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FINITE INTERSECTIONS OF PID OR FACTORIAL OVERRINGS

BY

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ABSTRACT. In this paper we study when an integral domain is a finite intersection of PID or factorial overrings. We show that any Krull domain is the intersection of a PID and a field. We give several sufficient conditions for a Krull domain to be an intersection of two PID or factorial overrings.

Introduction. An integral domain R is *locally factorial* if R_x is factorial for each nonzero, nonunit $x \in R$. In [1], we showed that a locally factorial Krull domain is an intersection of two factorial overrings. In particular, a locally factorial Dedekind domain is an intersection of two PID overrings. It is thus natural to ask when an integral domain is a finite intersection of factorial or PID overrings. Such an integral domain is necessarily a Krull domain. If R is a Krull domain, then R is a locally finite intersection of DVR's. Hence a Krull domain is always an intersection of PID overrings. Such a representation gives R as a finite intersection of DVR's if and only if R is a semilocal PID.

In Section 1, we show that if we do not restrict ourselves to overrings, then any Krull domain is an intersection of two PID's. In a like manner, we characterize integrally closed (resp., completely integrally closed) domains as being an intersection of two Bézout (resp., completely integrally closed Bézout) domains. We also show that if R is a Krull domain, then $R[X]$ is an intersection of two PID overrings. In the second section, we use divisor class group techniques to investigate when a Krull domain is a finite intersection of factorial or PID overrings. Our main result, Theorem 2.7, is that a Krull domain with countable divisor class group is an intersection of two factorial overrings. We know of no Krull domain which is not a finite intersection of factorial or PID overrings. We end the paper with several other related open questions.

Any undefined notions and basic facts about Krull domains may be found in [4] or [6]. Throughout, R will always denote an integral domain with quotient field K. Given a Krull domain R, we will denote its divisor class group by $C\ell(R)$, its set of height-one prime ideals by $X^{(1)}(R)$, and the class of a height-one prime ideal P in $C\ell(R)$ by $[P]$. As usual, an *overring* of R is subring of K which contains R. Finally, given $f(X) = a_0 + a_1X + \ldots + a_nX^n \in R[X]$, its *content* is the ideal $A_f = (a_0, \ldots, a_n)$ of R.

1. **Finite intersections of PID's**. If a DVR V and a field L are both subrings of a larger field F, then it is well-known that $V \cap L$ is again a DVR ([6], Theorem 19.16).

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More generally, suppose that R is a semilocal PID, say $R = V_1 \cap \ldots \cap V_n$, where each V_i is a DVR. Then $R \cap L = (V_1 \cap L) \cap \ldots \cap (V_n \cap L)$ is again a finite intersection of DVR's, and hence is also a semilocal PID. However, if R is a PID and L is a field, then $R \cap L$ need not be a PID, although it is a Krull domain. In fact, we next show that any Krull domain is an intersection of a PID and a field. For the basic facts about Kronecker function rings which we will use in our next several proofs, one may consult ([6], Sections 32, 34, and 44).

THEOREM 1.1. *Let R be an integral domain. Then the following statements are equivalent.*

(1) R is a Krull domain.

(2) R is an intersection of two PID's.

(3) R is an intersection of a PID and a field.

PROOF. Clearly (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (3), we note that the Kronecker function ring of R with respect to the v-operation, $R^v = \{f/g \mid f, g \in R[x], g \neq 0, (A_f)_v \subset (A_g)_v\}$, is a PID ([6], Corollary 44.12) and $R = R^v \cap K$ ([6], Theorem 32.7). □

REMARK 1.2. If R is a Krull domain, then R^v is actually a Euclidean domain ([3], Corollary 5.2 and Theorem 5.3). Also, R^v is a localization of $R[X]$, namely $R^v = R[X]_S$, where $S = \{f \in R[X] \mid (A_f)_v = R\}$ ([5], Theorem 2.5).

In a similar manner, the properties of being integrally closed or completely integrally closed may be characterized in terms of finite intersections of Bézout domains.

THEOREM 1.3. *Let R be an integral domain. Then:*

(1) The following three statements are equivalent.

(a) R is integrally closed.

(b) R is an intersection of two Bézout domains.

(c) R is an intersection of a Bézout domain and a field.

(2) The following three statements are equivalent.

(a) R is completely integrally closed.

(b) R is an intersection of two completely integrally closed Bézout domains.

(c) R is an intersection of a completely integrally closed Bézout domain and a field.

PROOF. (1) This follows as in Theorem 1.1 from the fact that if R is integrally closed, then the Kronecker function ring R^b is a Bézout domain and $R = R^b \cap K$ ([6], Theorem 32.7).

(2) We need only show that if R is completely integrally closed, then R^v is also completely integrally closed. For then, as in (1) above, R^v will be a completely integrally closed Bézout domain and $R = R^v \cap K$. So let $f, g \in R[X]$ with f/g almost integral over R^v. Then there is a nonzero $h \in R[X]$ such that $h(f/g)^n \in R^v$ for each $n \geq 1$. Hence $(A_{hf^n})_v \subset (A_{g^n})_v$. Now $A_h(A_f)_v^n \subset (A_h(A_f)_v^n)_v = (A_{hf^n})_v \subset (A_{g^n})_v = (A_g)_v^n$. Let $0 \neq r \in A_h$. Then $r((A_f)^n((A_g)^{-1})^n)_v \subset R$, so $r(A_f)^n\,(A_g)^{-n} \subset R$. Let $x \in A_f$ and $y \in (A_g)^{-1}$. Then $r(xy)^n \in R$ for each $n \geq 1$. Hence $xy \in R$ since R is completely

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integrally closed. Thus $A_f(A_g)^{-1} \subset R$. Since R is completely integrally closed, the set of v-ideals of R forms a group ([6], Theorem 34.3). Hence $(A_f)_v \subset (A_g)_v$, i.e., $f/g \in R^v$. □

For polynomial rings, the Kronecker function ring techniques yield better results.

THEOREM 1.4. *Let R be an integral domain. Then:*

(1) If R is integrally closed (resp., completely integrally closed), then $R[X]$ is an intersection of two Bézout (resp., completely integrally closed Bézout) domain overrings.

(2) If R is a Krull domain, then $R[X]$ is an intersection of two PID overrings. Moreover, each PID overring may be chosen to be a Euclidean domain which is a localization of $R[X]$.

PROOF. It is sufficient to show that $R[X] = R^* \cap K[X]$ for any e.a.b. $*$-operation on R (see [6], Section 32, for relevant definitions). For if R is integrally closed, then $R[X] = R^b \cap K[X]$; while if R is completely integrally closed, then $R[X] = R^v \cap K[X]$. As in the proof of Theorem 1.3, R^b is a Bézout domain and R^v is a completely integrally closed Bézout domain. If R is a Krull domain, the R^v is a PID. The "moreover" statement follows form Remark 1.2.

Clearly $R[X] \subset R^* \cap K[X]$. Conversely, let $h = c_0 + c_1X + \dots + c_nX^n \in R^* \cap K[X]$. Then $h = (a_0 + a_1X + \dots + a_nX^n)/b$ for some $a_0, a_1, \dots, a_n, b \in R$. Since R^* is well-defined, $(a_0, \dots, a_n) \subset (a_0, \dots, a_n)^* \subset (b)^* = (b)$. Thus each a_i is divisible by b, so $h \in R[X]$. □

It then follows immediately that for a Krull domain R, any localization or subintersection of $R[X]$ is also an intersection of two PID overrings. In particular, this holds for the two rings $R(X)$ and $R\langle X\rangle$. Recall that $R(X) = R[X]_S$ and $R\langle X\rangle = R[X]_U$, where $S = \{f \in R[X] | A_f = R\}$ and $U = \{f \in R[X] | f \text{ is monic}\}$.

REMARK 1.5. If R is a Krull domain, then we have seen that $R[X] = R[X]_S \cap R[X]_T$, where $S = \{f \in R[X] | (A_f)_v = R\}$ and $T = R\backslash\{0\}$ (so $R[X]_T = K[X]$), and each localization is a PID. A natural question is whether this result extends to power series rings. Let R be a Krull domain, $S = \{f \in R[[X]] | (A_f)_v = R\}$, and $T = R\backslash\{0\}$. Then it may be shown that $R[[X]] = R[[X]]_S \cap R[[X]]_T$ and that $R[[X]]_S$ is a PID. However, $R[[X]]_T$ need not be factorial, let alone a PID. For example, if R is factorial but $R[[X]]$ is not factorial (for such a Krull domain, see [4], page 118), then $R[[X]]_T$ can not be factorial by Nagata's Theorem (Theorem 2.1).

2. **Finite intersections of factorial overrings**. In this section, we investigate when a Krull domain R is an intersection of a finite number of factorial subintersections. Let R be a Krull domain with $X = X^{(1)}(R)$ its set of height-one prime ideals. For $Y \subset X$, $R_Y = \bigcap_{P \in Y} R_P$ is also a Krull domain, and is called a *subintersection* of R. Note that $R = R_{Y_1} \cap \dots \cap R_{Y_n}$ if and only if $X = Y_1 \cup \dots \cup Y_n$. This fact follows from the approximation theorem for Krull domains ([4], Theorem 5.8), and it will be used implicitly in the proofs of Theorem 2.5 and Theorem 2.10. Our main tool will be Nagata's Theorem ([4], Theorem 7.1), which relates $C\ell(R)$ to $C\ell(R_Y)$. For future

reference, we include Nagata's Theorem.

THEOREM 2.1. (Nagata's Theorem). *Let R be a Krull domain and R_Y a subintersection of R. Then the natural homomorphism $\phi:C\ell(R) \to C\ell(R_Y)$ is surjective and ker ϕ is generated by $\{[P]|P \in X - Y\}$.*

Recall that each localization R_S is a subintersection; in fact $R_S = R_Y$, where $Y = \{P|P \cap S = \phi\}$. Thus ker $(C\ell(R) \to C\ell(R_S))$ is generated by $\{[P]|P \in X$ and $P \cap S \neq \phi\}$. Also recall that if R is a Dedekind domain, then each overring of R is a subintersection and factorial overrings are just PID's. Our next lemma, an easy consequence of Nagata's Theorem, will prove very useful.

LEMMA 2.2. *Let R be a Krull domain and $P_1, \ldots, P_n \in X = X^{(1)}(R)$. Then $C\ell(R)$ is generated by the classes of $X \backslash \{P_1, \ldots, P_n\}$.*

PROOF. Let $Y = \{P_1, \ldots, P_n\}$. Then $R_Y = R_{P_1} \cap \ldots \cap R_{P_n}$ is a semilocal PID, and hence $C\ell(R_Y) = 0$. By Nagata's Theorem the kernel of $C\ell(R) \to C\ell(R_Y)$, which is $C\ell(R)$, is generated by the classes of $X \backslash Y$. □

As mentioned earlier, in ([1], Theorem 2.3 and Proposition 6.1) we showed that if R is a locally factorial Krull domain, then R is an intersection of two factorial overrings. Those proofs actually show that if R is a locally factorial Krull domain, then $R = R_x \cap R_y$ for some nonzero, nonunits $x, y \in R$ with R_x and R_y each factorial. Our next theorem shows to what extent this property characterizes locally factorial Krull domains.

THEOREM 2.3. *Let R be an integral domain. Then the following statements are equivalent.*

(1) R is a Krull domain and $C\ell(R)$ is finitely generated.

(2) There are nonzero $x, y \in R$ such that $R = R_x \cap R_y$ and R_x and R_y are each factorial.

(3) There are nonzero $x_1, \ldots, x_n \in R$ such that $R = R_{x_1} \cap \ldots \cap R_{x_n}$ and each R_{x_i} is factorial.

PROOF. Clearly (2) ⇒ (3). If R_x is factorial for some nonzero $x \in R$, then by Nagata's Theorem $C\ell(R)$ is finitely generated since x is contained in only a finite number of height-one prime ideals. Thus (3) ⇒ (1). So we show that (1) ⇒ (2). We may assume that R is not factorial, and hence R has infinitely many height-one prime ideals. Suppose that $[P_1], \ldots, [P_n]$ generate $C\ell(R)$. Choose $0 \neq x \in P_1 \cap \ldots \cap P_n$. We may assume that these are the only height-one prime ideals that contain x. Since $C\ell(R) = \langle [P]|P \in X^{(1)}(R) \backslash \{P_1, \ldots, P_n\} \rangle$ by Lemma 2.2, there are height-one prime ideals $Q_1, \ldots, Q_m$ of R, distinct from $P_1, \ldots, P_n$, whose classes generate $C\ell(R)$. Choose $y \in (Q_1 \cap \ldots \cap Q_m) \backslash (P_1 \cup \ldots \cup P_n)$. By Nagata's Theorem, R_x and R_y are each factorial. Since x and y belong to no common height-one prime ideals, we have $(x, y)_v = R$. Thus $R = R_x \cap R_y$ ([1], Lemma 2.1). □

REMARK 2.4. If R is a Dedekind domain, then $C\ell(R)$ is finitely generated if and only

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if either (2) or (3) holds with each R_{x_i} a PID. However, one may have $R = R_x \cap R_y$ with R_x and R_y each a PID, but R not a Dedekind domain. For example, let R be a two-dimensional local factorial integral domain.

We next determine conditions on $X^{(1)}(R)$ so that a Krull domain R will be an intersection of a finite number of factorial subintersections.

THEOREM 2.5. *Let R be a Krull domain with $X = X^{(1)}(R)$ and $G = C\ell(R)$. Then the following statements are equivalent.*

(1) $X = X_1 \cup \ldots \cup X_n$, $X_i \cap X_j = \phi$ for $i \neq j$, and $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$.

(2) $X = X_1 \cup \ldots \cup X_n$ and $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$.

(3) $R = R_1 \cap \ldots \cap R_n$, where each R_i, $i = 1, \ldots, n$, is a factorial subintersection of R.

Moreover, if R is a Dedekind domain, then in (3) each R_i is a PID overring of R.

PROOF. Clearly (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (3), let $R_i = R_{X_i}$ for each $i = 1, \ldots, n$. Then $R = R_1 \cap \ldots \cap R_n$ since $X = X_1 \cup \ldots \cup X_n$. By Nagata's Theorem, each R_i is factorial. Finally, suppose that (3) holds. Since each R_i is a subintersection, $R_i = R_{Y_i}$ for some $Y_i \subset X$. Then $X = Y_1 \cup \ldots \cup Y_n$ since $R = R_1 \cap \ldots \cap R_n$. Define $X_i = Y_i \backslash (X_1 \cup \ldots \cup X_{i-1})$ for each $i = 1, \ldots, n$. Then $X = X_1 \cup \ldots \cup X_n$, and the X_i's are pairwise disjoint. Since each $R_i = R_{Y_i}$ is factorial and $X_i \subset Y_i$, thus $R_{Y_i} \subset R_{X_i}$ and hence each R_{X_i} is factorial. Again, by Nagata's Theorem $G = \langle [P] | P \in X \backslash X_i \rangle$ for each $i = 1, \ldots, n$. □

Theorem 2.5 motivates our next observation.

PROPOSITION 2.6. *Suppose that R is an intersection of n factorial or PID subintersections of R. Then any subintersection of R is also an intersection of n such overrings. In particular, if a Dedekind domain R is an intersection of n PID overrings, then any overring of R is also an intersection of n PID overrings.*

PROOF. Suppose that $R = R_1 \cap \ldots \cap R_n$, where each $R_i = R_{Y_i}$ is a factorial subintersection of R. Let R_Y be a subintersection of R. Next, let $Y_i' = Y_i \cap Y$ and $R_i' = R_{Y_i'}$. Then $R_Y = R_1' \cap \ldots \cap R_n'$, and each R_i' is factorial since $Y_i' \subset Y_i$. □

If R is a Krull domain with finitely generated divisor class group, then by Theorem 2.3 R is an intersection of two factorial subintersections. Our next theorem extends this to Krull domains with countably generated divisor class groups.

THEOREM 2.7. *Let R be a Krull domain with $G = C\ell(R)$ countable. Then R is an intersection of two factorial subintersections. In addition, if R is a Dedekind domain, then R is an intersection of two PID overrings.*

PROOF. By Theorem 2.3, we may assume that G is infinite. Let $G = \{x_n\}_{n=1}^{\infty}$. Now $x_1 \in \langle [P] | P \in Y_1 \rangle$ for some finite $Y_1 \subset X = X^{(1)}(R)$. By Lemma 2.2, also $x_1 \in \langle [P] | P \in Z_1 \rangle$ for some finite $Z_1 \subset X$ with $Y_1 \cap Z_1 = \phi$. Continuing this process, for each positive integer n there are disjoint finite subsets Y_n and Z_n of X, each pairwise disjoint from the previously defined Y_i's and Z_i's, such that $x_n \in \langle [P] | P \in Y_n \rangle$ and $x_n \in \langle [P] | P \in Z_n \rangle$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$ and $Z = \bigcup_{n=1}^{\infty} Z_n$. Then $Y \cap Z = \phi$ and $G = \langle [P] | P \in Y \rangle = \langle [P] | P \in Z \rangle$. Next, let $X_1 = Y$ and $X_2 = X \backslash X_1 \supset Z$. Then $X = X_1 \cup X_2$, $X_1 \cap X_2 = \phi$, and $G = \langle [P] | P \in X_i \rangle$ for $i = 1, 2$. Hence by Theorem 2.5, R is an intersection of two factorial subintersections. □

REMARK 2.8. In particular, Theorem 2.7 is applicable if either R or $X^{(1)}(R)$ is countable. We know of no Krull domain which is not an intersection of two factorial (in fact, PID) subintersections. Thus the countable version of Claborn's construction of Dedekind domains ([2], Theorem 2.1, or [4], Theorem 15.18) can not be used to obtain a Dedekind domain which is not an intersection of two PID overrings.

Our next example shows that $C\ell(R)$ alone cannot be used to determine if there are any Krull domains which are not a finite intersection of PID overrings.

EXAMPLE 2.9. *Let G be any abelian group. Then for any positive integer n (or ∞), there is a Krull domain R of dimension n with $C\ell(R) = G$ which is an intersection of two PID overrings, each of which is a localization of R. We do the case when $n = 1$ (i.e., R is a Dedekind domain); the case when $n > 1$ then follows easily from Theorem 1.4. By Claborn's Theorem ([4], Theorem 14.10) there is a Dedekind domain A with $C\ell(A) = G$. Then $R = A\langle X \rangle$ is a Dedekind domain ([8], Proposition 2.3), and it may be shown that $C\ell(R) = G$. By the remark after Theorem 1.4, R is an intersection of two PID overrings, each of which is a localization of R. Alternately, ([7], Theorem 2.3) may be used to construct a Dedekind domain R with $C\ell(R) = G$ which is an intersection of two PID overrings.*

Our final two results may be viewed as companion theorems to Theorems 2.3 and 2.7.

THEOREM 2.10. *Let R be a Krull domain (resp., Dedekind domain) with $G = C\ell(R)$. Then the following statements are equivalent.*

(1) G is finitely generated.

(2) $R = R_1 \cap R_2$, where R_1 and R_2 are subintersections with R_1 factorial (resp., a PID) and R_2 a semilocal PID.

PROOF. (1) $\Rightarrow$ (2). Say that $[P_1], \ldots, [P_n]$ generate G. Let $R_2 = R_{P_1} \cap \ldots \cap R_{P_n}$ and $R_1 = R_Y$ for $Y = X^{(1)}(R) \backslash \{P_1, \ldots, P_n\}$. Clearly $R = R_1 \cap R_2$ and R_2 is a semilocal PID. Also, by Nagata's Theorem R_1 is factorial. (2) $\Rightarrow$ (1). Let $R_2 = R_{P_1} \cap \ldots \cap R_{P_n}$ and $R_1 = R_Z$ for some $P_1, \ldots, P_n \in X = X^{(1)}(R)$ and $Z \subset X$. Also, let $Y = X \backslash \{P_1, \ldots, P_n\}$. Then $Y \subset Z$, and thus R_Y is factorial. Hence $G = \langle [P_1], \ldots, [P_n] \rangle$ by Nagata's Theorem again. □

THEOREM 2.11. *Let R be a two-dimensional Krull domain with only a finite number of height-two maximal ideals, and let $G = C\ell(R)$. Then:*

(1) If G is finitely generated, then R is an intersection of a PID overring and a semilocal PID overring.

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(2) If G is countable, then R is an intersection of two PID overrings and a semilocal PID overring.

PROOF. Let $M_1, \ldots, M_n$ be the height-two maximal ideals of R. Choose $0 \neq x \in M_1 \cap \ldots \cap M_n$. Let $P_1, \ldots, P_k$ be the height-one prime ideals of R which contain x. Then $R = R_x \cap R_{P_1} \cap \ldots \cap R_{P_k}$. Now R_x is a Dedekind domain and $R_{P_1} \cap \ldots \cap R_{P_k}$ is a semilocal PID. Parts (1) and (2) now follow from Theorems 2.10 and 2.7, respectively. □

We close with several open questions.

QUESTION 1. Is each integrally closed (resp., completely integrally closed) domain an intersection of two Bézout (resp., completely integrally closed Bézout) domain overrings?

QUESTION 2. Is each Krull domain an intersection of a finite number of PID or factorial overrings?

QUESTION 3. If R is a finite intersection of PID or factorial overrings, is R actually an intersection of two such overrings?

There are also several other variants of the above questions. For example, we may ask if the overrings may be chosen to be localizations, subintersections, or flat overrings of R.

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