THE VALENCE OF SUMS AND PRODUCTS

A. W. GOODMAN

1. Introduction. A function $f(z)$ is said to be p-valent in a region $\mathscr{D}$, if it is regular in $\mathscr{D}$, if the equation

$$f(z) = w_0 \tag{1}$$

has p distinct roots in $\mathscr{D}$ for some particular w_0, and if for each complex w_0, equation (1) does not have more than p roots in $\mathscr{D}$. The function $f(z)$ is also said to have valence p in $\mathscr{D}$. In the case when $p = 1$, the function is said to be univalent in $\mathscr{D}$.

Given two functions f and g, there are various ways of composing them to form a new function $F = f \oplus g$. However, there seems to be little that is known about the valence of F in terms of the valence of f and g. In this note we examine the two simplest cases for $\oplus$, namely, $(f + g)/2$ and $(fg)^{1/2}$.

It is clear that any result relative to the valence of F is essentially independent of the nature of the domain. For suppose that $\phi(z)$ maps $\mathscr{D}_1$ conformally onto $\mathscr{D}_2$. Then any assertion about the valences in $\mathscr{D}_2$ for the functions in the equation

$$2F(z) = f(z) + g(z)$$

will also hold in $\mathscr{D}_1$ for the functions in the equation

$$2F(\phi(z)) = f(\phi(z)) + g(\phi(z)).$$

2. Sums. We first observe that the two functions

$$f(z) = z + z^n/n$$

and

$$g(z) = -z + z^n/n$$

are both univalent in the unit circle $E = \{z|\ |z| < 1\}$. But $f(z) + g(z) = 2z^n/n$ is n-valent in E. Thus, in order to form an interesting problem, we need some normalization that eliminates this type of example. Let $\mathscr{V}_E(1)$ be the set of all functions that are univalent in E and have a power series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Received January 13, 1967. This work was supported by the National Science Foundation, Research Grant GP-5689.

It has been conjectured[1] that if $f \in \mathscr{V}_E(1)$ and $g \in \mathscr{V}_E(1)$, then

$$F \equiv (f+g)/2 = z + \ldots$$

has valence at most 2 in E. The truth is quite different.

THEOREM 1. *There exist two functions $f \in \mathscr{V}_E(1)$ and $g \in \mathscr{V}_E(1)$ such that the function $F \equiv (f+g)/2$ has valence ∞ in E.*

Proof. For convenience, we transfer the domain of definition from the unit circle to the right-half plane, $H = \{z | \mathscr{R}z > 0\}$.

Let $\mathscr{V}_H(1)$ be the set of all functions that are univalent in H and have a power series of the form

$$f(z) = (z-1) + \sum_{n=2}^{\infty} a_n (z-1)^n \tag{3}$$

in the neighbourhood of $z = 1$. Clearly, $f(z) \in \mathscr{V}_E(1)$ if and only if

$$2f((z-1)/(z+1)) \in \mathscr{V}_H(1).$$

The particular function

$$f_1(z) \equiv \frac{1}{1+i} e^{(1+i)\ln z} - \frac{1}{1+i} \tag{4}$$

has a power series of the form (3). Further (using the proper branch) $\ln z$ carries H into the strip $-\pi/2 < \mathscr{I}(w) < \pi/2$. The factor $1 + i$ rotates this strip counterclockwise through an angle of $\pi/4$ and stretches it in such a way that each vertical line intersects the resulting strip in a segment of length 2π. It then follows that $f_1(z)$ is univalent in H. Similar considerations show that the function

$$g_1(z) \equiv \frac{1}{1-i} e^{(1-i)\ln z} - \frac{1}{1-i} \tag{5}$$

is also in the set $\mathscr{V}_H(1)$. Now let $F_1(z) \equiv (f_1(z) + g_1(z))/2$. A brief computation gives

$$F_1(z) = -\frac{1}{2} + \frac{1-i}{4} e^{(1+i)\ln z} + \frac{1+i}{4} e^{(1-i)\ln z}. \tag{6}$$

If we select $z_n > 0$ such that

$$\ln z_n = -\pi/4 + n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

it is clear that $F_1(z_n) = -1/2$ for each integer n. Hence $F_1(z)$ has valence ∞ in H.

---

[1]This conjecture, in a more general form, appeared in print in (4) in 1962, but I had already considered the problem at least ten years earlier.

https://doi.org/10.4153/CJM-1968-111-0 Published online by Cambridge University Press

3. Products. It has been conjectured (see **4**) that if $f \in \mathscr{V}_E(1)$ and $g \in \mathscr{V}_E(1)$, then the function fg has valence at most 2. Since the only zero of fg is the double zero at the origin, the function $(fg)^{1/2}$ is regular in E, and one might expect that it is univalent in E. If f and g are, in addition, star-like, then this is indeed the case. It seems to be well known that if λ and μ are any two positive numbers with $\lambda + \mu = 2$ and if f and g are star-like and univalent in E, then $(f^\lambda g^\mu)^{1/2}$ is star-like and univalent in E (see Hummel (**2**)). If we drop the star-like condition and assume only that f and g are normalized univalent, then $(fg)^{1/2}$ need not be univalent.

THEOREM 2. *There exist two functions $f \in \mathscr{V}_E(1)$ and $g \in \mathscr{V}_E(1)$ such that the function $F = (fg)^{1/2}$ has valence ∞ in E.*

Proof. Just as in § 2, we transfer the domain from E to H and in fact use the very same functions from $\mathscr{V}_H(1)$. Let

$$F_2^2(z) \equiv f_1(z)g_1(z),$$

where $f_1(z)$ and $g_1(z)$ are given by equations (4) and (5). Let $z_k = e^{x_k}$, where x_6 is any real root of the equation $e_6 = 2\cos x$. Then it is easy to see that

$$F_2^2(z_k) = \tfrac{1}{2}(e^{2x_k} - 2e^{x_k}\cos x_k + 1) = \tfrac{1}{2}.$$

Since the equation $e^x = 2\cos x$ has infinitely many real roots, the function $F_2^2(z)$ has valence ∞. The same is true of $F_2(z)$ no matter how the branch for the square root is selected.

What is the analogue of Theorem 2, if we change the normalization? Since each function in the set $\mathscr{V}_E(1)$ omits some value of unit modulus, a suitable linear transformation will give a new univalent function of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \qquad |a_1| = 1, \tag{7}$$

which is never zero in E. Let $\mathscr{V}_E^{(0)}(1)$ be the class of all functions $f(z)$ that are univalent in E, never zero in E, and for which $f(0) = 1$ and $|f'(0)| = 1$. It is a trivial matter to show that Theorem 2 is still true when the class $\mathscr{V}_E(1)$ is replaced by $\mathscr{V}_E^{(0)}(1)$. If $f \in \mathscr{V}_E^{(0)}(1)$, then $1/f \in \mathscr{V}_E^{(0)}(1)$. Hence, if we set $g = 1/f$, then certainly $F^2(z) \equiv f(z)g(z) = 1$ for all z in E and F has valence ∞. Although I have not been able to find a pair f, g in $\mathscr{V}_E^{(0)}(1)$ such that $fg \not\equiv$ constant and fg has infinite valence, it is easy to show that no upper bound can be placed on the valence of fg. Indeed, we select a sequence of positive constants, $a_1, a_2, \ldots, a_n$, such that

$$\sum_{k=2}^{n} k a_k < 1 \quad \text{and} \quad 1 > a_2 > a_3 > \ldots > a_n.$$

Then the function

$$f_2(z) \equiv 1 + z + \sum_{k=2}^{n} a_k z^k$$

https://doi.org/10.4153/CJM-1968-111-0 Published online by Cambridge University Press

is univalent and never zero in E (Kakeya-Eneström Theorem). Hence, $f_2(z)$ and

$$g_2(z) \equiv \left(1 + z + \sum_{k=2}^{n-1} a_k z^k\right)^{-1} = 1 - z + \dots$$

are in $\mathscr{V}_E^{(0)}(1)$. On the other hand, the product

$$f_2(z)g_2(z) = 1 + a_n z^n + \sum_{k=n+1}^{\infty} b_k z^k$$

has valence greater than $n - 1$.

Suppose that f and g are in $\mathscr{V}_E^{(0)}(1)$ and, in addition, $f'(0) = g'(0) = 1$. It seems likely that there are two such functions for which $f(z)g(z)$ has valence ∞ in E, but so far I have not been able to find such a pair.

4. Open questions. The results obtained in §§ 2 and 3 suggest a number of questions. Let f and g belong to the set $\mathscr{V}_E(1)$. It is easy to prove that there is some positive constant R_0 such that for every such pair the function $F = (f + g)/2$ is univalent in $|z| < R_0$. What[2] is the largest such R_0? More generally, let $\mathscr{V}_E(p)$ be the set of all functions $f(z)$ that are p-valent in E, and have a power series of the form

$$f(z) = z^{p_0} + \sum_{n=p_0+1}^{\infty} a_n z^n, \qquad 1 \leqq p_0 \leqq p. \tag{8}$$

Let $R(p, q, s)$ be the largest number with the property that if $f(z) \in \mathscr{V}_E(p)$ and $g(z) \in \mathscr{V}_E(q)$, then the function $F(z) \equiv f(z) + g(z)$ has valence less than or equal to s in $|z| < R(p, q, s)$. What can be said about the numbers $R(p, q, s)$ beyond the obvious and trivial facts?

What type functions can be obtained as the sum (or product) of functions $f \in \mathscr{V}_E(p)$ and $g \in \mathscr{V}_E(q)$? For example, is a Blaschke product ever the sum of two univalent functions, or the sum of two functions with finite valence? Further, we might re-examine Theorem 2 when *one* of the two functions is restricted to be star-like. More generally, let $f \in \mathscr{V}_E(p)$ and let $g \in \mathscr{V}_E(q)$, where g is generalized star-like in E (see **1** and **3**). What can be said about the valence of fg in E? Certainly, if both f and g are star-like, then fg is $(p + q)$-valent and star-like in E.

Suppose that both f and g are normalized, univalent, and convex in E. Then $(fg)^{1/2}$ is univalent star-like, but it is easy to show that $(fg)^{1/2}$ need not be convex. Under the same conditions, $(f + g)/2$ need not be convex, but I do not know whether it is always star-like, or even always univalent.[2]

If f and g are normalized star-like and univalent, then $(f + g)/2$ need not be univalent. But perhaps some upper bound can be put on the valence of $(f + g)/2$ under these conditions.

[2] *Added in proof.* After the paper was accepted, I learned that T. H. MacGregor had solved this problem. His results will appear in J. London Math. Soc.

References

1. A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. *68* (1950), 204–223.
2. J. A. Hummel, *The coefficient regions of starlike functions*, Pacific J. Math. *7* (1957), 1381–1389.
3. ——— *Multivalent starlike functions*, J. Analyse Math. *18* (1967), 133–160.
4. Problem 21, Classical function theory problems, Bull. Amer. Math. Soc. *68* (1962), 21–24.

University of South Florida,
Tampa, Florida