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THE n-DIMENSIONAL DISTRIBUTIONAL HANKEL TRANSFORMATION

E. L. KOH

1. Introduction. The Hankel transformation was extended to certain generalized functions of one dimension [**1**; **2**; **3**]. In this paper, we develop the n-dimensional case corresponding to [**1**]. The procedure in [**1**] is briefly as follows:

A test function space H_μ is constructed on which the μth order Hankel transformation h_μ defined by

$$h_\mu \phi = \int_0^\infty \phi(x)\ (xy)^{1/2} J_\mu(xy)dx, \quad \phi \in H_\mu$$

is an automorphism whenever $\mu \geqq -1/2$. The generalized transformation h_μ' is then defined on the dual H_μ' as the adjoint of h_μ through a Parseval relation, i.e.

$$\langle h_\mu' f, \phi \rangle = \langle f, h_\mu \phi \rangle, \qquad \phi \in H_\mu, f \in H_\mu'.$$

This definition coincides with the classical Hankel transformation when f is a regular distribution corresponding to an L_1 function.

We shall use the following notations. R^n and C^n are respectively the real and complex n-dimensional euclidean spaces. An n-tuple will be denoted by $z = \{z_1, \ldots, z_n\}$. For our purpose, we shall restrict x and y to the first orthant of R^n which we denote by I. Thus, $I = \{x \in R^n : 0 < x_\nu < \infty, \nu = 1, \ldots, n\}$. We shall use the usual euclidean norm, $|x| = [\sum_{\nu=1}^n x_\nu^2]^{1/2}$. A function on a subset of R^n shall be denoted by $f(x) = f(x_1, x_2, \ldots, x_n)$. By $[x]$, we mean the product $x_1 x_2 \ldots x_n$. Thus $[x^m] = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ where $m = \{m_1, m_2, \ldots, m_n\}$. The notations $x \leqq y$ and $x < y$ mean respectively $x_\nu \leqq y_\nu$ and $x_\nu < y_\nu$ ($\nu = 1, 2, \ldots, n$). The letters k and m shall denote nonnegative integers in R^n, i.e., k_ν and m_ν are nonnegative integers. Letting $(k) = k_1 + k_2 + \ldots + k_n, D_x^k$ shall denote

$$(1) \qquad \frac{\partial^{(k)}}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$$

while $(x^{-1}D_x)^k$ denotes

$$(2) \qquad \prod_{\nu=1}^n \left(x_\nu^{-1} \frac{\partial}{\partial x_\nu} \right)^{k_\nu}.$$

Other operators will be defined later when their uses arise.

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By a smooth function, we mean a function that possesses partial derivatives of all orders at all points of its domain.

2. The testing function space H_μ and its dual. Let μ be a fixed number in $(-\infty, \infty)$. We define H_μ to be the space of smooth complex-valued functions $\phi(x)$ which are defined on I and such that for each pair of nonnegative integers m and k in R^n

$$(3) \qquad \gamma^{\mu}_{m,k}\,(\phi(x)) = \sup_{x \in I} |[x^m](x^{-1}D_x)^k[x]^{-\mu-1/2}\,\phi(x)| < \infty .$$

Since $\phi(x)$ is smooth, the order of differentiation in $(x^{-1}D_x)^k$ is immaterial; thus

$$\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)\left(x_j^{-1}\frac{\partial}{\partial x_j}\right) = \left(x_j^{-1}\frac{\partial}{\partial x_j}\right)\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)$$

for all $i, j = 1, \ldots, n$.

H_μ is a vector space. Since $\gamma^{\mu}_{m,0}$ are norms, we have a separating collection of seminorms, i.e. a multinorm. (An equivalent topology for H_μ may be given by the multinorm $\{\rho_r^{\mu}\}$ with

$$\rho_r^{\mu}(\phi) = \max_{0 \leq m,k \leq r} \gamma^{\mu}_{m,k}\,(\phi), \quad r = (r_1, r_2, \ldots, r_n).)$$

As k and m traverse a countable index set, H_μ is, in fact, a countably multinormed space. We say that a sequence $\{\phi_\nu\}$ is Cauchy in H_μ if $\phi_\nu \in H_\mu$ for all ν and for every m, k, $\gamma^{\mu}_{m,k}(\phi_\nu - \phi_\lambda) \to 0$ as ν and $\lambda \to \infty$ independently.

LEMMA 1. *If $\phi(x) \in H_\mu$, $D_x^k\phi(x)$ is of rapid descent for each k.*

Proof. Since

$$\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)^{k_i} x_i^{-\mu-1/2}\phi(x_1, \ldots, x_i, \ldots, x_n) = x_i^{-2k_i}x_i^{-\mu-1/2}\sum_{j=0}^{k_i} b_j x_i^{\,j} \times \left(\frac{\partial}{\partial x_i}\right)^j \phi,$$

we have

$$(4) \qquad (x^{-1}D_x)^k[x]^{-\mu-1/2}\phi(x) = [x^{-2k}][x]^{-\mu-1/2}\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n} b_j[x^j] \times \frac{\partial^{j_1+\cdots+j_n}\phi}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}$$

where the b_j are appropriate constants. Now consider $\phi \in H_\mu$. By $\gamma^{\mu}_{m,0}(\phi) < \infty$, we have $\sup_I |[x^m][x]^{-\mu-1/2}\phi| < \infty$. Therefore, $[x^m]\phi \to 0$ as $|x| \to \infty$ for each

m. To show $[x^m]D_x\phi \to 0$ as $|x| \to \infty$, we observe that

$$\gamma^{\mu}_{m,1}(\phi) = \sup_{\substack{I \\ i=1,\dots,n}} \left| [x^m][x]^{-\mu-1/2} x_i^{-1} \frac{\partial}{\partial x_i} \phi \right| < \infty .$$

Finally, by induction on k and using (4), we have

$$\gamma^{\mu}_{m,k}(\phi) < \infty \Rightarrow [x^m]D_x^{\,k}\phi \to 0 \qquad \text{as } |x| \to \infty .$$

LEMMA 2. *If q is an even positive integer ($\in R^1$), then $H_{\mu+q} \subset H_\mu$ and convergence in $H_{\mu+q}$ implies convergence in H_μ.*

Proof. It is easy to show that

$$\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)^{k_i}(x_i^{-\mu-1/2}\phi) = \left(1 + \frac{x_i}{2k_i}\frac{\partial}{\partial x_i}\right)\left(2k_i\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)^{k_i-1} x_i^{-\mu-5/2}\phi\right).$$

Hence, we have the operational identity

$$(5) \qquad \prod_{i=1}^{n}\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)^{k_i} x_i^{-\mu-1/2} = \prod_{i=1}^{n}\left(1 + \frac{x_i}{2k_i}\frac{\partial}{\partial x_i}\right)\left(2k_i\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)^{k_i-1} x^{-\mu-5/2}\right).$$

Let

$$\Delta_i = \frac{x_i}{2k_i}\frac{\partial}{\partial x_i}.$$

The right hand side of (5) is a sum of 2^n terms of Δ_i multiplied by the operator $2^n[k](x^{-1}(\partial/\partial x))^{k-n}[x]^{-\mu-5/2}$, i.e.

$$\left(1 + \sum_{i=1}^{n}\Delta_i + \sum_{i\neq j}\Delta_i\Delta_j + \dots + \Delta_1\Delta_2\dots\Delta_n\right)\cdot 2^n[k] \times \left(x^{-1}\frac{\partial}{\partial x}\right)^{k-n}[x]^{-\mu-5/2}.$$

If now we evaluate $\gamma^{\mu}_{m,k}(\phi) = \sup_I |[x^m](x^{-1}D_x)^k[x]^{-\mu-1/2}\phi|$ we have

$$\gamma^{\mu}_{m,k}(\phi) \leqq 2^n[k]\gamma^{\mu+2}_{m,k-n}(\phi) + C_1\gamma^{\mu+2}_{m+1,k-n+1}(\phi) + \dots + C_j\gamma^{\mu+2}_{m+n,k}(\phi)$$

where $C_1, \dots, C_j$ are constants. The lemma follows by induction on q.

LEMMA 3. *H_μ is sequentially complete.*

Proof. Let $\{\phi_\nu\}_{\nu=1}^{\infty}$ converge in H_μ. Using the seminorms $\gamma^{\mu}_{0,k}$ and the relation (4), we have by induction on k_i that for each $k = \{k_1, \dots, k_n\}$ the sequence of partial derivatives $\{D_x^{\,k}\phi_\nu\}_{\nu=1}^{\infty}$ converges uniformly on every compact subset of I. Therefore, there exists a smooth function ϕ on I such that for each k and x, $D_x^{\,k}\phi_\nu(x) \to D_x^{\,k}\phi(x)$ as $\nu \to \infty$. Again, since $\{\phi_\nu\}$ is a Cauchy sequence, for each m and k and a given $\epsilon > 0$ there is a positive number $N_{m,k}$ such that for every $\nu, \eta > N_{m,k}$,

(6) $\quad \gamma^{\mu}_{m,k}(\phi_\nu - \phi_\eta) < \epsilon.$

Passing to the limit as $\eta \to \infty$, we have $\gamma^{\mu}_{m,k}(\phi_\nu - \phi) \leqq \epsilon$ for all $\nu > N_{m,k}$, i.e.

(7) $\quad \gamma^{\mu}_{m,k}(\phi_\nu - \phi) \to 0 \qquad \text{as } \nu \to \infty.$

To complete the proof, we show that $\phi \in H_\mu$ as follows: it is clear that

(8) $\quad \gamma^{\mu}_{m,k}(\phi) \leqq \gamma^{\mu}_{m,k}(\phi_\nu) + \gamma^{\mu}_{m,k}(\phi_\nu - \phi).$

By (7) and the fact that $\gamma^{\mu}_{m,k}(\phi_\nu) < \infty$ for all ν, it follows from (8) that $\gamma^{\mu}_{m,k}(\phi) < \infty$.

H_μ is therefore a Frèchet space, i.e. a complete countably multinormed space. Its dual is denoted by H_μ'. It follows that H_μ' is also complete [**4**, Theorem 1.8-3].

The following properties are immediate extensions of the one-dimensional case, using the relation (4) whenever called for.

1. $\mathscr{D}(I)$, the space of smooth functions with compact support on I, is a subspace of H_μ for every choice of μ. Convergence in $\mathscr{D}(I)$ implies convergence in H_μ. Thus, the restriction of any $f \in H_\mu'$ to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$. However $\mathscr{D}(I)$ is not dense in H_μ.

2. For each μ, H_μ is a subspace of $\mathscr{E}(I)$, the space of smooth functions on I. H_μ is dense in $\mathscr{E}(I)$. Moreover, the topology of H_μ is stronger than that induced on it by $\mathscr{E}(I)$. It follows that $\mathscr{E}'(I)$ is a subspace of H_μ'.

3. The complex number that $f \in H_\mu'$ assigns to $\phi \in H_\mu$ is denoted by $\langle f, \phi\rangle$. We assign to H_μ' the weak topology generated by the seminorms

$$\eta_\phi(f) = |\langle f, \phi\rangle| \qquad \text{where } \phi \in H_\mu.$$

For each $f \in H_\mu'$, there exist a positive constant C and a non-negative integer r such that

$$|\langle f, \phi\rangle| \leqq C\rho_r{}^\mu(\phi) \qquad \phi \in H_\mu.$$

Recall that $\rho_r{}^\mu = \max_{0 \leqq m,k \leqq r} \gamma^{\mu}_{m,k}(\phi)$.

4. Let $f(x)$ be a locally Lebesgue integrable function on I such that $f(x)$ is of slow growth as $|x| \to \infty$ and $[x]^{\mu+1/2}f(x)$ is absolutely integrable on $0 < x_\nu < 1$, $\nu = 1, 2, \ldots, n$. Then $f(x)$ generates a regular generalized function f in H_μ' defined by

$$\langle f, \phi\rangle = \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_n)\phi(x_1, \ldots, x_n)dx_1 dx_2 \ldots dx_n, \ \phi \in H_\mu.$$

This statement follows from the mean value theorem for n-dimensional integrals (see [**5**, p. 155]) and the fact that ϕ is of rapid descent.

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3. Operations on H_μ and H_μ'.

LEMMA 4. *For any positive or negative integer n and for any μ, the mapping $\phi(x) \to [x]^n\phi(x)$ is an isomorphism from H_μ onto $H_{\mu+n}$. Thus, the operator $f(x) \to [x]^n f(x)$ which is defined by*

$$\langle [x]^n f(x), \phi(x)\rangle = \langle f(x), [x]^n \phi(x)\rangle$$

is an isomorphism from $H_{\mu+n}'$ onto H_μ'.

Proof. If $\phi \in H_\mu$ then

$$\gamma_{m,k}^{\mu+n}([x]^n\phi) = \sup_I |[x^m](x^{-1}D_x)^k[x]^{-\mu-1/2-n}[x]^n\phi|$$
$$= \gamma_{m,k}^{\mu}(\phi).$$

We now define the following operators on H_μ:

$$N_{i\mu} = x_i^{\mu+1/2}\frac{\partial}{\partial x_i}x_i^{-\mu-1/2}$$

$$N_\mu = N_{1\mu}N_{2\mu}\ldots N_{n\mu} = [x]^{\mu+1/2}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{-\mu-1/2}$$

$$M_{i\mu} = x_i^{-\mu-1/2}\frac{\partial}{\partial x_i}x_i^{\mu+1/2}$$

$$M_\mu = M_{1\mu}M_{2\mu}\ldots M_{n\mu} = [x]^{-\mu-1/2}\frac{\partial^n}{\partial x_1\ldots\partial x_n}[x]^{\mu+1/2}.$$

Also, we define an inverse operator to N_μ as follows:

$$N_{1\mu}^{-1}\phi = x_i^{\mu+1/2}\int_\infty^{x_1} t^{-\mu-1/2}\phi(t, x_2, \ldots, x_n)dt$$

$$N_{2\mu}^{-1}\phi = x_2^{\mu+1/2}\int_\infty^{x_2} t^{-\mu-1/2}\phi(x_1, t, \ldots, x_n)dt$$

and so on.

$$N_\mu^{-1}\phi = N_{1\mu}^{-1}N_{2\mu}^{-1}\ldots N_{n\mu}^{-1}\phi$$
$$= [x]^{\mu+1/2}\int_\infty^{x_1}\ldots\int_\infty^{x_n}[t]^{-\mu-1/2}\phi(t)dt_n\ldots dt_1.$$

That N_μ^{-1} is truly the inverse to N_μ follows from the fact that ϕ is smooth and of rapid descent.

LEMMA 5. *$\phi \to N_\mu\phi$ is an isomorphism from H_μ onto $H_{\mu+1}$.*

Proof.

$$\gamma_{m,k}^{\mu+1}(N_\mu\phi) = \sup_I |[x^m](x^{-1}D_x)^k[x]^{-\mu-1-1/2}N_\mu\phi|$$

$$= \sup_I |[x^m](x^{-1}D_x)^{k+n}[x]^{-\mu-1/2}\phi| = \gamma_{m,k+n}^{\mu}(\phi).$$

This shows that N_μ is a continuous linear mapping of H_μ into $H_{\mu+1}$. To complete the proof, let $\phi \in H_{\mu+1}$. Let k be a fixed integer in R^n. Then

$$(x^{-1}D_x)^k[x]^{-\mu-1/2}N_\mu^{-1}\phi = (x^{-1}D_x)^k \int_\infty^{x_1} \dots \int_\infty^{x_n} [t]^{-\mu-1/2}\phi(t)dt_n \dots dt_1$$

$$= \left(x_1^{-1}\frac{\partial}{\partial x_1}\right)^{k_1-1} \dots \left(x_n^{-1}\frac{\partial}{\partial x_n}\right)^{k_n-1}[x]^{-\mu-3/2}\phi(x),$$

$$k_\nu \geqq 1.$$

Hence

$$(9) \qquad \gamma_{m,k}^{\mu}(N_\mu^{-1}\phi) = \gamma_{m,k-n}^{\mu+1}(\phi) \qquad \text{for } m = 0, 1, 2, \dots; k_\nu \geqq 1.$$

For $k_\nu = 0$, for all ν:

$$|x_1^{m_1}\dots x_n^{m_n}(x_1\dots x_n)^{-\mu-1/2}N_\mu^{-1}\phi| \leqq x_1^{m_1}\dots x_n^{m_n} \times \int_{x_1}^\infty \dots \int_{x_n}^\infty |(t_1\dots t_n)^{-\mu-1/2}\phi(t)|dt_n\dots dt_1$$

$$\leqq \int_{x_1}^\infty \dots \int_{x_n}^\infty \left| \frac{1}{t_1^2+1}(t_1^{m_1+1}+t_1^{m_1+3})\dots\frac{1}{t_n^2+1} \times (t_1^{m_n+1}+t_n^{m_n+3})(t_1\dots t_n)^{-\mu-3/2}\phi\right| dt_n \dots dt_1$$

$$\leqq \int_0^\infty \frac{dt_1}{t_1^2+1}\dots\int_0^\infty\frac{dt_n}{t_n^2+1}\sup_I |(t_1^{m_1+1}\dots t_n^{m_n+1}+\dots+t_1^{m_1+3}\dots t_n^{m_n+3}) \times (t_1\dots t_n)^{-\mu-3/2}\phi(t)|$$

Therefore,

$$(10) \quad \gamma_{m,0}^{\mu}(N_\mu^{-1}\phi) \leqq \frac{\pi^n}{2^n}[\gamma_{m+n,0}^{\mu+1}(\phi)+\dots+\gamma_{m+3n,0}^{\mu+1}(\phi)|, \quad m = 0, 1, 2, \dots.$$

Finally, for the case where some but not all k_ν are zero, a similar inequality to (10) can easily be obtained. It follows then that $\phi \to N_\mu^{-1}\phi$ is a continuous linear mapping of $H_{\mu+1}$ into H_μ. Since N_μ and N_μ^{-1} are inverses, these mappings are one-to-one. Therefore, N_μ is an isomorphism from H_μ onto $H_{\mu+1}$.

LEMMA 6. *$\phi \to M_\mu\phi$ is a continuous linear mapping of $H_{\mu+1}$ onto H_μ.*

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Proof. For $\phi \in H_{\mu+1}$ and each pair of m, k,

$$\gamma^{\mu}_{m,k}\,(M_\mu\phi) = \sup_I \left| [x^m](x^{-1}D_x)^k[x]^{-2\mu-1}\frac{\partial^n}{\partial x_1 \ldots \partial x_n}\,[x]^{\mu+1/2}\phi\right|$$

$$= \sup_I |(2\mu+2)^n[x^m](x^{-1}D_x)^k[x]^{-\mu-3/2}\phi$$

$$+ [x^m](x^{-1}D_x)^k[x]^2(x^{-1}D_x)[x]^{-\mu-3/2}\phi|$$

$$= \sup_I \left| (2\mu+2)^n[x^m](x^{-1}D_x)^k[x]^{-\mu-3/2}\phi \right.$$

$$\left. + [x^m]\prod_{i=1}^{n}\left(2k_i + x_i^2\left(x_i^{-1}\frac{\partial}{\partial x_i}\right)\right)(x^{-1}D_x)^k[x]^{-\mu-3/2}\phi\right|$$

(11) $$\leqq \{(2\mu+2)^n + 2^n[k]\}\gamma^{\mu+1}_{m,k}\,(\phi) + \sum_{i=1}^{n} 2^{n-i}C_i(k)\gamma^{\mu+1}_{m+2i,k+1}\,(\phi),$$

where $C_i(k)$ are appropriate sums of products of k_ν. For example, for $n = 3$:

$$\gamma^{\mu}_{m,k}(M_\mu\phi) \leqq \{(2\mu+2)^3 + 8k_1k_2k_3\}\gamma^{\mu+1}_{m,k}(\phi)$$
$$+ 4(k_1k_2 + k_1k_3 + k_2k_3)\gamma^{\mu+1}_{m+2,k+1}(\phi)$$
$$+ 2(k_1 + k_2 + k_3)\gamma^{\mu+1}_{m+4,k+2}(\phi).$$

Lemmas 5 and 6 imply

LEMMA 7.

$$M_\mu N_\mu = [x]^{-\mu-1/2}\frac{\partial^n}{\partial x_1 \ldots \partial x_n}[x]^{2\mu+1}\frac{\partial^n}{\partial x_1 \ldots \partial x_n}[x]^{-\mu-1/2} = \prod_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2} - \frac{4\mu^2-1}{4x_i^2}\right)$$

is a continuous linear mapping of H_μ into itself.

In the dual spaces, we define N_μ and M_μ as weak differential operators by

(12) $\langle N_\mu f, \phi\rangle = \langle f, (-1)^n M_\mu\phi\rangle \qquad f \in H_\mu', \phi \in H_{\mu+1}$

(13) $\langle M_\mu f, \phi\rangle = \langle f, (-1)^n N_\mu\phi\rangle \qquad f \in H_{\mu+1}', \phi \in H_\mu.$

Thus we also have

(14) $\langle M_\mu N_\mu f, \phi\rangle = \langle f, M_\mu N_\mu\phi\rangle \qquad f \in H_\mu', \phi \in H_\mu.$

These definitions are consistent with the usual meaning of weak derivatives. In view of lemmas 5, 6 and 7, we have

LEMMA 8. (i) *The weak differential operator N_μ, defined by* (12) *is a continuous linear mapping of H_μ' into $H_{\mu+1}'$.*

(ii) *The weak differential operator M_μ, defined by* (13) *is an isomorphism from $H_{\mu+1}'$ onto H_μ'.*

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(iii) *The weak differential operator* $M_\mu N_\mu$, *given by* (14) *is a continuous linear mapping of* H_μ *into itself.*

4. The n-dimensional Hankel transformation. We shall define the n-dimensional classical μth order Hankel transformation h_μ by

$$(h_\mu\phi)(y) = \int_0^\infty \ldots \int_0^\infty \phi(x_1, \ldots x_n) \left(\prod_{i=1}^n (x_i y_i)^{1/2} J_\mu(x_i y_i)\right) dx_1 \ldots dx_n.$$

For $\mu \geqq -1/2$, the Hankel transform $(h_\mu\phi)(y)$ exists for every $\phi \in H_\mu$. This is due to the facts that ϕ is smooth and of rapid descent as $|x| \to \infty$ while $(x_i y_i)^{1/2} J_\mu(x_i y_i) = 0(x_i^{\mu+1/2})$ as $x_i \to 0^+$ and it remains bounded as $x_i \to \infty$. These properties of $\phi(x_1, \ldots, x_n)$ also ensure the validity of the classical inversion theorem [**6**, Theorem 19] when extended to n-dimensions.

THEOREM 1. *For* $\mu \geqq -1/2$, *the Hankel transformation* h_μ *is an automorphism on* H_μ.

Proof. Let $\Phi(y) = h_\mu(\phi(x))$. Then

$$[y^m](y^{-1}D_y)^k[y]^{-\mu-1/2}\Phi(y)$$

$$= \int_0^\infty \ldots \int_0^\infty \phi(x)(-1)^{(k)}[x]^{1/2} \times \prod_{\nu=1}^n x_\nu^{k_\nu} y_\nu^{-\mu-k_\nu+m_\nu} J_{\mu+k_\nu}(x_\nu y_\nu) dx_1 \ldots dx_n \tag{15}$$

$$= \int_0^\infty \ldots \int_0^\infty \phi(x)(-1)^{(k)}[x]^{-\mu+1/2} \times \prod_{\nu=1}^n \left(x_\nu^{-1}\frac{\partial}{\partial x_\nu}\right)^{m_\nu} y_\nu^{-\mu-k_\nu} x_\nu^{\mu+k_\nu+m_\nu} J_{\mu+k_\nu+m_\nu}(x_\nu y_\nu) dx_1 \ldots dx_n \tag{16}$$

$$= (-1)^{(k)+(m)} \int_0^\infty \ldots \int_0^\infty \left(\prod_{\nu=1}^n x_\nu^{2\mu+2k_\nu+m_\nu+1}\right) \times ((x^{-1}D_x)^m[x]^{-\mu-1/2}\phi(x)) \prod_{\nu=1}^n (x_\nu y_\nu)^{-\mu-k_\nu} J_{\mu+k_\nu+m_\nu}(x_\nu y_\nu) dx_1 \ldots dx_n. \tag{17}$$

Equation (15) is obtained by differentiating under the integral sign and a repeated use of

$$\frac{\partial}{\partial y} y^{-\mu} J_\mu(xy) = -xy^{-\mu} J_{\mu+1}(xy). \tag{18}$$

Equation (16) follows from m_ν-times application of the identity

$$yx^{\mu+1} J_\mu(xy) = \frac{\partial}{\partial x} x^{\mu+1} J_{\mu+1}(xy) \tag{19}$$

and equation (17) is obtained by integration by parts through each variable $x_1, \ldots, x_n$. The limit terms vanish since $\phi(x)$ is of rapid descent for large x while $x_i^{1/2} J_{\mu+1}(x_i y_i) = O(x_i)$, $\phi(x) = O(1)$ as $x_i \to 0$.

As $z^{-\mu-k_\nu} J_{\mu+k_\nu+m_\nu}(z)$ is bounded on $0 < z < \infty$, by say B_ν, the integral in (17) converges uniformly for all $y \in I$ so that $\Phi(y)$ is smooth on I.

If p_ν is an integer no less than $\mu + k_\nu + \frac{1}{2}(m_\nu + 1)$, then

$$x_\nu^{2\mu+2k_\nu+m_\nu+1} < (1 + x_\nu^2)^{p_\nu} \quad \text{for} \quad x_\nu > 0.$$

Hence, equation (17) yields

$$\gamma^\mu_{m,k}(\Phi) \leqq \int_0^\infty \ldots \int_0^\infty \prod_{\nu=1}^n (1 + x_\nu^2)^{p_\nu+1} |(x^{-1}D_x)^m [x]^{-\mu-1/2} \phi(x)| \times \prod_{\nu=1}^n \frac{B_\nu}{(1 + x_\nu^2)} dx_1 \ldots dx_n$$

$$\leqq \left(\frac{\pi}{2}\right)^n [B] \sum_{j=0}^Q C_j(p_\nu) \gamma^\mu_{2j,m}(\phi)$$

where Q is some integer and $C_j(p\nu)$ are appropriate constants involving p_ν. This proves that $\Phi \in H_\mu$ whenever $\phi \in H_\mu$, and that the linear mapping h_μ is also continuous from H_μ onto H_μ. The classical inversion theorem together with the fact that $h_\mu^{-1} = h_\mu$ [**6**] ensure that h_μ is one-to-one, whenever $\mu \geqq -1/2$. Hence h_μ is an automorphism on H_μ.

We may now define the n-dimensional distributional Hankel transformation h_μ' on H_μ' as the adjoint of h_μ on H_μ. Let $\mu \geqq -1/2$. For $\Phi \in H_\mu$ and $f \in H_\mu'$, the Hankel transform $F = h_\mu' f$ is defined by

$$\langle h_\mu' f, \Phi \rangle = \langle f, h_\mu \Phi \rangle.$$

THEOREM 2. *For $\mu \geqq -1/2$, the distributional Hankel transformation h_μ' is an automorphism on H_μ'.*

Proof. See [**4**, Theorem 1.10-2] and Theorem 1 above.

We now establish some transform formulas on H_μ and H_μ'.

LEMMA 9. *Let $\mu \geqq -1/2$. If $\phi \in H_\mu$, then*

(20) $h_{\mu+1}([-x]\phi(x)) = N_\mu h_\mu \phi(x)$

(21) $h_{\mu+1}(N_\mu \phi) = [-y] h_\mu \phi$

(22) $h_\mu([x]^2 \phi) = (-1)^n M_\mu N_\mu h_\mu \phi$

(23) $h_\mu(M_\mu N_\mu \phi) = (-1)^n [y]^2 h_\mu \phi.$

If $\phi \in H_{\mu+1}$, then

(24) $h_\mu([x]\phi) = M_\mu h_{\mu+1} \phi$

(25) $h_\mu(M_\mu \phi) = [y] h_{\mu+1} \phi.$

Proof. Let $\Phi = (h_\mu \phi)(y)$, where $\phi \in H_\mu$. Then

$$(26)\quad \frac{\partial^n}{\partial y_1 \ldots \partial y_n} [y]^{-\mu-1/2}\Phi(y) = \int_0^\infty \ldots \int_0^\infty \phi(x)[x]^{1/2} \frac{\partial^n}{\partial y_1 \ldots \partial y_n} \times \left\{ \prod_{i=1}^n y_i^{-\mu} J_\mu(x_i y_i) \right\} dx_1 \ldots dx_n.$$

By the identity (18), the right hand side of (26) becomes

$$(-1)^n \int_0^\infty \ldots \int_0^\infty \phi(x)[x]^{3/2}[y]^{-\mu} \left(\prod_{i=1}^n J_{\mu+1}(x_i y_i) \right) dx_1 \ldots dx_n.$$

We may differentiate under the integral sign in (26) because for $\mu \geqq -1/2$, $\prod_{i=1}^n J_{\mu+1}(x_i y_i)$ is a smooth bounded function on I and $\phi(x)[x]^{3/2}$ is of rapid descent. Thus, (26) is a uniformly convergent integral on every compact subset of I. Hence

$$N_\mu h_\mu \phi = [y]^{\mu+1/2} \frac{\partial^n}{\partial y_1 \ldots \partial y_n} [y]^{-\mu-1/2}\Phi(y) = h_{\mu+1}([-x]\phi(x))$$

which is (20).

To prove (21), we use the formula (19) together with integration by parts. Thus,

$$\begin{aligned} h_{\mu+1}(N_\mu \phi) &= [y]^{1/2} \int_0^\infty \ldots \int_0^\infty \left(\frac{\partial^n}{\partial x_1 \ldots \partial x_n} [x]^{-\mu-1/2}\phi(x) \right) \times \prod_{i=1}^n x_i^{\mu+1} J_{\mu+1}(x_i y_i) dx_n \ldots dx_1 \\ &= [y]^{1/2} \int_0^\infty \ldots \int_0^\infty \frac{\partial^n}{\partial x_1 \ldots \partial x_{n-1}} \Big\{ \phi(x)(x_1 \ldots x_{n-1})^{-\mu-1/2} \times x_n^{1/2} J_{\mu+1}(x_n y_n) \Big|_{x_n = 0}^{x_n = \infty} \\ &\quad - \int_0^\infty [x]^{-\mu-1/2}\phi(x) y_n x_n^{\mu+1} J_\mu(x_n y_n) dx_n \Big\} \times \prod_{i=1}^{n-1} x_i^{\mu+1} J_{\mu+1}(x_i y_i) dx_{n-1} \ldots dx_1. \end{aligned}$$

The limit terms vanish since $\phi(x)$ is of rapid descent as $x_n \to \infty$ and $x_n^{1/2} J_{\mu+1}(x_n y_n) = O(x_n)$ while $\phi(x) = O(1)$ as $x_n \to 0$. Continuing the integration by parts through the succeeding components $x_{n-1}, \ldots, x_2, x_1$, we obtain the result (21).

Formulas (24) and (25) are proved in a manner analogous to the proofs for (20) and (21). Combining (20) and (24), we obtain (22). Indeed

$$M_\mu N_\mu h_\mu \phi = M_\mu h_{\mu+1}([-x]\phi(x)) = h_\mu((-1)^n[x]^2\phi(x)).$$

Similarly, (23) follows from (21) and (25):

$$h_\mu(M_\mu N_\mu \phi) = [y]h_{\mu+1}(N_\mu \phi) = (-1)^n[y]^2 h_\mu \phi.$$

Lemma 9 enables us to prove the following theorem, whose proof follows analogous arguments to Theorem 3 of [**1**] using the appropriate definition of weak operators (12), (13), and (14).

THEOREM 3. *Let* $\mu \geqq -1/2$. *If* $f \in H_\mu'$, *then*

$$\begin{aligned} h_{\mu+1}'((-1)^n[x]f) &= N_\mu h_\mu' f \\ h_{\mu+1}'(N_\mu f) &= (-1)^n[y]h_\mu' f \\ h_\mu'((-1)^n[x]^2 f) &= M_\mu N_\mu h_\mu' f \\ h_\mu'(M_\mu N_\mu f) &= (-1)^n[y]^2 h_\mu' f. \end{aligned}$$

If $f \in H_{\mu+1}'$, *then*

$$\begin{aligned} h_\mu'([x]f) &= M_\mu h_{\mu+1}' f \\ h_\mu'(M_\mu f) &= [y]h_{\mu+1}' f. \end{aligned}$$

Remarks. (i) The results in the present work reduce to the one-dimensional case in [**1**] when $n = 1$.

(ii) By a similar device as in this work, it might be possible to extend the n-dimensional Hankel transformation to generalized functions of exponential descent [**2**] and certain distributions of rapid growth [**3**].

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University of Regina,
Regina, Saskatchewan