}{8\pi^{n+1}}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{4}s}\frac{\nu(s)ds}{[\gamma(z,s)-it]^n}, \tag{2.1}$$

and the following formal formula for its Fourier transform

$$\widehat{K}_\alpha(\xi,\tau) = 2^n\sum_{\mathbf{k}\in\mathbb{Z}_+^{2n}}\frac{1}{\alpha\tau + \sum_{j=1}^{2n}2a_j|\tau|(4k_j+1)}\prod_{j=1}^{2n}\frac{\phi_{2k_j}(\xi_j/\sqrt{2a_j|\tau|})}{2^{2k_j}k_j!}, \tag{2.2}$$

where $\phi_k(x)$ is the Hermite function, $\gamma(z,s) = \sum_{j=1}^n a_j|z_j|^2\coth(a_js)$ and $\nu(s) = \prod_{j=1}^n\frac{a_j}{\sinh(a_js)}$. See also [1] for the study of fundamental solutions of general Kohn type sub-Laplacian. We recall that the Fourier transform on $\mathbb{R}^m$ is normalized by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^m} e^{-i(\xi,x)}f(x)dx.$$

The integral formula (2.1) for $\alpha = 0$ can also be obtained from a more general formula in [1, Theorem 3]. It is elementary to see that the integral (2.1) is convergent if and only if $|\mathrm{Re}(\alpha)| < 4\sum_{j=1}^n a_j$. There arises therefore the question of analytic continuation of this integral and the proper justification of convergence of the series (2.2).

Let $L_k^{(\beta)}$ be the Laguerre polynomial and write $L_k = L_k^{(0)}$. Then the Laguerre polynomials and the Hermite functions are related by

$$L_k^{(-\frac{1}{2})}(|\xi|^2)e^{-\frac{|\xi|^2}{2}} = (-1)^k\frac{1}{2^{2k}k!}\phi_{2k}(\xi),$$

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and for $\xi = (\xi_1, \xi_2) \in \mathbb{C} = \mathbb{R}^2$

$$L_k(|\xi|^2)e^{-\frac{|\xi|^2}{2}} = (-1)^k \sum_{k_1+k_2=k} \frac{1}{2^{2k_1+2k_2}k_1!k_2!}\phi_{2k_1}(\xi_1)\phi_{2k_2}(\xi_2),$$

which can be easily deduced from the following generating function formula of Laguerre polynomials

$$\sum_{k=0}^{\infty} r^k L_k^{(\alpha)}(x) = (1-r)^{-\alpha-1}e^{-rx/1-r}, \qquad |r| < 1. \tag{2.3}$$

The sum (2.2) over $\mathbb{Z}_+^{2n}$ can then be rewritten as one over $\mathbb{Z}_+^n$ in terms of the Laguerre polynomials,

$$\widehat{K}_\alpha(\xi,\tau) = 2^n \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \frac{1}{\alpha\tau + \sum_{j=1}^n 4a_j|\tau|(2k_j+1)}(-1)^{|\mathbf{k}|}\prod_{j=1}^n L_{k_j}\left(\frac{|\xi|^2}{2|\tau|}\right)e^{-\frac{|\xi|^2}{4|\tau|}}, \tag{2.4}$$

where $|\mathbf{k}| = \sum_{j=1}^n k_j$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n = \mathbb{R}^{2n}$.

Denote

$$\Lambda = \left\{ \pm 4\sum_{j=1}^n a_j(2k_j+1) : \mathbf{k} = (k_1, \dots, k_n) \in Z_+^n \right\}.$$

We have the following result on the meromorphic dependence of K_α on α, which will also be used in the proof of Theorem 3.3.

Theorem 2.1 *The function K_α defines a tempered distribution for $|Re(\alpha)| < 4\sum_{j=1}^n a_j$ and has a meromorphic continuation to the whole complex plane with simple poles at Λ.*

Proof We will only prove the result for $n = 1$. The general case can be proved by the same method except that the notations are more complicated. In this case we can put $a = 1$.

For $g \in \mathcal{S}(\mathbb{R}^3)$, the Schwartz class, we let $f = \hat{g}$, then $f \in \mathcal{S}(\mathbb{R}^3)$. We have

$$\begin{aligned}
(K_\alpha, g) &= (\widehat{K}_\alpha, f) \\
&= 2\int_0^\infty \int_{\mathbb{R}^2} \sum_{k=0}^\infty \frac{(-1)^k}{\alpha + 4(2k+1)} L_k\left(\frac{|\xi|^2}{2|\tau|}\right)e^{-\frac{|\xi|^2}{4|\tau|}} f(\xi,\tau)\frac{1}{\tau}d\xi d\tau \\
&\quad + 2\int_{-\infty}^0 \int_{\mathbb{R}^2} \sum_{k=0}^\infty \frac{(-1)^k}{\alpha - 4(2k+1)} L_k\left(\frac{|\xi|^2}{2|\tau|}\right)e^{-\frac{|\xi|^2}{4|\tau|}} f(\xi,\tau)\frac{1}{\tau}d\xi d\tau \\
&= I_1 + I_2.
\end{aligned}$$

Now we estimate $I_1 = I_1(f)$ in terms of the family of seminorms defining tempered distributions on $f \in \mathcal{S}(N) = \mathcal{S}(\mathbb{R}^3)$. Recall [11, Chapter I] that these seminorms are

$$\rho_{\beta,\gamma}(f) = \sup_{y\in\mathbb{R}^3} |y^\beta \partial^\gamma f(y)|$$

and

$$\rho_{s,t}(f) = \sum_{|\beta|=s,\ |\gamma|=t} \rho_{\beta,\gamma}(f),$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices and we use the usual convention for y^β and ∂^β. We divide I_1 into two parts,

$$I_1 = \int_0^1 \int_{\mathbb{R}^2} + \int_1^\infty \int_{\mathbb{R}^2} = I_{11} + I_{12}.$$

Recall also ([13, Chapter 1, Lemma 1.5.4]), that

$$\int_{\mathbb{R}^2} L_k(|\xi|^2) e^{-\frac{|\xi|^2}{2}} d\xi \le C(k+1)^{\frac{1}{2}}. \tag{2.5}$$

The term $|I_{11}|$ is thus an absolutely convergent series, and it can be estimated by

$$|I_{11}| \le C\,\rho_{0,0}(f) \sum_k \frac{1}{|\alpha + 4(2k+1)| k^{\frac{1}{2}}} \le C\,\rho_{0,0}(f).$$

To treat I_{12} we write each integral as

$$\int_1^\infty \int_{\mathbb{R}^2} L_k(|\xi|^2) e^{-\frac{|\xi|^2}{2}} f(\sqrt{2|\tau|}\xi, \tau) d\xi d\tau \tag{2.6}$$

by a change of variables. However, the Laguerre functions $L_k(|\xi|^2)e^{-\frac{|\xi|^2}{2}}$ satisfy the harmonic oscillator equation

$$\left(|\xi|^2 - \frac{\partial^2}{\partial \xi_1^2} - \frac{\partial^2}{\partial \xi_2^2} \right) (L_k(|\xi|^2) e^{-\frac{|\xi|^2}{2}}) = 2(2k+1) L_k(|\xi|^2) e^{-\frac{|\xi|^2}{2}},$$

and we can perform partial integration in (2.6) to get

$$\frac{1}{2(2k+1)} \int_{-\infty}^\infty \int_{\mathbb{R}^2} L_k(|\xi|^2) e^{-\frac{|\xi|^2}{2}} \left(|\xi|^2 - \partial_{\xi_1}^2 - \partial_{\xi_2}^2\right) \left(f(\sqrt{2|\tau|}\xi, \tau)\right) d\xi d\tau.$$

We estimate the term $(|\xi|^2 - \partial_{\xi_1}^2 - \partial_{\xi_2}^2)\big(f(\sqrt{2|\tau|}\xi, \tau)\big)$ in the integrand. By the definition of $\rho_{s,t}(f)$,

$$|\xi|^2 f(\sqrt{2|\tau|}\xi, \tau) \le \rho_{3,0}(f) \frac{1}{\tau^2}.$$

Also,

$$|(\partial_{\xi_1}^2 + \partial_{\xi_2}^2) f(\sqrt{2|\tau|}\xi, \tau)| \le \rho_{3,2}(f) \frac{2}{\tau^2},$$

since

$$(\partial_{\xi_1}^2 + \partial_{\xi_2}^2)\big(f(\sqrt{2|\tau|}\xi, \tau)\big) = 2|\tau|(\partial_1^2 f + \partial_2^2 f)(\sqrt{2|\tau|}\xi, \tau).$$

Using the previous estimate (2.5) again, we get

$$|I_{12}| \le C\,\rho_{3,2}(f)\sum_k \frac{(k+1)^{\frac{1}{2}}}{|\alpha+4(2k+1)|(2k+1)}$$
$$\le C\,\rho_{3,2}(f)\sum_k \frac{1}{|\alpha+4(2k+1)|(k+1)^{\frac{1}{2}}} \le C\,\rho_{3,2}(f).$$

Putting those together we obtain

$$|I_1| \le C\cdot(\rho_{3,2}(f)+\rho_{0,0}(f))\sum_k \frac{1}{|\alpha+4(2k+1)|k^{\frac{1}{2}}} \le C(\rho_{3,2}(f)+\rho_{0,0}(f)).$$

The estimate for $|I_2|$ is the same.

Finally, we find

$$|(\widehat{K}_\alpha, f)| \le C(\rho_{3,2}(f)+\rho_{0,0}(f)),$$

and the temperedness follows from the definition [11, Chapter I,Theorem 3.11]. The meromorphic continuation follows from the absolute convergence of the series for $\alpha \notin \Lambda$. ■

When $a_1 = a_2 = \cdots = a_n = 1$, an explicit formula for K_α is given in ([10, Chapter XIII, Theorem 1]); see the formula (2.7) below. Even in this case, it seems easier to prove the tempered property of K_α by using the expansion formula (2.4) as done above.

One can also prove, by dealing with the integral of the form

$$\int_0^\infty e^{\beta s}\prod_{j=1}^n \frac{e^{-a_j s}}{\sinh(a_j s)}\,ds,$$

that the integral (2.1) as a function of $(z,t) \ne 0$ has meromorphic continuation with poles at Λ.

2.2 Partial Fundamental Solution for Singular α

Notice that if $\alpha = \alpha_0 = \pm 4\sum_{j=1}^n a_j(2k_j+1) \in \Lambda$, then $\mathcal{L}_\alpha$ has a kernel; thus it does not have a fundamental solution. We compute then the partial fundamental solution. For that purpose we consider the expansion of the distribution K_α near α_0. It is of the form

$$K_\alpha = \frac{K_{\alpha_0}^{(-1)}}{\alpha-\alpha_0} + K_{\alpha_0}^{(0)} + (\alpha-\alpha_0)K_{\alpha_0}^{(1)} + \cdots$$

by Theorem 2.1. Recall the notation $|z|_a^2 = \sum_{j=1}^n a_j|z|_j^2$.

Theorem 2.2 *Let $\alpha_0 = \pm 4\sum_{j=1}^n a_j(2k_j^0+1)$, $K_{\alpha_0}^{(0)}$, and $K_{\alpha_0}^{(-1)}$ be as above. The following formula holds as tempered distribution on N,*

$$\mathcal{L}_{\alpha_0}K_{\alpha_0}^{(0)} - iTK_{\alpha_0}^{(-1)} = \delta_0.$$

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Moreover, $K_{\alpha_0}^{(-1)}$ is explicitly given by

$$K_{\alpha_0}^{(-1)} = \mp\frac{2^{n-1}a_1\cdot\cdot a_n}{\pi^{n+1}}(|z|_a^2 \pm it)^{-n}$$
$$\times\sum_{\mathbf{m}}(-1)^{\sum_{j=1}^n m_j}\sum_{l_1\le m_1,\dots,l_n\le m_n}(|z|_a^2\pm it)^{(-\sum l_i)}\ \frac{(\sum l_i+n-1)!}{l_i!}$$
$$\times\prod_{j=1}^n\binom{m_j}{l_j}(-2a_j|z_j|^2)^{l_i},$$

where the first sum is over all $\boldsymbol{m}\in\mathbb{Z}^n$ such that $\sum_{j=1}^n a_j(m_j-k_j^0)=0$. The Fourier transform of $K_{\alpha_0}^{(0)}$ is

$$\widehat{K_{\alpha_0}^{(0)}}(\xi,\tau) = 2^n\sum\frac{1}{\alpha_0\tau+\sum_{j=1}^n 4a_j|\tau|(2k_j+1)}(-1)^{|\boldsymbol{k}|}\prod_{j=1}^n L_{k_j}\left(\frac{|\xi|^2}{2|\tau|}\right)e^{-\frac{|\xi|^2}{4|\tau|}},$$

where the sum is over all $\boldsymbol{k}\in\mathbb{Z}^n$ such that $\sum_{j=1}^n a_j(2k_j+1)\ne\sum_{j=1}^n a_j(2k_j^{(0)}+1)$.

Proof We start with the formula

$$\mathcal{L}_\alpha K_\alpha = \delta_0$$

in a punctured neighborhood of α_0. The Laurent expansion of the left-hand side of the formula is then

$$\frac{\mathcal{L}_{\alpha_0}K_{\alpha_0}^{(-1)}}{\alpha-\alpha_0} + (\mathcal{L}_{\alpha_0}K_{\alpha_0}^{(0)} - iTK_{\alpha_0}^{(-1)}) + O(|\alpha-\alpha_0|) = \delta_0.$$

Thus

$$\mathcal{L}_{\alpha_0}K_{\alpha_0}^{(-1)} = 0,\qquad \mathcal{L}_{\alpha_0}K_{\alpha_0}^{(0)} - iTK_{\alpha_0}^{(-1)} = \delta_0.$$

We compute now the residue $K_{\alpha_0}^{(-1)}$. Using equation (2.2) and the fact [3, (15)] that Laguerre functions are eigenfunctions of the Fourier transform

$$\widehat{e^{-\frac{|x|^2}{2}}L_k(|x|^2)}(\xi) = (-1)^k\sqrt{2\pi}e^{-\frac{|\xi|^2}{2}}L_k(\xi),$$

we find

$$K_{\alpha_0}^{(-1)} = \mathrm{Res}[K_\alpha(z,t),\pm4\sum_{j=1}^n a_j(2k_j^0+1)] = \mp\frac{2^{n-1}a_1\cdot\cdot a_n}{(\pi)^{n+1}}$$
$$\sum_{\sum_{j=1}^n a_j(m_j-k_j^0)=0}(-1)^{\sum_{j=1}^n m_j}\int_0^\infty e^{(-\sum_{j=1}^n a_j\tau|z_j|^2\mp it\tau)}\prod_{j=1}^n L_{m_j}^{(0)}(2a_j\tau|z_j|^2)\tau^{n-1}d\tau.$$

This integration can be evaluated by the definition of the polynomials $L_k^{(0)}$ and their Laplace transforms as in [12], and we find the formula as claimed.

The formula for the constant term $K_{\alpha_0}^{(0)}$ follows directly from (2.4). ∎

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When $\alpha_0 = 4\sum_{j=1}^{n} a_j$ is the first positive pole, we get the following well-known result

Corollary 2.3 *Let $\alpha_0 = 4\sum_{j=1}^{n} a_j$. Then*

$$K_{\alpha_0}^{(-1)} = -\frac{2^{n-1}(n-1)!a_1 \cdot \cdot a_n}{\pi^{n+1}} \left(\sum_{j=1}^{n} a_j|z|^2 + it\right)^{-n},$$

and

$$-iTK_{\alpha_0}^{(-1)} = \frac{2^{n-1}n!}{\pi^{n+1}} \times (\sum_{j=1}^{n} a_j|z|^2 + it)^{-n-1}$$

is the Cauchy-Szegö kernel for the domain:

$$\Omega_{n+1} = \left\{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : Im(z_{n+1}) > \sum_{j=1}^{n} a_j|z_j|^2\right\}.$$

We specialize our result to the case of (isotropic) Heisenberg groups.

Corollary 2.4 *Let $a_1 = a_2 = \cdots = a_n = 1$. Then*

$$K_{4(n+2k)}^{(-1)} = -\frac{2^{n-1}}{\pi^{n+1}} \frac{(n+k-1)!}{k!} \times \frac{(|z|^2 - it)^k}{(|z|^2 + it)^{n+k}},$$

and

$$-iTK_{4(n+2k)}^{(-1)} = \frac{2^{n-1}}{\pi^{n+1}} \times \left[\frac{(n+k)!}{k!} \cdot \frac{(|z|^2 - it)^k}{(|z|^2 + it)^{n+k+1}}\right] \times \left[1 + \frac{k}{n+k} \cdot \frac{|z|^2 + it}{|z|^2 - it}\right].$$

Proof The residue can now be directly computed using (2.2) and Fourier inversion in the z variable,

$$K_{4(n+2k)}^{(-1)}(z,t) = \frac{-2^{n-1}}{(\pi)^{n+1}}(-1)^k \int_0^\infty e^{(-\tau|z|^2)} L_k^{(n-1)}(2\tau|z|^2)\tau^{n-1}e^{-it\tau}d\tau.$$

This integral was evaluated in [12] by using identity (2.3), and it is

$$K_{4(n+2k)}^{(-1)} = -\frac{2^{n-1}(n+k-1)!}{\pi^{n+1}k!} \cdot \frac{(|z|^2 - it)^k}{(|z|^2 + it)^{n+k}}.$$

Meanwhile,

$$\begin{aligned} -iTK_{4(n+2k)}^{(-1)} &= \frac{2^{n-1}(n+k-1)!}{\pi^{n+1}k!}\left[\frac{k(|z|^2 - it)^{k-1}}{(|z|^2 + it)^{n+k}} + \frac{(n+k)(|z|^2 - it)^k}{(|z|^2 + it)^{n+k+1}}\right] \\ &= \frac{2^{n-1}(n+k)!}{(\pi)^{n+1}k!}\left[\frac{(|z|^2 - it)^k}{(|z|^2 + it)^{n+k+1}}\right] \cdot \left[1 + \frac{k}{n+k}\left(\frac{|z|^2 + it}{|z|^2 - it}\right)\right]. \end{aligned}$$ ■

Remark 2.5 When $a_1 = \cdots = a_n = 1$ the kernel K_α can be explicitly evaluated by using formula (2.2),

$$K_\alpha = \frac{2^{n-1}}{8\pi^{n+1}} \Gamma\Big(\frac{n}{2} - \frac{\alpha}{8}\Big) \Gamma\Big(\frac{n}{2} + \frac{\alpha}{8}\Big) \big(|z|^2 + it\big)^{-(\frac{n}{2}+\frac{\alpha}{8})} \big(|z|^2 - it\big)^{-(\frac{n}{2}-\frac{\alpha}{8})}. \tag{2.7}$$

A partial fundamental solution $K^{(0)}_{\alpha_0}$ (which is not unique) and the kernel $-iTK^{(-1)}_{\alpha_0}$ at $\alpha_0 = -4n$ in Λ, *i.e.*, the first negative singular point, were found in [10] using the above formula. We may find them at any point $\alpha_0 = \pm 4(n + 2k)$. Indeed $K^{(-1)}_{\alpha_0}$ is as given in Corollary 2.4, and

$$\begin{aligned} K^{(0)}_{\pm 4(n+2k)} = \frac{(-1)^{k+1} 2^{n-4} (n + k - 1)!}{\pi^{n+1} k!} &(|z|^2 \mp it)^k (|z|^2 \pm it)^{-n-k} \\ &\times \Big\{ \Big(\sum_{j=k+1}^{n+k-1} \frac{1}{j} \Big) \mp \Big(\log \frac{|z|^2 + it}{|z|^2 - it} \Big) \Big\}. \end{aligned}$$

However, the function $(|z|^2 \mp it)^k (|z|^2 \pm it)^{-n-k}$ is in the kernel of $\mathcal{L}_{\alpha_0}$, and disregarding the constant multiplier, the function

$$(|z|^2 \mp it)^k (|z|^2 \pm it)^{-n-k} \log \frac{|z|^2 + it}{|z|^2 - it}$$

is also a partial fundamental solution. This kernel is also the integral kernel of the projection onto the k-th "Heisenberg fan" studied by Strichartz [12].

3 H-type Groups

3.1 H-type Groups and Sub-Laplacians on Clifford-valued Functions

We recall that a step two nilpotent algebra $\mathfrak{n} = V \oplus \mathfrak{t}$, $[V, V] \subset \mathfrak{t}$ is of Heisenberg type, or simply of H-type, if there is an inner product $(\cdot, \cdot)$ in N such that the linear map $J \colon \mathfrak{t} \to \mathrm{End}(V)$ defined by $(J_t(u), v) = \frac{1}{2}(t, [u, v])$ satisfies $J_t^2 = -|t|^2 I$ for all $t \in \mathfrak{t}$. Here I is the identity map on V. Then the dimension $p = \dim V$ is even, and we write $p = 2n$ and denote $q = \dim \mathfrak{t}$.

The corresponding Lie group N will be identified with the Lie algebra $\mathfrak{n}$, and the group product is

$$(x, t) \cdot (y, s) = (x + y, t + s + \tfrac{1}{2}[x, y]).$$

Groups of H-type were introduced by Kaplan in [8], and they have been studied in several contexts for different motivations, see [4, 5, 7]. We will introduce certain Dirac operator and Kohn sub-Laplacians on vector-valued functions and find their fundamental solutions.

Let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$. We equip $\mathfrak{t}^*$ with the induced inner product $(\cdot, \cdot)$ by identifying $v \in \mathfrak{t}$ with the element $w \to (v, w)$ in $\mathfrak{t}^*$. Let W be any Clifford module of $\mathfrak{t}^*$. Recall that a Euclidean space W is a Clifford module if there is a linear map $\rho : \mathfrak{t}^* \to \mathrm{End}(W)$ such that

$$\rho(T)^2 = -\|T\|^2 I, \quad T \in \mathfrak{t}^*$$

where I is the identity map on W. For any $W^{\mathbb{C}}$-valued function f, we define the Dirac operator

$$\rho(\partial_2)f(v,t) = \sum_{j=1}^{q} \rho({T_j}^*)(\partial_{T_j} f). \tag{3.1}$$

It is easy to see that it is well defined, namely, it is independent of the choice of the orthonormal basis.

Note that when N is the Heisenberg group $\mathbb{R}^{2n} \oplus \mathbb{R}$, a $2m$-dimensional Clifford module W of $\mathbb{R}$ is given by a skew symmetric matrix ρ with $\rho^2 = -I$. The complexification of W is $W^{\mathbb{C}} = \mathbb{C}^m_+ + \mathbb{C}^m_-$ with ρ acting by $\pm i$. The Dirac operator is then

$$\rho(\partial_2)f(v,t) = i\partial_t f_+(v,t) - i\partial_t f_-(v,t),$$

where $f = f_+ + f_-$ and $f_\pm$ are the decomposition of f as $\mathbb{C}^m_\pm$-valued functions. For W being the dual of the subspace $\mathbb{R}^{2n}$, those are the $(1,0)$ and $(0,1)$-forms on the Heisenberg group with the given CR-structure [10].

Let $\{e_j\}_1^p$ and $\{T_k\}_1^q$ be an orthonormal basis of V, respectively $\mathfrak{t}$. We let $\{E_j\}$ and $\{T_k\}$ be the corresponding left-invariant differential operators acting on vector-valued functions. They are given by $T_k f(v,t) = \partial_{T_k} f(v,t)$ and

$$E_j f(v,t) = \partial_{e_j} f + \tfrac{1}{2}\partial_{[v,e_j]} f = \partial_{e_j} f + \sum_{k=1}^{q} (J_{T_k} v, e_j) T_k f,$$

in terms of the operator J. Let

$$\mathfrak{L}_\alpha = -\sum_{j=1}^{2n} E_j^2 + \alpha\rho(\partial_2) \tag{3.2}$$

be the Kohn type sub-Laplacian acting on $W^{\mathbb{C}}$-valued functions. We are interested in finding the fundamental solution of $\mathfrak{L}_\alpha$, namely, an $\mathrm{End}(W^{\mathbb{C}})$-valued distribution such that $\mathfrak{L}_\alpha f = \delta I$.

3.2 Kohn Sub-Laplacians on Horizontal Differential Forms

In this subsection we shall find a formula for the Kohn sub-Laplacian acting on horizontal j-forms on the H-type group $H = V \times \mathfrak{t}$; for $j = 1$ this is a special case considered in the previous subsection with the defining Clifford action of $\mathfrak{t}^*$ on V^*.

Let $T(N)$ be the tangent bundle of the H-type group N and $T^*(N)$ the dual tangent bundle. For each $x \in N$ we identify the tangent space $T_x(N)$ with $\mathfrak{n}$ via the (differential of the) left multiplication $l_x \colon \mathfrak{n} = T_0(N) \to T_x(N)$. We let $T_{H,x}(N)$ be the subspace $T_{H,x}(N) = l_x V$ and $T^*_{H,x}(N)$ the dual space; we shall identify $T^*_{H,x}(N)$ with V^*, with $\{dv_1, \dots, dv_p\}$ being a dual basis to a fixed orthonormal basis $\{\partial_{v_1}, \dots, \partial_{v_p}\}$ of V. Let $T_H(N)$ and $T^*_H(N)$ be the corresponding vector bundles, which we shall call the horizontal tangent, respectively cotangent, bundles. We consider the exterior

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product $\wedge^j T_H^*(N)$ for $1 \le j \le p$, and their smooth sections, which will be called the horizontal differential j-forms; see [6]. Any section of the bundle is of the form

$$f = \sum_I f_I dv^I$$

where $dv^I = dv_{i_1} \wedge \cdots \wedge dv_{i_j}$, $I = (i_1, \ldots, i_j)$ with $1 \le i_1, \ldots, i_j \le p$. We define the following horizontal differentiation

$$d_H f = \sum_I \Big(\sum_{k=1}^{p} E_k f_I dv_k \Big) \wedge dv^I$$

and d_H^* its formal adjoint.

Definition 3.1 We define the Kohn sub-Laplacian on the horizontal differential forms by

$$\Box_H = d_H^* d_H + d_H d_H^*. \tag{3.3}$$

Let M_l be the multiplication operator by dv_l, $M_l f = dv_l \wedge f$ and ι_l the (negative of) dual of M_l,

$$\iota_l dv_{i_1} \wedge \cdots \wedge dv_{i_j} = \begin{cases} 0, & l \notin I \\ (-1)^{i_k - 1} dv_{i_1} \wedge \cdots \widehat{dv_{i_k}} \cdots \wedge dv_{i_j}, & l = i_k \in I \end{cases};$$

see [10].

We also define a Dirac operator $\rho(\partial_2)$ on $\wedge^j V^*$-valued functions by

$$\rho(\partial_2) f := \sum_I \sum_{s=1}^{q} \sum_{k,l=1}^{p} (\partial_s f_I)(J_{T_s} e_k, e_l) i_k M_l dv^I.$$

The operator $\rho(\partial_2)$ cannot be formulated as in (3.1). For $j = 1$ this coincides with (3.1); see below. However, for $j > 1$ the induced action $\mathfrak{t}^*$ on $\wedge^j V^*$ does not form a Clifford module.

Proposition 3.2 *The Kohn sub-Laplacian* (3.3) *is of the form*

$$\Box_H = -\Big(\sum_{j=1}^{p} E_j^2 + 2\rho(\partial_2) \Big).$$

For $j = 1$ the Dirac operator $\rho(\partial_2)$ coincides with the Dirac operator $\rho(\partial_2)$ in Section 3.1 *with the Clifford action of $\mathfrak{t}^*$ on V^* given by the dual action.*

Proof Let $f = \sum_I f_I dv^I$. By definition,

$$-\Box_H f = \sum_I \Big(\sum_{k=1}^{p} E_k^2 f_I dv^I + \sum_{k,l=1}^{p} ([E_k, E_l] f_I) \iota_k M_l dv^I \Big).$$

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Let $c^s_{k,l}$ be the structural constants, *i.e.*, $[E_k, E_l] = \sum_{r=1}^{q} c^s_{k,l} T_s$ with

$$c^s_{k,l} = (T_s, [E_k, E_l]) = 2(J_{T_s} E_k, E_l).$$

Namely,

$$[E_k, E_l] f = 2 \sum_{r=1}^{q} (J_{T_s} E_k, E_l) T_s f = 2 \sum_{r=1}^{q} (J_{T_s} E_k, E_l) \partial_s f.$$

The second term can then be written as

$$\sum_{k,l=1}^{p} ([E_k, E_l] f_I) \iota_k M_l dv^I = 2 \sum_{s=1}^{q} \sum_{k,l=1}^{p} (\partial_s f_I)(J_{T_s} E_k, E_l) \iota_k M_l dv^I.$$

This proves the first claim.

For $j = 1$ one observes that the formula

$$T^* \mapsto \sum_{k,l=1}^{p} (J_T e_k, e_l) i_k M_l,$$

where $T \in \mathfrak{t}$ is the dual element of $T^* \in \mathfrak{t}^*$ defined $T^*(v) = (v, T)$, defines a Clifford action, and it coincides with the dual action of $\mathfrak{t}$ on V, which can be seen easily by choosing an orthonormal basis $\{e_1, \dots, e_n; e_{n+1}, \dots, e_{2n}\}$ such that $J_T e_k = |T| e_{n+k}$, $J_T e_{n+k} = -|T| e_k$, $j = 1, \dots, n$. ■

The above proposition is a generalization of the known formula for the Kohn sub-Laplacian on $(0, q)$-forms on the Heisenberg group $\mathbb{C}^n + \mathbb{R}$; see *e.g.*, [10, Proposition 2.2, Chapter XIII].

3.3 Fundamental Solution

We now compute the fundamental solution of the operator $\mathfrak{L}_\alpha$ in (3.2) for any Clifford module W of $\mathfrak{t}^*$ (identified with $\mathfrak{t}$). Let $W_{\alpha,\beta}(x)$ be the Whittaker function; see *e.g.*, [14].

Theorem 3.3 *Let W be a Clifford module of $\mathfrak{t}$ as in Section 3.1. Then for $|Re(\alpha)| < 2n$ the fundamental solution of the equation $\mathfrak{L}_\alpha f = \delta I$ is given by $f = \frac{1}{2}(f_+ + f_-)$, where*

$$f_+(v,t) = f_0^{(\alpha)}(v,t) I + \rho(\partial_2) f_1^{(\alpha)}(v,t),$$
$$f_-(v,t) = f_0^{(-\alpha)}(v,t) I - \rho(\partial_2) f_1^{(-\alpha)}(v,t),$$

and

$$f_0^{(\alpha)}(v,t) = \frac{\Gamma(\frac{n}{2} \pm \frac{\alpha}{4})}{(2\pi)^q 4\pi^n |v|^n} \int_{\mathfrak{t}} |\tau|^{n/2-1} W_{\mp\alpha/4, n-1/2}(|\tau| \cdot |v|^2) e^{i(t,\tau)} d\tau,$$
$$f_1^{(\alpha)}(v,t) = \frac{\Gamma(\frac{n}{2} \pm \frac{\alpha}{4})}{(2\pi)^q 4\pi^n |v|^n} \int_{\mathfrak{t}} |\tau|^{n/2-2} W_{\mp\alpha/4, n-1/2}(|\tau| \cdot |v|^2) e^{i(t,\tau)} d\tau.$$

Moreover, f defines a tempered distribution for $|Re(\alpha)| < 2n$ and has meromorphic continuation to the whole plane with simple poles at $\{\pm 2(n + 2k) : k \in Z_+\}$.

Proof We seek the Fourier transform g of the fundamental solution f, namely

$$f(v,t) = \frac{1}{(2\pi)^q}\int_{\mathfrak{t}} e^{i(t,\tau)} g(v,\tau)\,d\tau.$$

The formal computations below will be justified in the end. The distribution $\delta(v,t)$ in (v,t) is the Fourier transform of the function $\delta(v)$ in v,

$$\delta(v,t) = \frac{1}{(2\pi)^q}\int_{\mathfrak{t}} e^{i(t,\tau)}\delta(v)\,d\tau.$$

Let the Kohn sub-Laplacian $\mathfrak{L}_\alpha$ act on f,

$$\mathfrak{L}_\alpha f(v,t) = \frac{1}{(2\pi)^q}\int_{\mathfrak{t}} e^{i(t,\tau)}\Big\{-\sum_{j=1}^{2n}(\partial_j + i(J_\tau v, e_j))^2 + \alpha\rho(\tau)\Big\} g(v,\tau)\,d\tau,$$

where $\{e_j\}_1^p$ is an orthonormal basis of V. Thus we look for a solution $g(v,\tau)$ of the equation

$$\Big[\sum_{j=1}^{2n}\partial_j^2 - |\tau|^2|v|^2 + 2i(J_\tau v, e_j)\partial_j + i\alpha\rho(\tau)\Big] g(v,\tau) = \delta(v)I.$$

By rotation invariance, we can require that $(J_\tau v, e_j)\partial_j g(v,\tau) = 0$, since the previous equation for g is rotation invariant. Then the function $g(v,\tau)$ satisfies

$$\Big\{-\Big[\sum_{j=1}^{2n}\partial_j^2 - |\tau|^2|v|^2\Big] + i\alpha\rho(\tau)\Big\} g(v,\tau) = \delta(v)I.$$

The operator $\rho(\hat\tau)$, $\hat\tau = \frac{\tau}{|\tau|}$ defines a complex structure of W, and its complexification is

$$W^{\mathbb{C}} = W_+^{\mathbb{C}} \oplus W_-^{\mathbb{C}}, \quad W_\pm^{\mathbb{C}} = Ker(\rho(\hat\tau) \pm i),$$

with $\frac{1}{2}(I \pm i\rho(\hat\tau))$ the projection onto $W_\pm^{\mathbb{C}}$ parallel to $W_\mp^{\mathbb{C}}$. Accordingly, we write the functions g and I in terms of the decomposition as

$$\begin{aligned} g(v,\tau) &= \tfrac{1}{2}\big(g_+(v,\tau) + g_-(v,\tau)\big),\\ I &= \tfrac{1}{2}\big((I + i\rho(\hat\tau)) + \big(I - i\rho(\hat\tau)\big)\big),\\ g_\pm(v,t) &= \tfrac{1}{2}(I \pm i\rho(\hat\tau)\big)\, g. \end{aligned}$$

The equation for g then breaks into two equations,

$$\Big(-\Big[\sum_{j=1}^{2n}\partial_j^2 - |\tau|^2|v|^2\Big] + \alpha|\tau|\Big) g_+(v,\tau) = \delta(v)\big(I + i\rho(\hat\tau)\big)$$

and

$$\Big(-\Big[\sum_{j=1}^{2n}\partial_j^2 - |\tau|^2|v|^2\Big] - \alpha|\tau|\Big) g_-(v,\tau) = \delta(v)\big(I - i\rho(\hat\tau)\big).$$

But they are the (matrix-valued) Hermite oscillator equations in $2n$-variables studied in [3] with $\alpha \to \pm\alpha|\tau|$, $\lambda_j \to |\tau|$, so the solution is, for $\alpha|\tau| \notin |\tau|\{-(2n+4k) : k \in Z_+\}$, *i.e.*, $\alpha \notin \{\pm 2(n+2k) : k \in Z_+\}$, formally given by

$$g_+(\nu,\tau) = \frac{|\tau|^{n/2-1}\Gamma(\frac{n}{2}+\frac{\alpha}{4})W_{-\alpha/4,n-1/2}(|\tau|\cdot|\nu|^2)}{4\pi^n|\nu|^n}(I + i\rho(\hat{\tau})),$$

and

$$g_-(\nu,\tau) = \frac{|\tau|^{n/2-1}\Gamma(\frac{n}{2}-\frac{\alpha}{4})W_{\alpha/4,n-1/2}(|\tau|\cdot|\nu|^2)}{4\pi^n|\nu|^n}(I - i\rho(\hat{\tau})).$$

See [3, (12)][1] with our $2n$ corresponding to their n. The solution f is then

$$f = \frac{1}{2}(f_+ + f_-), \quad f_\pm(\nu,t) = \frac{1}{(2\pi)^q}\int_{\mathfrak{t}} e^{i(t,\tau)} g_\pm(\nu,\tau)d\tau.$$

The function f_+ can be written, using the formula for g_+, as a sum of a scalar multiple of I and an integration of the matrix $\rho(\hat{\tau})$, namely

$$f_+(\nu,t) = f_0^{(\alpha)}(\nu,t) + \frac{\Gamma(\frac{n}{2}+\frac{\alpha}{4})}{(2\pi)^q 4\pi^n|\nu|^n}\int_{\mathfrak{t}} |\tau|^{n/2-1}W_{-\alpha/4,n-1/2}(|\tau|\cdot|\nu|^2)i\rho(\hat{\tau})\, e^{i(t,\tau)}d\tau$$

where the scalar function $f_0^{(\alpha)}$ is given as stated. The second integration of $\rho(\hat{\tau})$, apart from the constant factors, can then be expressed as the operator $\rho(\partial_2)$ acting on an integration, namely

$$\int_{\mathfrak{t}} |\tau|^{n/2-1}W_{-\alpha/4,n-1/2}(|\tau|\cdot|\nu|^2)i\rho(\hat{\tau})\, e^{i(t,\tau)}d\tau = \rho(\partial_2)\int_{\mathfrak{t}} |\tau|^{n/2-2}W_{-\alpha/4,n-1/2}(|\tau|\cdot|\nu|^2)\, e^{i(t,\tau)}d\tau.$$

Here we have used the trivial fact that $|\tau|\rho(\hat{\tau}) = \rho(\tau)$ and that $\rho(\partial_2)e^{i(t,\tau)} = i\rho(\tau)e^{i(t,\tau)}$. This proves the formula for f_+, and that for f_- is the same.

The functions g_+ and g_- defines a tempered distribution in (ν,τ) for $|\mathrm{Re}(\alpha)| < 2n$ and has meromorphic continuation to the complex plane with simple poles as indicated, which can be proved by similar methods as the proof of Theorem 2.1 using expansion in terms of Hermite polynomials. Thus the functions f_+ and f_- are also well-defined tempered distributions. This completes the proof of the Theorem 3.3. ∎

We note that there is a discrepancy between the pole set $\Lambda = \{\pm 4(n+2k)\}$ in Section 2 for the Heisenberg group and the pole set $\{\pm 2(n+2k)\}$ in Section 3. This is due to the different normalizations, the J_t operator for the Heisenberg group satisfies $J_t^2 = -4t^2$ instead of $-t^2$.

[1] Note that there is a typo in the formula [3, (12)]. The Whittaker function $W_{-\frac{\alpha}{4\lambda},\frac{n}{2}-\frac{1}{2}}$ should be in the numerator instead of the denominator.

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