SUPERMAGIC COMPLETE GRAPHS

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In our paper "Magic graphs" **(1)** we showed that every complete graph K_n with $n \geqslant 5$ is "magic," i.e., if the vertex set is indicated $\{v_i\}$ and if e_{ij} is the edge joining v_i and v_j, $i \neq j$, then there exists a function $\alpha(e_{ij})$ such that the set $\{\alpha(e_{ij})\}$ consists of distinct positive rational integers and the vertex sums

$$(1) \qquad \sigma^k = \sigma(v_k) = \sum_{1 \leqslant i < k}' \alpha(e_{ik}) + \sum_{k < j \leqslant n}'' \alpha(e_{kj})$$

have a constant value $\sigma(\alpha)$ for $k = 1, 2, \ldots, n$. We noted that K_2 is magic and showed that K_3 and K_4 are not magic.

We raised the question whether K_n is "supermagic" for $n \geqslant 5$, i.e., does there exist an α under which K_n is magic with the additional property that the set $\{\alpha(e_{ij})\}$ consists of *consecutive* integers? Since K_n is regular, the supermagic problem reduces to using the particular set $\{1, 2, \ldots, E\}$, where

$$E = n(n-1)/2.$$

We showed that K_n is not supermagic when $n \equiv 0 \bmod 4$. We claimed (without details) that K_5 is not supermagic; and we showed that K_6 and K_7 are supermagic.

In this paper we shall show that K_n is supermagic for $n \geqslant 5$ if and only if $n > 5$ and $n \not\equiv 0 \bmod 4$.

THEOREM 1. *K_n is not supermagic when $n \equiv 0 \bmod 4$.*

Proof. If K_n is supermagic, a necessary relation is obtained by summing $\alpha(e_{ij})$ over all the edges of K_n, namely:

$$(2) \qquad n\sigma(\alpha) = E(E+1),$$

which reduces to the form

$$(2') \qquad 4\sigma(\alpha) = (n-1)(n^2 - n + 2).$$

When $n \equiv 0 \bmod 4$, the relation (2′) cannot be satisfied, since $0 \not\equiv 2 \bmod 4$.

THEOREM 2. *K_5 is not supermagic.*

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Proof. We shall show that the integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 cannot be assigned to the edges of K_5 in such a way that each vertex sum is 22. Certainly 10 must be assigned to one edge, say e_{12}. In order that the vertex sums at v_1 and v_2 be 22, we must select from the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 two disjoint sets A_1 and A_2, each containing three integers having the sum 12.

If 9 is used in A_1, it must be used with 1 and 2, and the only companion set A_2 is 3, 4, 5. Suppose $\alpha(e_{13}) = 9$; then the other three edges incident to V_3 must have assignments which total 13. But the remaining numbers are 6, 7, 8; hence $\alpha(e_{34}) + \alpha(e_{35})$ already totals 13. Hence if K_5 is supermagic under α, the sets A_1 and A_2 must not contain 9.

If 8 is used in A_1, it must be used with 1 and 3; since 9 may not be used in A_2, the only companion set A_2 is 2, 4, 6. Suppose $\alpha(e_{13}) = 8$; then $\alpha(e_{23}) = 6, 4$, or 2, and $\alpha(e_{34})$ and $\alpha(e_{35})$ must be chosen from 5, 7, 9. If $\alpha(e_{23}) = 6$, then $\sigma^3 = 22$ implies that $\alpha(e_{34}) + \alpha(e_{35}) = 8$, which is impossible. If $\alpha(e_{23}) = 4$, then $\sigma^3 = 22$ implies that $\alpha(e_{34}) + \alpha(e_{35}) = 10$, which is impossible. But if $\alpha(e_{23}) = 2$, then $\sigma^3 = 22$ implies that $\alpha(e_{34}) + \alpha(e_{35}) = 12$, which is possible with $\alpha(e_{34})$ and $\alpha(e_{35})$ chosen from 5, 7. However, this requires $\alpha(e_{45}) = 9$. Then $\sigma^4 = 22$ implies that $\alpha(e_{14}) + \alpha(e_{24}) + \alpha(e_{34}) = 13$. Since $\alpha(e_{24})$ is even, being chosen from 4, 6, and since $\alpha(e_{14}) + \alpha(e_{34})$ is even, with summands being chosen from 1, 3, 5, 7, the sum 13 cannot be realized. Hence if K_5 is supermagic under α, the sets A_1 and A_2 must contain neither 9 nor 8.

From the remaining numbers 1, 2, 3, 4, 5, 6, 7 there are only five sets of three with the sum 12, namely: $\{1, 4, 7\}$, $\{2, 3, 7\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$, $\{3, 4, 5\}$. Only two of these sets are disjoint, namely: $A_1 = \{2, 3, 7\}$ and $A_2 = \{1, 5, 6\}$. The assignments $\alpha(e_{34})$, $\alpha(e_{35})$, $\alpha(e_{45})$ must be chosen from 4, 8, 9. Suppose $\alpha(e_{13}) = 7$. Then $\alpha(e_{13}) + \alpha(e_{34}) + \alpha(e_{35})$ is either 19, 20, or 24. Since $\alpha(e_{23})$ must be chosen from A_2, it is impossible to make $\sigma^3 = 22$. This concludes the proof of Theorem 2.

Because of Theorems 1 and 2, an induction proof that K_n is supermagic for $n > 5$ and $n \not\equiv 0 \bmod 4$ can only be successful if we set $n = 4k + r$ and discuss each of the cases $r = 1, 2, 3$ with an induction on k, beginning with appropriate values: if $r = 1$, $k \geqslant 2$; if $r = 2$, $k \geqslant 0$; if $r = 3$, $k \geqslant 1$.

We shall describe an α for K_n by giving the entries $\alpha(e_{ij})$ in the upper triangle, $i < j$, of a matrix. Because of the restriction $i < j$, the triangle contains $n - 1$ rows, numbered $i = 1, 2, \ldots, n - 1$, and $n - 1$ columns, numbered $j = 2, 3, \ldots, n$.

For $k = 2, 3, \ldots, n - 1$ we note that the vertex sum σ^k will be found by adding the entries in the k-column and the k-row of the triangle. This is indicated in (1) by separating σ^k into the parts $\sum'$ and $\sum''$. In figures we shall use a right-angled arrow, along the diagonal of the triangle, to remind the viewer how to form the sum σ^k. Of course, for $k = 1$, σ^1 consists of $\sum''$ only, determined by the 1-row of the triangle; and for $k = n$, σ^n consists of $\sum'$ only, determined by the n-column of the triangle.

In the inductions we shall obtain a triangle describing an α' for K_{n+4} by adjoining four columns to the triangle describing an α for K_n, so we shall have use for the following lemma.

LEMMA 1. *Given a_i and $\bar{a}_i = Q - a_i$, $i = 1, 2, \ldots, 8$, satisfying the conditions*

$$a_1 + a_4 = a_2 + a_3, \qquad a_5 + a_8 = a_6 + a_7, \tag{3}$$

then the matrix

$$\begin{matrix} a_1 & \bar{a}_1 & a_5 & \bar{a}_5 \\ \bar{a}_2 & a_2 & \bar{a}_6 & a_6 \\ \bar{a}_3 & a_3 & \bar{a}_7 & a_7 \\ a_4 & \bar{a}_4 & a_8 & \bar{a}_8 \end{matrix} \tag{3'}$$

has the property that each row and column sum is $2Q$.

Proof. Since $a_i + \bar{a}_i = Q$ for $i = 1, 2, \ldots, 8$, each row sum in (3′) is obviously $2Q$. Since $a_1 + a_4 = a_2 + a_3$, it is easy to check for the first two columns that

$$a_1 + (Q - a_2) + (Q - a_3) + a_4 = 2Q = (Q - a_1) + a_2 + a_3 + (Q - a_4).$$

A similar argument applies to the last two columns.

Our induction construction of a triangle describing an α' for K_{n+4} from knowledge of a triangle describing an α for K_n will vary according to the value of r, but will always hinge on comparing $\sigma' = \sigma(\alpha')$ and E' for K_{n+4} with $\sigma = \sigma(\alpha)$ and E for K_n.

We know that $E = n(n-1)/2$ and $E' = (n+4)(n+3)/2$; hence $E' - E = 2(2n+3)$. From (2) we have

$$\sigma = (E+1)(n-1)/2, \quad \sigma' = (E'+1)(n+3)/2;$$

hence

$$\sigma' = \sigma + (E' - E)(n-1)/2 + 2(E'+1), \tag{4}$$

$$\sigma' = \sigma + (n-1)(2n+3) + 2(E'+1). \tag{4'}$$

THEOREM 3. *If n is odd and $n > 5$, then K_n is supermagic.*

Proof. The degree of each vertex of K_n is $n - 1$, so that comparison with (4′) suggests the following induction plan. Assume that α makes K_n supermagic using the assignments $1, 2, \ldots, E$. Increase each edge assignment by the amount $2n + 3$. Then the new α_1 has for each vertex of K_n the sum $\sigma_1 = \sigma + (n-1)(2n+3)$ and uses the consecutive assignments

$$2n+4, 2n+5, \ldots, E + 2n + 3 = E' - (2n+3).$$

We shall show that K_{n+4} is supermagic by producing an α' based on α_1. We shall need to use the integers $\{x\}$, where $1 \leqslant x \leqslant 2n + 3$, and the complementary integers $\{\bar{x}\}$, where $\bar{x} = (E' + 1) - x$, which fill the interval $E' + 1 - (2n + 3) \leqslant \bar{x} \leqslant E'$. For the vertices in K_{n+4} which are also in K_n this looks hopeful, since (4′) shows that $\sigma' = \sigma_1 + 2(E' + 1)$ and assignments such that

$$\alpha'(e_{i,n+1}) + \alpha'(e_{i,n+2}) + \alpha'(e_{i,n+3}) + \alpha'(e_{i,n+4}) = 2(E' + 1)$$

for $i = 1, 2, \ldots, n$ will be found readily using two pairs $x, \bar{x}, y, \bar{y}$.

However, there are six other new edges, joining the four new vertices, to which assignments must be made. Furthermore, the vertex sums for the four new vertices must also be σ'. This is the stage of the argument where the cases with n odd are easier than the case where n is even, because when n is odd, the fraction $(n - 1)/2$ appearing in (4) is an integer.

Case 1. If $n = 4k + 3 \geqslant 7$, the degree for K_{n+4} is $n + 3 \geqslant 10$, so it makes sense to make a separate description of the last ten rows of the last four columns of the triangle for α'. For the other rows ($k \geqslant 2$) we have Lemma 1 and (3) and (3′) in mind as we make the entries which follow:

(5)

	$n+1$	$n+2$	$n+3$	$n+4$
$4t+1$	$8t+1$	$\overline{8t+1}$	$8t+2$	$\overline{8t+2}$
$4t+2$	$\overline{8t+3}$	$8t+3$	$\overline{8t+4}$	$8t+4$
$4t+3$	$\overline{8t+5}$	$8t+5$	$\overline{8t+6}$	$8t+6$
$4t+4$	$8t+7$	$\overline{8t+7}$	$8t+8$	$\overline{8t+8}$

for $t = 0, 1, \ldots, k - 2$. We note that this will use all the x and $\bar{x}$ with $1 \leqslant x \leqslant 8k - 8$. In the last ten rows we enter

Row:					
$n-6$	$8k-7$	$\overline{8k-7}$	$8k-6$	$\overline{8k-6}$	$\to 2(E'+1)$
$n-5$	$\overline{8k-5}$	$8k-5$	$\overline{8k-4}$	$8k-4$	$\to 2(E'+1)$
$n-4$	$\overline{8k-3}$	$8k-3$	$\overline{8k-2}$	$8k-2$	$\to 2(E'+1)$
$n-3$	$8k-1$	$\overline{8k}$	$8k$	$\overline{8k-1}$	$\to 2(E'+1)$
$n-2$	$\overline{8k+1}$	$8k+2$	$\overline{8k+2}$	$8k+1$	$\to 2(E'+1)$
$n-1$	$8k+3$	$\overline{8k+4}$	$8k+4$	$\overline{8k+3}$	$\to 2(E'+1)$
n	$8k+5$	$\overline{8k+6}$	$8k+6$	$\overline{8k+5}$	$\to 2(E'+1)$
$n+1$	↳	$\overline{8k+7}$	$\overline{8k+8}$	$8k+8$	$\to 5(E'+1)$
$n+2$		↳	$8k+9$	$8k+7$	$\to 5(E'+1)$
$n+3$			↳	$\overline{8k+9}$	$\to 5(E'+1)$
				↳	$\to 5(E'+1)$

These entries use all the remaining x and $\bar{x}$ with

$$8k - 7 \leqslant x \leqslant 8k + 9 = 2n + 3,$$

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exactly as planned. From (4′) we check for the n "old" vertices that $\sigma_1 + 2(E' + 1) = \sigma'$. Of course as part of our check we use the fact that (5) is a special instance of (3′) with each row sum being $2(E' + 1)$. But the choices in (5) also satisfy (3). Hence for the four "new" vertices we use the result in Lemma 1 concerning column sums, for the $k - 1$ cases,

$$t = 0, 1, \ldots, k - 2,$$

together with our special arrangement in the last ten rows, to check that

$$(k - 1)2(E' + 1) + 5(E' + 1) = (2k + 3)(E' + 1) = (E' + 1)(n + 3)/2 = \sigma'.$$

If we can produce a solution for $n = 7$, this will complete the induction proof for the case $n = 4k + 3 \geqslant 7$. As a basis for the induction we exhibit a triangle describing an α for K_7:

11	1	21	12	2	19	$\rightarrow 66 = \sigma^1$
↳	18	5	3	20	9	$\rightarrow 66 = \sigma^2$
	↳	10	17	16	4	$\rightarrow 66 = \sigma^3$
		↳	7	8	15	$\rightarrow 66 = \sigma^4$
			↳	14	13	$\rightarrow 66 = \sigma^5$
				↳	6	$\rightarrow 66 = \sigma^6$
					└	$\rightarrow 66 = \sigma^7$

As an illustration of the induction step we show the solution for K_{11} derived from the above solution for K_7. However, the situation is not quite typical for (5) is vacuous, and only the specially arranged ten rows appear.

Add $2n + 3 = 17$ to α for K_7						Apply induction with $E' + 1 = 56$			
28	18	38	29	19	36	1	55	2	54
	35	22	20	37	26	53	3	52	4
		27	34	33	21	51	5	50	6
			24	25	32	7	48	8	49
				31	30	47	10	46	9
					23	11	44	12	45
						13	42	14	43
							41	40	16
								17	15
									39

Case 2. If $n = 4k + 1 \geqslant 9$, the degree for K_{n+4} is $n + 3 \geqslant 12$. We begin with (5) for $t = 0, 1, \ldots, k - 3$. This uses all the x and $\bar{x}$ with

$$1 \leqslant x \leqslant 8k - 16.$$

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In the last twelve rows we enter

Row:					
$n-8$	$8k-15$	$\overline{8k-15}$	$\overline{8k-14}$	$8k-14$	$\rightarrow 2(E'+1)$
$n-7$	$\overline{8k-13}$	$8k-13$	$8k-12$	$\overline{8k-12}$	$\rightarrow 2(E'+1)$
$n-6$	$\overline{8k-11}$	$8k-11$	$\overline{8k-10}$	$8k-10$	$\rightarrow 2(E'+1)$
$n-5$	$8k-9$	$\overline{8k-8}$	$8k-8$	$\overline{8k-9}$	$\rightarrow 2(E'+1)$
$n-4$	$\overline{8k-7}$	$\overline{8k-6}$	$8k-6$	$8k-7$	$\rightarrow 2(E'+1)$
$n-3$	$8k-5$	$8k-4$	$\overline{8k-4}$	$\overline{8k-5}$	$\rightarrow 2(E'+1)$
$n-2$	$\overline{8k-3}$	$8k-3$	$8k-2$	$\overline{8k-2}$	$\rightarrow 2(E'+1)$
$n-1$	$8k-1$	$\overline{8k-1}$	$\overline{8k}$	$8k$	$\rightarrow 2(E'+1)$
n	$8k+1$	$\overline{8k+2}$	$8k+2$	$\overline{8k+1}$	$\rightarrow 2(E'+1)$
$n+1$	↳	$\overline{8k+5}$	$\overline{8k+4}$	$8k+4$	$\rightarrow 6(E'+1)$
$n+2$		↳	$8k+5$	$8k+3$	$\rightarrow 6(E'+1)$
$n+3$			↳	$\overline{8k+3}$	$\rightarrow 6(E'+1)$
				└────	$\rightarrow 6(E'+1)$

These entries use all the remaining x and $\bar{x}$ with

$$8k - 15 \leqslant x \leqslant 8k + 5 = 2n + 3,$$

exactly as planned. From the above arrangement and from (5) we check, using (4′), for each of the n "old" vertices, that $\sigma_1 + 2(E' + 1) = \sigma'$. From Lemma 1, applied to $k - 2$ cases, $t = 0, 1, \ldots, k - 3$, and from the special arrangement in the last twelve rows, we check for each of the four "new" vertices that

$$(k-2)2(E'+1) + 6(E'+1) = (2k+2)(E'+1)$$
$$= (E'+1)(n+3)/2 = \sigma'.$$

This concludes the induction proof for the case $n = 4k + 1 \geqslant 9$, except for exhibiting a basis for the induction in the form of an α under which K_9 is supermagic. One such α is described by the following triangle:

19	17	31	7	10	27	24	13
	20	8	30	5	35	3	28
		16	21	12	25	36	1
			26	18	32	6	11
				22	4	15	23
					14	33	34
						2	9
							29

THEOREM 4. *If $n = 4k + 2$, then K_n is supermagic for $k \geqslant 0$.*

Proof. The case $n = 2$, which furnishes a basis for the induction, is trivial, since K_2 has only two vertices and one edge, for which the assignment

$\alpha(e_{12}) = 1$ fits the requirements. The plan used in the proof of Theorem 3 is not applicable, since $\sigma = (E+1)(n-1)/2$ does not have $(n-1)/2$ as an integer. Instead, we have

$$\sigma = 2k(E+1) + (E+1)/2. \tag{6}$$

Our induction plan will vary slightly according to the size of k and will vary according as k is odd or even.

Our solutions for $n = 4k+2$ will have a special form in the sense that we prescribe

$$\begin{cases} \alpha(e_{4t+1,4t+2}) = (E+1)/2 - t, & t = 0, 1, \ldots, k; \\ \alpha(e_{4t-1,4t}) \;\;= (E+1)/2 + t, & t = 1, 2, \ldots, k. \end{cases} \tag{7}$$

The entries in (7) will provide the $(E+1)/2$ noted in (6), for just one of these terms appears in each vertex sum. Furthermore, we shall assume that our solution has the following special property:

(8) $\begin{cases} \text{for each vertex sum, the terms other than those listed in (7) have } 2k \\ \text{of them greater than } (E+1)/2 \text{ and } 2k \text{ of them less than } (E+1)/2. \end{cases}$

To construct an α' for K_{n+4}, retaining the properties (7) and (8), we make the following preliminary change. We assume K_n is supermagic under an α with properties (7) and (8) and begin constructing a new α_1 for K_n as follows:

$$\begin{cases} \alpha_1(e_{4t+1,4t+2}) = (E'+1)/2 - t = \alpha(e_{4t+1,4t+2}) + (E'-E)/2, \\ \alpha_1(e_{4t-1,4t}) \;\;= (E'+1)/2 + t = \alpha(e_{4t-1,4t}) \;\;+ (E'-E)/2, \end{cases} \tag{7'}$$

for $t = 0, 1, \ldots, k$ and $t = 1, 2, \ldots, k$, respectively. Since

$$(E'-E)/2 = 8k+7$$

we may think first of adding $8k+7$ to every edge assignment in α. However, since we shall want α' to have property (7) we anticipate having

$$\begin{cases} \alpha'(e_{4k+5,4k+6}) = (E'+1)/2 - (k+1), \\ \alpha'(e_{4k+3,4k+4}) = (E'+1)/2 + (k+1). \end{cases} \tag{7''}$$

Hence we shall leave a place for these two integers and define

$$\begin{cases} \alpha_1(e_{ij}) = \alpha(e_{ij}) + 8k + 6, & \text{if } \alpha(e_{ij}) < (E+1)/2, \\ \alpha_1(e_{ij}) = \alpha(e_{ij}) + 8k + 8, & \text{if } \alpha(e_{ij}) > (E+1)/2, \end{cases} \tag{8'}$$

where either $i < j$ and i is even, or $i < j$ and i is odd, but $j \neq i+1$ (the case where i is odd and $j = i+1$ is already defined in (7$'$)). Combining (7$'$), (7$''$), and (8$'$) we see that the integers remaining to be used in the description of α' are the two sets of consecutive integers $\{x\}$ and $\{\bar{x}\}$ where

$$\bar{x} = (E'+1) - x$$

and

$$1 \leqslant x \leqslant 8k+6. \tag{9}$$

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We note that α_1 makes K_n magic with
$\sigma_1 = \sigma + 2k(8k + 6 + 8k + 8) + (E' - E)/2 = \sigma + (E' - E)(n - 1)/2.$
Then from (4) we obtain

$$(4'') \qquad \sigma' = \sigma_1 + 2(E' + 1).$$

Thus for the "old" vertices of K_{n+4} we can hope to use two pairs $x, \bar{x}, y, \bar{y}$ to make up the proper vertex sum σ' for α'. For the first $4k$ rows of the last four columns of the triangle describing α', we hope to use Lemma 1 and (3′) for k sets. For the last five rows we plan to use some variation of the following pattern, which incorporates (7″):

$$\begin{array}{ccccl}
\underline{1} & \overline{u+k+1} & \bar{1} & u+k+1 & \rightarrow 2(E'+1) \\
\overline{k+2} & u & \bar{u} & k+2 & \rightarrow 2(E'+1) \\
\hookrightarrow & (E'+1)/2+(k+1) & v & \bar{v} & \rightarrow 5(E'+1)/2 \\
 & \hookrightarrow & u+k+2-v & \overline{u+k+2-v} & \rightarrow 5(E'+1)/2 \\
(10) & & \hookrightarrow & (E'+1)/2-(k+1) & \rightarrow 5(E'+1)/2 \\
 & & & \hookrightarrow & \rightarrow 5(E'+1)/2
\end{array}$$

In each use of (10) we must check that the integers

$$1, \quad u, \quad v, \quad u + k + 2 - v, \quad k + 2, \quad u + k + 1$$

are distinct and within the set (9), and that the remaining integers in (9) can be distributed in $2k$ sets of four satisfying the requirement (3) in Lemma 1. If we can satisfy the requirements (3) and (9) for every $k \geqslant 0$, then from (4″) we can check that the "old" vertices have $\sigma_1 + 2(E' + 1) = \sigma'$. For the four "new" vertices we have Lemma 1 for k cases and the special sums in (10), so that

$$\begin{aligned} k2(E' + 1) + 5(E' + 1)/2 &= (E' + 1)(4k + 5)/2 \\ &= (E' + 1)(n + 3)/2 = \sigma'. \end{aligned}$$

So we shall be able to pass from an α for which K_n is supermagic to an α' for which K_{n+4} is supermagic. Since n corresponds to k and $n + 4$ corresponds to $k + 1$, the proof will be complete by induction on k. Of course we must check that if α has properties (7) and (8), then the α' which is obtained also has both properties (7) and (8), so that it will be suitable for the next step of the induction.

It remains to describe for each k how to choose u, v in (10) and how to choose the $2k$ sets x_1, x_2, x_3, x_4 to satisfy (3) and (8) and (9).

When $k = 0$ the simple solution $\alpha(e_{12}) = 1$ has property (7), and has property (8) vacuously. From (7′) we obtain $\alpha_1(e_{12}) = \alpha(e_{12}) + 7 = 8$. The

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choice $u = 5$, $v = 3$ satisfies (9), and (3) is satisfied vacuously. We apply (10) and obtain a solution for K_6, with property (8), as follows:

8	1	10	15	6	→ 40
↳	14	5	11	2	→ 40
	↳	9	3	13	→ 40
		↳	4	12	→ 40
			↳	7	→ 40
				└	→ 40

When $k = 1$, we start from the above solution for K_6. In the edge assignments for e_{12}, e_{34}, e_{56} we make the increment $8k + 7 = 15$. For the other edges, if $\alpha(e_{ij}) < 8$, we make the increment $8k + 6 = 14$; but if $\alpha(e_{ij}) > 8$, we make the increment $8k + 8 = 16$, exactly as in (7′) and (8′). The choice $u = 4$, $v = 2$, and the choice 7, 8, 9, 10; 11, 12, 13, 14 for a_1 to a_8 will satisfy (3) and (9). We apply (10) and (3′) and obtain a solution α' for K_{10}, with property (8). Before simplification α' appears as

8 + 15	1 + 14	10 + 16	15 + 16	6 + 14	7	46 − 7	11	46 − 11	(3′)
	14 + 16	5 + 14	11 + 16	2 + 14	46 − 8	8	46 − 12	12	
		9 + 15	3 + 14	13 + 16	46 − 9	9	46 − 13	13	
			4 + 14	12 + 16	10	46 − 10	14	46 − 14	
α_1				7 + 15					
					1	46 − 6	46 − 1	6	(10)
					46 − 3	4	46 − 4	3	
						23 + 2	2	46 − 2	
							5	46 − 5	
								23 − 2	

After simplification α' appears as

23	15	26	31	20	7	39	11	35	→ 207
↳	30	19	27	16	38	8	34	12	→ 207
	↳	24	17	29	37	9	33	13	→ 207
		↳	18	28	10	36	14	32	→ 207
			↳	22	1	40	45	6	→ 207
				↳	43	4	42	3	→ 207
					↳	25	2	44	→ 207
						↳	5	41	→ 207
							↳	21	→ 207
								└	→ 207

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When $k = 2$, we start from the above solution for K_{10} and follow the planned procedure. We use

$$u = 3, \quad v = 2, \quad k + 2 = 4, \quad u + k + 2 - v = 5, \quad u + k + 1 = 6$$

which agrees with (9). The sixteen remaining numbers are easily arranged in sets of four numbers which satisfy (3) for if a_1, a_2, a_3, a_4 are consecutive integers a, $a + 1$, $a + 2$, $a + 3$, then $a_1 + a_4 = a_2 + a_3$. So we may use the sets (7, 8, 9, 10), (11, 12, 13, 14), (15, 16, 17, 18), (19, 20, 21, 22) to obtain a solution α' for K_{14} with α' having properties (7) and (8).

For every odd $k \geqslant 3$, if there is a solution known for K_{4k+2}, we shall follow the planned procedure to obtain a solution for K_{4k+6}. We set $k = 2K + 1$. We use $u = 2$ and $v = 3$ and find that

$$u + k + 2 - v = 2K + 2, \quad k + 2 = 2K + 3, \quad u + k + 1 = 2K + 4.$$

This agrees with (9) providing $K \geqslant 1$ so that $2K + 2 \geqslant 4$. This explains the condition $k = 2K + 1 \geqslant 3$.

If K is odd, then $2K - 2 \equiv 0 \bmod 4$, so that the remaining integers in (9) fall into an even number of sets of four consecutive integers:

$$(4, 5, 6, 7), \ldots, (2K - 2, 2K - 1, 2K, 2K + 1),$$
$$(2K + 5, 2K + 6, 2K + 7, 2K + 8), \ldots, (8k + 3, 8k + 4, 8k + 5, 8k + 6).$$

If K is even, then $2K \equiv 0 \bmod 4$, so that the remaining integers in (9) fall with one exception into sets of four consecutive integers. The exceptional set $(2K, 2K + 1, 2K + 5, 2K + 6)$ retains the property $a_1 + a_4 = a_2 + a_3$ in (3). In both cases the entire construction is successful. Consideration of (3′) and (10) shows that if α has property (8), then α' will also have property (8).

For every even $k \geqslant 2$, we proceed as follows. We set $k = 2K$. We use $u = 3$ and $v = 2$ and find that

$$k + 2 = 2K + 2, \quad u + k + 2 - v = 2K + 3, \quad u + k + 1 = 2K + 4.$$

This agrees with (9) since $2K + 2 > 3$ for $k = 2K \geqslant 2$. The remaining integers in (9) are $4, 5, \ldots, 2K + 1$; $2K + 5$, $2K + 6, \ldots, 8k + 6$. Hence their distribution into an even number of sets of four integers satisfying (3) is exactly as in the case k odd.

Combining the cases $k = 0$, $k = 1$, even $k \geqslant 2$, odd $k \geqslant 3$, we see by induction on k that K_{4k+2} is supermagic for all $k \geqslant 0$.

We have noted in Theorem 1 that K_{4k} is not supermagic. As a near substitute we have the following theorem.

THEOREM 5. *If $k \geqslant 2$, then K_{4k} is magic under an α which uses the assignments* $1, 2, \ldots, E + 1$, *omitting the middle integer* $(E + 2)/2$.

Proof. We shall outline the proof, for it is a duplicate of the proof in Theorem

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4 in almost every respect, except that $E + 2$ replaces $E + 1$. However, we must modify (7) to eliminate $(E + 2)/2$, so we use

$$(7^*) \qquad \begin{cases} \alpha(e_{4t-3,4t-2}) = (E+2)/2 + t, & t = 1, 2, \ldots, k; \\ \alpha(e_{4t-1,4t}) = (E+2)/2 - t, & t = 1, 2, \ldots, k. \end{cases}$$

The construction of α_1 is like that in Theorem 4 with an increment of $2n + 3 = 8k + 3$ for the edges in (7*); and an increment of $2n + 2 = 8k + 2$ for the other edges if $\alpha(e_{ij}) < (E + 2)/2$; but an increment of $2n + 4 = 8k + 4$ if $\alpha(e_{ij}) > (E + 2)/2$. The pattern (10) becomes (10*) with $E' + 2$ replacing $E' + 1$. For odd $k = 2K + 1 \geqslant 3$, the choice is $u = 2$, $v = 3$. For even $k = 2K \geqslant 2$, the choice is $u = 3$, $v = 2$. The integers $\{x\}$ and $\{\bar{x}\}$, with $\bar{x} = (E' + 2) - x$, which remain to be used are $4 \leqslant x \leqslant 2K + 1$,

$$2K + 5 \leqslant x \leqslant 8k + 2.$$

The integers $\{x\}$ in the ranges $4 \leqslant x \leqslant 2K + 1$, $2K + 5 \leqslant x \leqslant 8k - 10$ (vacuous only when $k = 2$) can be distributed into an even number of sets of four integers satisfying (3) very much as in Theorem 4, with slightly different plans according as K is odd or even. The remaining twelve integers (never vacuous), $8k - 9 \leqslant x \leqslant 8k + 2$, cannot be distributed into an even number of sets of four. Instead, we use the following pattern:

$$(11^*) \qquad \begin{array}{cccc} 8k-9 & \overline{8k-9} & 8k-7 & \overline{8k-7} \\ 8k-2 & \overline{8k-2} & 8k-3 & \overline{8k-3} \\ 8k+1 & \overline{8k+1} & 8k-1 & \overline{8k-1} \\ \overline{8k-8} & 8k-8 & \overline{8k-6} & 8k-6 \\ \overline{8k-4} & 8k-4 & \overline{8k-5} & 8k-5 \\ \overline{8k+2} & 8k+2 & \overline{8k} & 8k \end{array}$$

which has $2(E' + 2)$ for each row sum and has $3(E' + 2)$ for each column sum. The patterns (3*), (10*), and (11*) preserve properties (7*) and (8*), where (3*) and (8*) are the analogues of (3) and (8) with $E + 2$ in place of $E + 1$. To complete the proof of Theorem 5, by induction on $k \geqslant 2$, we must produce a solution α for K_8 having properties (7*) and (8*). Witness the following:

$$\begin{array}{ccccccc|l} 16 & 7 & 22 & 4 & 26 & 12 & 18 & \rightarrow 105 \\ \hookrightarrow & 23 & 8 & 25 & 3 & 21 & 9 & \rightarrow 105 \\ & \hookrightarrow & 14 & 5 & 27 & 1 & 28 & \rightarrow 105 \\ & & \hookrightarrow & 24 & 2 & 29 & 6 & \rightarrow 105 \\ & & & \hookrightarrow & 17 & 10 & 20 & \rightarrow 105 \\ & & & & \hookrightarrow & 19 & 11 & \rightarrow 105 \\ & & & & & \hookrightarrow & 13 & \rightarrow 105 \\ & & & & & & \llcorner\!\!\longrightarrow & 105 \end{array}$$

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To illustrate the induction procedure (except that (3*) is vacuous) we show how α_1, (10*), and (11*) are used to find α' (omitting 34) for K_{12} from the above α (omitting 15) for K_8. Since $k = 2$ is even, we use $u = 3$, $v = 2$.

35	25	42	22	46	30	38	7	61	9	59	
	43	26	45	21	41	27	14	54	13	55	
		33	23	47	19	48	17	51	15	53	(11*)
			44	20	49	24	60	8	58	10	
				36	28	40	56	12	57	11	
					39	29	50	18	52	16	
α_1						32	1	62	67	6	
							64	3	65	4	
								37	2	66	(10*)
									5	63	
										31	

Reference

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